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Hopf Cyclic Cohomology and Chern Character of Equivariant K-Theories

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Abstract. We extend the Chern character construction of Neshveyev and Tuset to a map whose values lie in Hopf cyclic homology with coefficients, generalizing their definition of K-theory as well. We also introduce the sheaf of equivariant K-theory (with and without coefficients) similar to the equivariant cohomology of Block and Getzler. This construction is much more geometric (it is defined only for the case in which the Hopf algebra and the Hopf-module algebra are both algebras of functions on some spaces). Thus, we give a geometric definition of the corresponding Chern character, which takes values in a version of Block–Getzler's sheaf of equivariant cohomology.

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INTRODUCTION

In recent years, many papers dealing with the Hopf-cyclic cohomology of algebras and coalgebras were published. The main motivation for this study is a result due to Connes and Moscovici, who gave a construction of the characteristic map taking the Hopf-cyclic cohomology of a Hopf algebra with invertible antipode to the cyclic cohomology of an algebra on which the Hopf algebra acts. An important property of this map, justifying further investigations, is that the image of the map contains characteristic classes associated with certain differential operators (in the case treated by Connes and Moscovici, this was the transversal signature operator of a foliation). Later on, the ideas of Connes and Moscovici were developed by Crainic, Hajac, Khalkhali, Rangipour, Sommerhäuser, and other authors to a fully-fledged theory of Hopf-cyclic (co)homology of algebras and coalgebras, which made it possible to use a rather wide spectrum of coefficient modules, the so-called stable anti-Yetter–Drinfeld modules (SAYD for short). One can regard this cohomology theory as an algebraic (noncommutative) analog of the equivariant cohomology, namely, in some elementary cases (e.g., in the case of a finite group Hopf algebra), these cohomology spaces can be rather easily identified with a version of equivariant cohomology (although the general situation is not quite clear). The interested reader can find further details in [10].

As was mentioned above, from the very beginning, the Hopf-cyclic cohomology was the range of a characteristic map. However, it was never clear what the domain of this map should be. In the original papers of Connes and Moscovici, the characteristic map was associated with a pseudo-elliptic differential operator. In the subsequent papers, this question was answered by introducing the dual Hopf-cyclic homology theory, so that the characteristic map became a characteristic pairing. If we want to extend the analogy with equivariant cohomology, it seems that the most adequate domain of such a map should be given by an equivariant K-theory. However, it is still unclear what kind of equivariant K-theory should be considered. Possible constructions of equivariant K-theory of a module algebra over some Hopf algebra and the corresponding Chern character for the equivariant cyclic homology were proposed by Neshveyev and Tuset [8] and Akbarpour and Khalkhali [1]. Both constructions are quite similar and coincide if the Hopf algebra is quasitriangular. They are built on the notion of Hopf-equivariant modules over Hopf-module algebras and, although they work pretty well for K^0, it is not evident whether or not one can extend this algebraic construction to higher algebraic equivariant K-theory. In addition, the definition of these authors uses only the Hopf algebra as coefficients (both on the level of K-theory and in the corresponding Hopf-cyclic cohomology), and it is not evident how it can be replaced by a more general SAYD.

In the present paper, we suggest two different ways to answer the above questions: what is a Hopf-equivariant K-theory with coefficients and how can one introduce a Chern character on it with values in the Hopf-cyclic theory. First, we extend the Chern character construction of Neshveyev and Tuset to a map whose values lie in the Hopf cyclic homology with coefficients, generalizing their definition of K-theory as well. Second, we introduce a sheaf of equivariant K-theory (with and without coefficients), similar to the equivariant cohomology of Block and Getzler. This construction is much more geometric (it is defined only in the case when the Hopf algebra and the Hopf-module algebra are both algebras of functions on some spaces). So we give a geometric definition of the corresponding Chern character, which takes values in a version of Block–Getzler's sheaf of equivariant cohomology. Although we know that this sheaf can be described algebraically as the Hopf-cyclic cohomology with coefficients (the corresponding algebraic construction is described in simple cases in [10]), there is an unsolved question: Are these two constructions related?

The paper is organized as follows. In Sec. 1 we recall the definition of Hopf-cyclic homology with coefficients, as well as the definition of Block–Getzler's sheaf of equivariant forms, and formulate theorems analogous to the Hochschild–Kostant–Rosenberg theorem for them. Section 2 is devoted to the constructions of equivariant K-theories; in Sec. 2.1 we introduce a λ-stable K-theory that generalizes the equivariant K-theory of Neshveyev and Tuset [8], and in Sec. 2.2 we define the sheaf of K-theories à la Block–Getzler. The Chern character constructions that map λ-stable K-theory of a module algebra to its Hopf cyclic homology with coefficients and the sheaf of K-theories to the corresponding Block–Getzler equivariant cohomology are described in Sec. 3.

1. HOPF CYCLIC HOMOLOGY

1.1. The algebraic construction

Throughout the paper, $\mathcal{H}$ stands for a fixed Hopf algebra with invertible antipode S over a fixed characteristic zero field $\Bbbk$ and $\mathcal{M}$ is a *stable anti-Yetter–Drinfeld (SAYD)* module over $\mathcal{H}$. Assume that $\mathcal{M}$ is a left $\mathcal{H}$-comodule and right module. Throughout the text, we use the standard (Sweedler's) notation with superscripts for the comultiplications and coactions, e.g., $\Delta(h) = h^{(1)} \otimes h^{(2)}$ for all $h \in \mathcal{H}$ and $\Delta_{\mathcal{M}}(m) = m^{(-1)} \otimes m^{(0)}$ for every $m \in \mathcal{M}$. Under these assumptions, the anti-Yetter–Drinfeld condition takes the form

$$(mh)^{(-1)} \otimes (mh)^{(0)} = S(h^{(3)})m^{(-1)}h^{(1)} \otimes m^{(0)}h^{(2)}, \tag{1}$$

and the stability of $\mathcal{M}$ means that $m^{(0)}m^{(-1)} = m$ for all $m \in \mathcal{M}$.

Recall now the definition of cyclic (co)homology of an algebra with coefficients in stable anti-Yetter–Drinfeld modules. Let A be a (left) Hopf-module algebra over $\mathcal{H}$, i.e., there is an action $\mathcal{H} \otimes A \to A$ such that, for all $a,\ b \in A$ and all $h \in \mathcal{H}$, one has $h(ab) = h^{(1)}(a)h^{(2)}(b)$. Following [6], consider the cyclic module $C_*^{\mathcal{H}}(A, \mathcal{M})$, $C_n^{\mathcal{H}}(A, \mathcal{M}) = \mathcal{M} \otimes_{\mathcal{H}} A^{\otimes n+1}$. The cyclic operations are given by the formulas (we do not distinguish between the $\Bbbk$-linear and $\mathcal{H}$-linear tensor products in our notation):

$$\delta_i(m \otimes a_0 \otimes \cdots \otimes a_n) = \begin{cases} m \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n), & 0 \leqslant i \leqslant n-1, \\ m^{(0)} \otimes S^{-1}(m^{(-1)})(a_n)a_0 \otimes \cdots \otimes a_n), & i = n, \end{cases}$$

$$\sigma_j(m \otimes a_0 \otimes \cdots \otimes a_n) = m \otimes a_0 \otimes \cdots \otimes 1 \otimes \cdots \otimes a_n,\ 0 \leqslant j \leqslant n+1,$$

where 1 stands in the j-th place and

$$\tau_n(m \otimes a_0 \otimes \cdots \otimes a_n) = m^{(0)} \otimes S^{-1}(m^{(-1)})(a_n) \otimes a_0 \otimes \cdots \otimes a_{n-1}.$$

One can see that, due to stability and the anti-Yetter–Drinfeld condition on $\mathcal{M}$, these formulas determine a well-defined cyclic structure. The Hochschild, cyclic, and periodic cyclic homology (see [7]) of this cyclic module is called the *Hochschild, cyclic, and periodic Hopf-cyclic homology of A with coefficients in $\mathcal{M}$*, respectively. The cohomology is defined by the dualization of this construction (see [6] for details).

Example 1. Let $\mathcal{M} = \mathcal{H}_m$ be the SAYD module which coincides with $\mathcal{H}$ as a linear space. The action of $\mathcal{H}$ on $\mathcal{H}_m$ is the usual multiplication in $\mathcal{H}$, and the coaction $\Delta_{\mathcal{M}}$ is given by the formula $\Delta_{\mathcal{M}}(m) = S(m^{(3)})m^{(1)} \otimes m^{(2)}$. Then $\mathcal{H}_m$ is a SAYD-module. One can check that the maps $\Phi\colon C_*^{\mathcal{H}}(A, \mathcal{H}_m) \to C_*(A)$, where $\Phi(m \otimes a_0 \otimes \ldots a_n) = m \cdot (a_0 \otimes \cdots \otimes a_n)$, induce an isomorphism between the Hopf cyclic homology $HC_*^{\mathcal{H}}(A, \mathcal{H}_m)$ and the ordinary cyclic homology $HC_*(A)$.

Example 2. Let $\mathcal{M} = \mathcal{H}_\Delta$ be the SAYD module which coincides with $\mathcal{H}$ as a linear space. The coaction $\Delta_\mathcal{M}$ of $\mathcal{H}$ on $\mathcal{H}_\Delta$ is the usual comultiplication Δ in $\mathcal{H}$ and the action of $\mathcal{H}$ is given by the formula $m \cdot h = S(h^{(2)})mh^{(1)}$, $m \in \mathcal{H}_\Delta = \mathcal{H}$, $h \in \mathcal{H}$. Then $\mathcal{H}_\Delta$ is a SAYD-module. The Hopf cyclic homology $HC_*^{\mathcal{H}}(A, \mathcal{H}_\Delta)$ coincides with the equivariant cyclic homology of $\mathcal{H}$-module algebra A defined by Neshveyev and Tuset [8].

1.2. The Block–Getzler sheaf of equivariant forms

Let a Lie group G act on a manifold X and on itself with conjugations. This fixed, the family of open sets in G should be stable under the conjugations by elements of G. Let g be an element of G and let X^g be the set of points in X stabilized by g, i.e., $X^g = \{x \in X | g(x) = x\}$. Let $G^g = \{h \in G | gh = hg\}$ be the centralizer of g, and let $\mathfrak{g}^g$ be the Lie algebra of G^g, or, equivalently, $\mathfrak{g}^g = \{\xi \in \mathfrak{g} | Ad_g\xi = \xi\}$. Observe that $h(X^g) = X^g$ for all $h \in G^g$. Thus, X^g is a G^g-equivariant manifold. According to [2], define the equivariant differential forms on X as the sheaf on G whose stalk at g is given by $\Omega^\bullet_g(X, G) = \Omega^\bullet_{G^g}(X^g)$, where $\Omega^\bullet_{G^g}(X^g)$ stands for the completion of the usual G^g-equivariant forms on X^g in Cartan's sense: $\Omega^\bullet_{G^g}(X^g) = C_0^\infty(\mathfrak{g}^g, \Omega^\bullet(X^g))^{G^g}$, i.e., the G^g-invariant part of the germs at 0 of the smooth maps from $\mathfrak{g}^g$ to $\Omega^\bullet(X^g)$. There are two differentials d and ι in $\Omega^\bullet_{G^g}(X^g)$, $(d\omega)(X) = d(\omega(X))$ and $(\iota\omega)(X) = \iota_X(\omega(X))$, for arbitrary $X \in \mathfrak{g}^g$. Here d is the de Rham differential in $\Omega(X^g)$ and ι_X is the contraction of a differential form on X^g with the vector field induced by X. It follows from the Cartan identities that $(d + \iota)^2 = 0$. The cohomology of $\Omega^\bullet_{G^g}(X^g)$ with respect to the differential $d_G = d + \iota$ (Cartan differential) is called the *equivariant cohomology* of X^g.

To define the *sheaf of equivariant differential forms on* G, we must define the sections of the sheaf on open subsets (see Sec. 1 of [2]). To this end, we first let the group G act on the disjoint union of stalks $\bigcup_{g\in G} \Omega^\bullet_{G^g}(X^g)$ by translations, namely, for any $k \in G$, we put $\omega \mapsto k \cdot \omega \in \Omega^\bullet_{k\cdot G^g}(X^{k\cdot g})$. Here $k \cdot g$ and $k \cdot G^g$ stand for the conjugation of g and G^g by k. This action commutes with the equivariant differentials on both sides. One says (see [2]) that a point $h = g \exp \xi$ ($\xi \in \mathfrak{g}^g$, and hence $h \in G^g$) is *near* $g \in G$, if $X^h \subseteq X^g$ and $G^h \subset G^g$. By Lemma 1.3 of [2], the set of all points in G^g that are *near* g is an open equivariant neighborhood of g in G^g. Finally, one defines the local sections of our sheaf as the sections $\omega \in \Gamma(U; \bigcup_{g\in G} \Omega^\bullet_{G^g}(X^g))$ (U is an invariant open set) such that, for any h *near* g, $\omega_g|_{X^h\times\mathfrak{g}^h} = \omega_h$ (the restriction here is just the restriction of germs).

As was shown in [2], there is a natural isomorphism between the sheaf of equivariant cohomology groups associated to the above sheaf of equivariant differential forms and the sheaf of equivariant cyclic cohomology of the sheaf of functions on X which, for every Ad_G-equivariant open subset U of G, is just the Hopf-cyclic homology of $\mathcal{O}(G)$-comodule algebra $C^\infty(U)$ with coefficients in the SAYD module $\mathcal{O}(G)$. Observe that one should consider, instead of the tensor products of $C^\infty(X)$ and $\mathcal{O}(U)$ that appear in definitions of cyclic modules, their appropriate completions or plainly the functions on $U \times X^{\times n}$. Then all cyclic operations can be expressed in the terms of maps of this space.

Let now $\mathcal{M}$ be the module of coefficients over the Hopf algebra of functions on G. Assume that it is equal to the module of sections of a smooth equivariant vector bundle E on G. We can form the sheaf on G of equivariant E-valued differential forms on X, similarly to the sheaf of ordinary differential forms.

First of all, for every $g \in G$, we consider the exponential map $\exp_g\colon \mathfrak{g}^g \to G$, $X \mapsto g\exp(X)$, where $X \in \mathfrak{g}^g$ and $\exp\colon \mathfrak{g} \to G$ is the ordinary exponent map of G. Let $\widetilde{E}^g = \exp_g^* E \to \mathfrak{g}^g$ be the pullback bundle. The group G^g acts on its Lie algebra by the adjoint action Ad and the bundle $\widetilde{E}^g$ is equivariant with respect to this action if we induce the action on the bundle from the action of G on E, namely. just observe that the conjugation by any $h \in G^g$ takes g to itself and induces the adjoint action on $\mathfrak{g}^g$.

So we define the stalks of the sheaf $\Omega^\bullet(X, G; E)$ at g by

$$\Omega^\bullet_g(X, G; E) = \Omega^\bullet_{G^g}(X^g; E) = C_0^\infty(\mathfrak{g}^g, \Omega^\bullet(X^g) \otimes \widetilde{E}^g)^{G^g},$$

i.e., the G^g-equivariant differential forms on X^g with coefficients in the germs of sections of $\widetilde{E}^g$ at 0 (we let G^g act on this space combining the actions on X^g and $\widetilde{E}^g$). Cartan's differential on differential forms can be extended to these modules by putting $d_G(e) = 0$ for $e \in \widetilde{E}^g$. One further defines the space of sections of the sheaf $\Omega^\bullet(X, G; E)$ on an equivariant open subset $U \subseteq G$. As above, it is defined in such a way that $\omega_g|_{X^h\times\mathfrak{g}^h} = \omega_h$ for all h *near* g, where the term *near* means, as above, that $G^h \subseteq G^g$ and $X^h \subseteq X^h$ and, on the left-hand side, we consider the restriction of the germs of sections of $\widetilde{E}_g$ to the subspace $\mathfrak{g}^h$.

Finally, consider the sheaf $C^\bullet(G, X; \mathcal{M})$ of Hopf-cyclic complexes on G. Put

$$\Gamma(U; C^\bullet(G, X; \mathcal{M})) = C^\bullet_{\mathcal{O}(G)}(C^\infty(X), \mathcal{E}(U)) = \oplus_{n\geqslant 0}(\mathcal{E}(U) \otimes C^\infty(X)^{\otimes n+1})^G,$$

where $\mathcal{E}(U)$ stands for the SAYD $\mathcal{O}(G)$-module of sections of the restricted bundle $E|_U$ and the superscript G on the right-hand side for the space of G-invariants. As in the paper of Block and Getzler, one should take the topological point of view, so that $\Gamma(U; C^\bullet(G, X; \mathcal{M}))$ should be regarded as the space of (smooth) sections of the bundle $p^*(E|_U)$, where $p\colon U \times X^{\times n} \to U$ is the projection.

Then, as we have shown in [10], the Hochschild–Kostant–Rosenberg map of Block and Getzler can be extended to a quasi-isomorphism of the sheaf of equivariant differential forms with coefficients and the sheaf of Hopf-cyclic cohomology modules.

2. EQUIVARIANT K-THEORIES

2.1. λ-stable K-theory

Choose a linear map $\lambda\colon \mathcal{H} \to \Bbbk$.

Let V be a finite-dimensional $\mathcal{H}$-module. The action of $\mathcal{H}$ on V is determined by a homomorphism $\pi\colon \mathcal{H} \to End(V)$. The product $B = End(V) \otimes A$ has the structure of an $\mathcal{H}$-module algebra with the multiplication $(T_1 \otimes a_1)(T_2 \otimes a_2) = T_1T_2 \otimes a_1a_2$, $a_1, a_2 \in A$, $T_1, T_2 \in End(V)$, and an action of $\mathcal{H}$, $h \triangleright (T \otimes a) = \pi(h^{(1)})T\pi(S(h^{(3)})) \otimes h^{(2)} \cdot a$, $a \in A$, $T \in End(V)$, $h \in \mathcal{H}$.

Definition 1. Suppose that B is a left $\mathcal{H}$-module. An element $b \in B$ is said to be λ-*stable* if $\lambda(h)b = \lambda(h^{(1)})h^{(2)} \cdot b$ for each $h \in \mathcal{H}$.

Example 3. Let $\lambda = \varepsilon$ be the counit of the Hopf algebra $\mathcal{H}$. Then an element $b \in B$ is λ-stable if and only if b is $\mathcal{H}$-invariant, i.e., $h \cdot b = \varepsilon(h)b$ for any $h \in \mathcal{H}$.

Consider some general properties of λ-stable elements.

Proposition 1.

(1) *An element $b \in B$ is λ-stable if and only if $\lambda(h)b = \lambda(h^{(1)})S^{-1}(h^{(2)}) \cdot b$ for any $h \in \mathcal{H}$;*
(2) *if B is a $\mathcal{H}$-module algebra, then the set of λ-stable elements $B_\lambda = \{b \in B \,|\, b$ is λ-stable$\}$ is a subalgebra of B;*
(3) *if $b \in B$ is invertible and λ-stable, then b^{-1} is λ-stable.*

Proof. 1. Let b be λ-stable. Then

$$\begin{aligned}\lambda(h^{(1)})S^{-1}(h^{(2)}) \cdot b &= S^{-1}(h^{(2)}) \cdot \big(\lambda(h^{(1)})b\big) = S^{-1}(h^{(3)}) \cdot \big(\lambda(h^{(1)})h^{(2)} \cdot b\big)\\ &= \lambda(h^{(1)})\big(S^{-1}(h^{(3)})h^{(2)}\big) \cdot b = \lambda(h^{(1)})\varepsilon(h^{(2)})b = \lambda(h^{(1)}\varepsilon(h^{(2)}))b = \lambda(h)b.\end{aligned}$$

On the other hand, if $\lambda(h)b = \lambda(h^{(1)})S^{-1}(h^{(2)}) \cdot b$ for any $h \in \mathcal{H}$, then

$$\begin{aligned}\lambda(h^{(1)})h^{(2)} \cdot b &= h^{(2)} \cdot \big(\lambda(h^{(1)})b\big) = h^{(3)} \cdot \big(\lambda(h^{(1)})S^{-1}(h^{(2)}) \cdot b\big)\\ &= \lambda(h^{(1)})\big(h^{(3)}S^{-1}(h^{(2)})\big) \cdot b = \lambda(h^{(1)})\varepsilon(h^{(2)})b = \lambda(h^{(1)}\varepsilon(h^{(2)}))b = \lambda(h)b.\end{aligned}$$

2. Let a and b be λ-stable. Then

$$\lambda(h^{(1)})h^{(2)} \cdot (ab) = (\lambda(h^{(1)})h^{(2)} \cdot a)(h^{(3)} \cdot b) = \lambda(h^{(1)})a(h^{(2)} \cdot b) = \lambda(h)ab \quad \text{for any} \quad h \in \mathcal{H}.$$

3. Let b be λ-stable and invertible. Then

$$\begin{aligned}b\big(\lambda(h^{(1)})h^{(2)} \cdot b^{-1}\big) &= \lambda(h^{(1)})\big[\big(h^{(3)}S^{-1}(h^{(2)})\big) \cdot b\big](h^{(4)} \cdot b^{-1})\\ &= \big[h^{(3)} \cdot \big(\lambda(h^{(1)})S^{-1}(h^{(2)}) \cdot b\big)\big](h^{(4)} \cdot b^{-1}) = \lambda(h^{(1)})(h^{(2)} \cdot b)(h^{(3)} \cdot b^{-1})\\ &= \lambda(h^{(1)})h^{(2)} \cdot (bb^{-1}) = \lambda(h^{(1)})\varepsilon(h^{(2)}) = \lambda(h).\end{aligned}$$

Hence $\lambda(h)b^{-1} = b^{-1}b\,\big(\lambda(h^{(1)})h^{(2)} \cdot b^{-1}\big) = \lambda(h^{(1)})h^{(2)} \cdot b^{-1}$.

Proposition 2. *Let $\mathcal{H}_\lambda$ be the subalgebra in $\mathcal{H}$ generated by the elements $\lambda(h^{(1)})h^{(2)}$, $h \in \mathcal{H}$. Then an element b in a $\mathcal{H}$-module algebra B is λ-stable if and only if b is invariant under $\mathcal{H}_\lambda$.*

Proof. If b is λ-stable, then we have $\lambda(h^{(1)})h^{(2)} \cdot b = \lambda(h)b = \varepsilon\left(\lambda(h^{(1)})h^{(2)}\right) b$ for any $h \in \mathcal{H}$, which means that b is invariant under the element $\lambda(h^{(1)})h^{(2)}$. Hence, b is invariant under the subalgebra generated by the elements of this kind.

For an $\mathcal{H}$-module algebra A, consider all possible λ-stable idempotents $p \in End(V) \otimes A$, where V runs over all the finite-dimensional $\mathcal{H}$-modules. Two idempotents $p \in End(V) \otimes A$ and $p' \in End(V') \otimes A$ are said to be equivalent if there exist λ-stable elements $\gamma \in Hom(V, V') \otimes A$ and $\gamma' \in Hom(V', V) \otimes A$ such that $p = \gamma'\gamma$ and $p' = \gamma\gamma'$. The set of equivalence classes of λ-stable idempotents $\bar{C}^{\mathcal{H},\lambda}(A)$ is an Abelian semigroup.

We can equivalently describe the semigroup $\bar{C}^{\mathcal{H},\lambda}(A)$ as the set of λ-stable idempotents $p \in End(V) \otimes A$ modulo the equivalence relation generated by

(1) (conjugacy) $p \sim \gamma p \gamma^{-1}$, where $\gamma \in End(V) \otimes A$ is an invertible λ-stable element;
(2) (stabilization) $p \sim p \oplus 0 \in End(V \oplus V') \otimes A$.

Definition 2. The *λ-stable K-theory* $K^{\mathcal{H},\lambda}(A)$ of a $\mathcal{H}$-module algebra A is defined as the Grothendieck group of the monoid $\bar{C}^{\mathcal{H},\lambda}(A)$.

Example 4. If $\lambda = \varepsilon$ is the counit of the Hopf algebra $\mathcal{H}$, then λ-stable K-theory $K^{\mathcal{H},\lambda}(A)$ is generated by equivalence classes of $\mathcal{H}$-invariant idempotents; so it coincides with the equivariant K-theory $K_0^{\mathcal{H}}(A)$ defined in [8].

If $\lambda = 0$ is trivial, then $K^{\mathcal{H},\lambda}(A) = K(A)$ is the ordinary K-theory of the algebra A, because the condition of λ-stability is trivial here.

Let G be a finite group and let $\mathcal{H} = \mathbb{C}[G]$ be the group algebra of G. Assume that G acts on a smooth manifold X. Then $A = C^\infty(X)$ is an $\mathcal{H}$-module algebra. Let $\lambda\colon \mathcal{H} \to \mathbb{C}$ be a linear map. It is determined by its restriction $\lambda\colon G \to \mathbb{C}$ to G.

Proposition 3. $K^{\mathcal{H},\lambda}(A) = K^{G_\lambda}(A) = K^{G_\lambda}(X)$, *where $G_\lambda = \langle g | \lambda(g) \neq 0\rangle$ is the subgroup generated by the support of λ.*

Proof. The condition of λ-stability for a group-like element g of Hopf algebra looks as follows: $\lambda(g)a = \lambda(g)g \cdot a$, where $a \in A$. If $\lambda(g) = 0$, then the equality is obviously true; otherwise the element a must be g-invariant, $g \cdot a = a$. Hence, λ-stable elements are the same as elements which are invariant under the group elements from the set $\{g | \lambda(g) \neq 0\}$, and thus under G_λ.

The group $K^{\mathcal{H},\lambda}(A)$ is generated by G_λ-invariant idempotents $p \in End(V) \otimes A$, where V is a finite-dimensional representation of G, whereas $K^{G_\lambda}(A)$ is generated by the idempotents in $End(W) \otimes A$, where W runs over all the finite-dimensional representations of G_λ. Therefore, for the coincidence of the K-groups, it suffices to show that one can complement any finite-dimensional representation W of G_λ by another finite-dimensional representation W' of G_λ in such a way that the action of G_λ on $W \oplus W'$ can be extended to an action of G. This certainly holds for finite groups. Indeed, any finite-dimensional representation of G_λ is completely reducible, any irreducible representation of G_λ is a direct summand of the regular representation of G_λ, and the regular representation of the group $G_\lambda \subset G$ is a direct summand of the regular representation of G.

2.2. The sheaf of equivariant K-theories

Let G be a compact Lie group acting on a space X; X is assumed to be a smooth closed manifold. Let $E \to G$ be an equivariant vector bundle (where we let G act on itself by conjugations), or, more generally, let $\mathcal{E}$ be a G-equivariant sheaf of modules over the functions on G. In the case of a finite group G, this is the same as a family of representations of G indexed by the conjugacy classes in G. We are going to define a version of G-equivariant K-theory on X with coefficients in the G-equivariant module $\mathcal{E}$ of sections of E (more generally, we can speak about an arbitrary sheaf $\mathcal{E}$).

Let $h \in G$ be an arbitrary element. Recall that X^h is the subspace of elements preserved by h and $G^h = \{k \in G \mid hk = kh\}$. Thus, G^h is a subgroup of G, namely, the centralizer of h. **Here and below, we assume that the centralizers of elements of G are compact.** For a G-equivariant vector bundle E on G (with respect to the adjoint action of G on itself), we shall denote by $E^h \subseteq E$ the subbundle of elements preserved by g; this is a G^h-equivariant bundle over G^h whose sections will be denoted by $\mathcal{E}^h$ (similarly, we can speak about the subsheaf preserved by h).

We are now able to give the definition mentioned above. Let us begin with the definition of an extraordinary equivariant cohomology without coefficients (compare [2, 10] and Sec. 1.2 above).

Let h^* be an extraordinary cohomology theory with values in commutative $\mathbb{R}$-algebras. We shall use the term *G-equivariant extraordinary (co)homology of X associated with h^** for the graded G-equivariant sheaf $\mathcal{H}^*_G(X)$ defined as follows: the stalk of $\mathcal{H}^*_G(X)$ at a point $g \in G$ is equal to $h^*_{G^g}(X^g)$, where h^*_G stands for the usual equivariant cohomology of Borel. The group G permutes the stalks of this sheaf; to see this, just observe that the action by g sends X^h to $X^{ghg^{-1}}$ for any G-space X, and the adjunction by g on G is an isomorphism of G^h and $G^{ghg^{-1}}$. In particular, G^h acts on $h^*_{G^h}(X^h)$. These observations are sufficient to define an equivariant sheaf for a discrete group. In case of a Lie group, we need to explain the local structure of sections.

Introduce a topology in the union of stalks $\bigcup_{g\in G} h^*_{G^g}(X^g)$ by defining the set of possible sections. We say that a point $h = g\exp\xi$, for a vector $\xi \in \mathfrak{g}^g$ such that $h \in G^g$, is *near* $g \in G$ if $X^h \subseteq X^g$ and $G^h \subset G^g$. According to Lemma 1.3 of [2] again, the set of all points in G^g which are *near* g form an open neighborhood of g in G^g. Define a *section* $\omega \in \Gamma(h^*_{G^g}(X^g))$ for any adjoint-invariant open subset $U \subseteq G$ as a family of the elements $\omega_g \in h^*_{G^g}(X^g)$, $g \in U$ such that, for any h which is *near* g, we have $\phi^{G^h}_{X^h}(\omega_g) = \omega_h$, where ϕ^H_Y is the natural map of the equivariant cohomology induced by an equivariant map $Y \to X$, $H \to G$ for a G-space X and an H-space Y.

Observe that $\mathcal{H}^*_G(X)$ is an equivariant sheaf of modules over the sheaf of rings with stalks $h^*_G(pt) = \mathcal{O}^h_G$, which is in its turn a module over the equivariant sheaf $\mathcal{O}_G$ of functions on G (it is a subsheaf of the sheafification of the union of the straight lines $\mathbb{R}$ over the points of G with respect to the equivariant topology of sections, see above). Let now $\mathcal{E}$ be an equivariant sheaf of $\mathcal{O}_G$-modules on G; define the *equivariant cohomology of X with coefficients in $\mathcal{E}$* to be equal to the equivariant sheafification of the union of tensor products $(\mathcal{H}^*_G(X)_g \otimes_{\mathcal{O}_g} \mathcal{E}_g)^{G^g}$ (the subscript g indicates the stalk at g, while the superscript G, where G is a group, indicates the subspace of elements invariant under the action of the whole group). This is the same as the equivariant tensor product of sheaves.

Thus, for any equivariant vector bundle $E \to G$ (more generally, for an equivariant sheaf on G), we have a graded equivariant sheaf $\mathcal{H}^*_G(X, E)$. In the case under consideration in the paper, we put $h^* = K^*_{\mathbb{R}}$ and denote the corresponding sheaf by $\mathcal{K}^*_G(X, E)$. Our prime interest is in $\mathcal{K}^0(X, E)$.

In the particular case $h^* = H^*(-, \mathbb{R})$ (and when X is a manifold), we obtain the Block–Getzler sheaf of equivariant de Rham cohomology in this way. To see this, recall that the corresponding definition in their paper is given at the level of equivariant differential forms by the sheafification of the corresponding set of stalks of equivariant forms. Since the sequence of sheaves is exact if and only if it is exact on the level of stalks, we conclude that the corresponding sheaf of cohomology is equal to the sheafification of the equivariant cohomology of stalks.

It is easy to see that the usual statements concerning equivariant theories (such as exact sequences, equivariant homotopy invariance, etc.) hold here as well, since the constructions used here are functorial with respect to the G-equivariant maps (we also use the fact that the exactness of all sequences of sheaves follows from their exactness on the level of stalks).

Next, let $F \to G$ be another G-equivariant vector bundle and let $\phi\colon F \to E$ be an equivariant map. As one can immediately see, ϕ induces a map on the level of stalks, and hence a map $\phi_*\colon \mathcal{K}^*_G(X, F) \to \mathcal{K}^*_G(X, E)$. Moreover, if $0 \to F_0 \to E \to F_1 \to 0$ is a short exact sequence, then the same exact sequence occurs on the level of stalks (since we assume that G is a compact group). The same conclusions follow for an arbitrary exact sequence of sheaves of $\mathcal{O}$-modules on G.

Now let us consider a few particular cases.

Example 5.

(1) Let $G = 1$ and $E = \mathbb{R}$. Then $\mathcal{K}^*_G(X, E) = K^*_{\mathbb{R}}(X) = K^*(X) \otimes \mathbb{R}$. More generally, if $E = \mathbb{R}^n$,

then $\mathcal{K}^*_G(X, E) = (K^*(X) \otimes \mathbb{R})^{\oplus n}$.

(2) Let the action of G on X be free. Then $X^g = \begin{cases} \varnothing, & g \neq 1 \\ X, & g = 1. \end{cases}$ So $\mathcal{K}^*_G(X, E)$ is the skyscraper sheaf whose stalk at $1 \in G$ is equal to $(K^*(X) \otimes E_1)^G$ (and 0 otherwise).

(3) Let $E = \mathbb{R}$ with the trivial action. Then the stalk of $\mathcal{K}^*_G(X, E)$ at a point $h \in G$ is equal to $K^*_{\mathbb{R}}(X^h)^{G^h}$, where $K^*_{G^h}(X^h)$ is the Grothendieck group of G^h-equivariant vector bundles on X^h.

(4) Let $\mathbb{E}$ be the skyscraper sheaf at $1 \in G$ whose stalk is given by a linear representation V of G. Then $\mathcal{K}^*_G(X, E)$ is also a skyscraper sheaf, and its only nontrivial stalk is equal to $K^*_G(X) \otimes_{\mathbb{Z}} V$.

3. CHERN CHARACTER

3.1. λ-stable Chern character

We define the *Chern character* in the λ-stable K-theory to the Hopf cyclic homology with coefficients in a SAYD module $\mathcal{M}$ in the form $ch^\lambda_n\colon K^{\mathcal{H},\lambda}(A) \to Hom_{\Bbbk}\left(\mathcal{M}, HC^{\mathcal{H}}_{2n}(A, \mathcal{M})\right)$.

Let $p = \sum_i T_i \otimes a_i \in End(V) \otimes A$ be a λ-stable idempotent and $m \in \mathcal{M}$. The result $ch^\lambda_n([p])(m)$ of application of the Chern character to $[p] \in K^{\mathcal{H},\lambda}(A)$ is, by definition, the homology class of the element

$$\sum_{i_0, i_1, \ldots, i_{2n}} \lambda(m^{(-2)}) \mathrm{Tr}\left(\pi(m^{(-1)}) T_{i_0} T_{i_1} \ldots T_{i_{2n}}\right) m^{(0)} \otimes a_{i_0} \otimes \cdots \otimes a_{i_{2n}} \in C^{\mathcal{H}}_{2n}(A, \mathcal{M}).$$

Theorem 3.1. *The map ch^λ_n is well defined.*

Proof. Let $p = \sum_i T_i \otimes a_i \in End(V) \otimes A$ be a λ-stable idempotent and let $m \in \mathcal{M}$. The map ch^λ_n can be regarded as a part of the following commutative diagram:

$$\begin{array}{ccccc} p \in K(B_\lambda) & \longrightarrow & & \longrightarrow & K^{\mathcal{H},\lambda}(A) \\ \Big\downarrow ch_n & & & & \Big\downarrow ch^\lambda_n \\ HC_{2n}(B_\lambda) & \xrightarrow{\Psi^{\lambda,m}_*} & HC^{\mathcal{H}}_{2n}(B, \mathcal{M}) & \xrightarrow{\Phi_*} & HC^{\mathcal{H}}_{2n}(A, \mathcal{M}) \end{array}$$

where $B = End(V) \otimes A$, B_λ is the subalgebra of λ-stable elements of B, and $\Psi^{\lambda,m}_*$ and Φ_* are the homomorphisms of cyclic homologies induced by the maps $\Psi^{\lambda,m}$ and Φ defined below.

The map $\Psi^{\lambda,m}\colon C_*(B_\lambda) \to C^{\mathcal{H}}_*(B, \mathcal{M})$ can be defined for any $\mathcal{H}$-module algebra B; it is given by the formula $\Psi^{\lambda,m}(b_0 \otimes b_1 \otimes \cdots \otimes b_n) = \lambda(m^{(-1)}) m^{(0)} \otimes b_0 \otimes b_1 \otimes \cdots \otimes b_n$, where $b_0 \otimes b_1 \otimes \cdots \otimes b_n$ is an element of $C_n(B_\lambda)$.

Proposition 5. *The map $\Psi^{\lambda,m}$ is a morphism of cyclic modules, and thus it defines a homomorphism $\Psi^{\lambda,m}_*\colon HC_*(B_\lambda) \to HC^{\mathcal{H}}_*(B, \mathcal{M})$ of cyclic homologies.*

Proof. We just need to check that $\Psi^{\lambda,m}$ commutes with the operators of cyclic module structure. This can be done by straightforward calculations. For example, $\Psi^{\lambda,m} \circ \tau = \tau \circ \Psi^{\lambda,m}$ since

$$\begin{aligned} (\Psi^{\lambda,m} \circ \tau)(b_0 \otimes b_1 \otimes \cdots \otimes b_n) &= \Psi^{\lambda,m}(b_n \otimes b_0 \otimes \cdots \otimes b_{n-1}) \\ &= \lambda(m^{(-1)}) m^{(0)} \otimes b_n \otimes b_0 \otimes \cdots \otimes b_{n-1}, \\ (\tau \circ \Psi^{\lambda,m})(b_0 \otimes b_1 \otimes \cdots \otimes b_n) &= \tau(\lambda(m^{(-1)}) m^{(0)} \otimes b_0 \otimes b_1 \otimes \cdots \otimes b_n) \\ &= \lambda(m^{(-1)}) (m^{(0)})^{(0)} \otimes S^{-1}((m^{(0)})^{(-1)}) \cdot b_n \otimes b_0 \otimes \cdots \otimes b_{n-1} \\ &= m^{(0)} \otimes \lambda(m^{(-2)}) S^{-1}(m^{(-1)}) \cdot b_n \otimes b_0 \otimes \cdots \otimes b_{n-1} \\ &= \lambda(m^{(-1)}) m^{(0)} \otimes b_n \otimes b_0 \otimes \cdots \otimes b_{n-1}, \end{aligned}$$

where the last equality follows from the first statement of Proposition 1.

The map

$$\Phi\colon C_*^{\mathcal{H}}(End(V)\otimes A, M)\to C_*^{\mathcal{H}}(A,M)$$

is defined by the formula

$$\Phi(m\otimes(T_0\otimes a_0)\otimes\dots(T_n\otimes a_n)) = \mathrm{Tr}\left(\pi(m^{(-1)})T_0\dots T_n\right)m^{(0)}\otimes a_0\otimes\dots\otimes a_n,$$

where $m\otimes(T_0\otimes a_0)\otimes\dots(T_n\otimes a_n)\in C_n^{\mathcal{H}}(End(V)\otimes A, M)$. It does not depend on λ or m.

Proposition 6. *The map Φ is a morphism of cyclic modules.*

Proof. One must check that Φ commutes with the operators of cyclic module. For example,

$$\begin{aligned}
&(\Phi\circ\tau)(m\otimes(T_0\otimes a_0)\otimes\dots(T_n\otimes a_n))\\
&=\Phi(m^{(0)}\otimes S^{-1}(m^{(-1)})\triangleright(T_n\otimes a_n)\otimes(T_0\otimes a_0)\otimes\dots(T_{n-1}\otimes a_{n-1}))\\
&=\Phi(m^{(0)}\otimes(\pi(S^{-1}(m^{(-1)}))T_n\pi(m^{(-3)})\otimes S^{-1}(m^{(-2)})\cdot a_n)\otimes(T_0\otimes a_0)\otimes\dots(T_{n-1}\otimes a_{n-1}))\\
&=\mathrm{Tr}(\pi(m^{(-1)})\pi(S^{-1}(m^{(-2)}))T_n\pi(m^{(-4)})T_0\dots T_{n-1})m^{(0)}\otimes S^{-1}(m^{(-3)})\cdot a_n\otimes a_0\otimes\dots\otimes a_{n-1}\\
&=\mathrm{Tr}(T_n\pi(m^{(-2)})T_0\dots T_{n-1})m^{(0)}\otimes S^{-1}(m^{(-1)})\cdot a_n\otimes a_0\otimes\dots\otimes a_{n-1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(\tau\circ\Phi)(m\otimes(T_0\otimes a_0)\otimes\dots(T_n\otimes a_n)) = \tau(\mathrm{Tr}(\pi(m^{(-1)})T_0\dots T_n)m^{(0)}\otimes a_0\otimes\dots\otimes a_n)\\
&=\mathrm{Tr}(\pi(m^{(-1)})T_0\dots T_n)(m^{(0)})^{(0)}\otimes S^{-1}((m^{(0)})^{(-1)})\cdot a_n\otimes a_0\otimes\dots\otimes a_{n-1}\\
&=\mathrm{Tr}(T_n\pi(m^{(-2)})T_0\dots T_{n-1})m^{(0)}\otimes S^{-1}(m^{(-1)})\cdot a_n\otimes a_0\otimes\dots\otimes a_{n-1}
\end{aligned}$$

The other commutation relations can be verified similarly.

Thus, the element $ch_n^\lambda(p)(m)$ can be regarded as a composition of the conventional Chern character and homomorphisms of cyclic homologies. Hence, $ch_n^\lambda(p)(m)$ is well defined provided that the idempotent p is fixed. Moreover, the result is not changed if the idempotent p is replaced by its conjugate $\gamma p\gamma^{-1}$, where γ is an invertible element of B_λ.

On the other hand, by definition, $ch_n^\lambda(p) = ch_n^\lambda(p\oplus 0)$. Thus, the element $ch_n^\lambda(p)(m)$ does not depend on the choice of a representative idempotent in the equivalence class $[p]\in K^{\mathcal{H},\lambda}(A)$.

Proposition 3.4. *If $\lambda\colon\mathcal{H}\to\Bbbk$ is a character, then the Chern character ch_n^λ maps the group $K^{\mathcal{H},\lambda}(A)$ to the subspace $Hom_{\mathcal{H}}\left(\mathcal{M}, HC_{2n}^{\mathcal{H}}(A,\mathcal{M})\right)\subset Hom_{\Bbbk}\left(\mathcal{M}, HC_{2n}^{\mathcal{H}}(A,\mathcal{M})\right)$, where $\mathcal{H}$ acts on $HC_{2n}^{\mathcal{H}}(A,\mathcal{M})$ trivially, i.e., $\alpha\cdot h=\varepsilon(h)\alpha$, $\alpha\in HC_{2n}^{\mathcal{H}}(A,\mathcal{M})$, $h\in\mathcal{H}$.*

Proof. Let $p=\sum_i T_i\otimes a_i\in End(V)\otimes A$ be a λ-stable idempotent, $m\in\mathcal{M}$, and $h\in\mathcal{H}$. Then

$$\begin{aligned}
&ch_n^\lambda([p])(mh) = \sum_{i_0,i_1,\dots,i_{2n}}\lambda((mh)^{(-2)})\mathrm{Tr}\left(\pi((mh)^{(-1)})T_{i_0}T_{i_1}\dots T_{i_{2n}}\right)(mh)^{(0)}\otimes a_{i_0}\otimes\dots\otimes a_{i_{2n}}\\
&=\sum_{i_0,i_1,\dots,i_{2n}}\lambda(S(h^{(5)})m^{(-2)}h^{(1)})\mathrm{Tr}\left(\pi(S(h^{(4)}))\pi(m^{(-1)})\pi(h^{(2)})T_{i_0}T_{i_1}\dots T_{i_{2n}}\right)m^{(0)}h^{(3)}\otimes a_{i_0}\otimes\dots\otimes a_{i_{2n}}\\
&=\sum_{i_0,i_1,\dots,i_{2n}}\lambda(S(h^{(5)}))\lambda(m^{(-2)})\lambda(h^{(1)})\mathrm{Tr}\left[\pi(m^{(-1)})\pi(h^{(2)})T_{i_0}T_{i_1}\dots T_{i_{2n}}\pi(S(h^{(4)}))\right]\\
&\qquad m^{(0)}\otimes h^{(3)}\cdot(a_{i_0}\otimes\dots\otimes a_{i_{2n}})\\
&=\sum_{i_0,i_1,\dots,i_{2n}}\lambda(S(h^{(2n+5)}))\lambda(m^{(-2)})\lambda(h^{(1)})\mathrm{Tr}\left[\pi(m^{(-1)})\pi(h^{(2)})T_{i_0}T_{i_1}\dots T_{i_{2n}}\pi(S(h^{(2n+4)}))\right]\\
&\qquad m^{(0)}\otimes h^{(3)}\cdot a_{i_0}\otimes\dots\otimes h^{(2n+3)}\cdot a_{i_{2n}}\\
&=\sum_{i_0,i_1,\dots,i_{2n}}\lambda(S(h^{(2n+7)}))\lambda(m^{(-2)})\cdot\\
&\qquad\mathrm{Tr}\left[\pi(m^{(-1)})\lambda(h^{(1)})\pi(h^{(2)})T_{i_0}\pi(S(h^{(4)}))\pi(h^{(5)})T_{i_1}\dots T_{i_{2n}}\pi(S(h^{(2n+6)}))\right]\cdot\\
&\qquad m^{(0)}\otimes h^{(3)}\cdot a_{i_0}\otimes h^{(6)}\cdot a_{i_1}\dots\otimes h^{(2n+5)}\cdot a_{i_{2n}}
\end{aligned}$$

$$= \sum_{i_0,i_1,\dots,i_{2n}} \lambda(S(h^{(2n+4)}))\lambda(m^{(-2)})\lambda(h^{(1)})\cdot$$
$$\mathrm{Tr}\left[\pi(m^{(-1)})T_{i_0}\pi(h^{(2)})T_{i_1}\dots T_{i_{2n}}\pi(S(h^{(2n+3)}))\right]\cdot$$
$$m^{(0)}\otimes a_{i_0}\otimes h^{(3)}\cdot a_{i_1}\dots\otimes h^{(2n+2)}\cdot a_{i_{2n}} = \dots$$
$$\sum_{i_0,i_1,\dots,i_{2n}} \lambda(S(h^{(2)}))\lambda(m^{(-2)})\lambda(h^{(1)})\mathrm{Tr}\left[\pi(m^{(-1)})T_{i_0}T_{i_1}\dots T_{i_{2n}}\right] m^{(0)}\otimes a_{i_0}\otimes a_{i_1}\dots\otimes a_{i_{2n}}$$
$$= \sum_{i_0,i_1,\dots,i_{2n}} \lambda(h^{(1)}S(h^{(2)}))\lambda(m^{(-2)})\mathrm{Tr}\left[\pi(m^{(-1)})T_{i_0}T_{i_1}\dots T_{i_{2n}}\right] m^{(0)}\otimes a_{i_0}\otimes a_{i_1}\dots\otimes a_{i_{2n}}$$
$$= \varepsilon(h)ch_n^{\lambda}([p])(m).$$

Here we have used, $2n+1$ times, the equality

$$\sum_i \lambda(h^{(1)})\pi(h^{(2)})T_i\pi(S(h^{(4)}))\otimes h^{(3)}\cdot a_i = \sum_i \lambda(h)T_i\otimes a_i,$$

which means that the element p is λ-stable.

Remark 1. We can reformulate the Chern character as a pairing of the K-group with the Hopf cyclic cohomology $\langle\cdot,\cdot\rangle^{\lambda}\colon K^{\mathcal{H},\lambda}(A)\times HC^{2n}_{\mathcal{H}}(A,\mathcal{M})\to Hom_{\Bbbk}(\mathcal{M},\Bbbk)$ or, if λ is a character, as a pairing $\langle\cdot,\cdot\rangle^{\lambda}\colon K^{\mathcal{H},\lambda}(A)\times HC^{2n}_{\mathcal{H}}(A,\mathcal{M})\to Hom_{\mathcal{H}}(\mathcal{M},\Bbbk)$.

When $\lambda=\varepsilon$ and $\mathcal{M}=\mathcal{H}_{\Delta}$, one obtains the Chern character considered in [8]. In this case, $Hom_{\mathcal{H}}(\mathcal{M},\Bbbk)$ is the space $R(\mathcal{H})$ of $\mathcal{H}$-invariant linear functionals on $\mathcal{H}$.

Example 6. Let G be a finite group that acts on a smooth manifold X. Let $\mathcal{H}=\mathbb{C}[G]$ be the group algebra of G and let $A=C^{\infty}(X)$ be the algebra of smooth functions. A SAYD module over $\mathcal{H}$ can be regarded as sections of an equivariant sheaf $\mathcal{M}=\bigoplus_{g\in G}\mathcal{M}_g$ over G. Denote by ${}_GX=\{(g,x)\in G\times X\mid gx=x\}$ the Baum–Schneider space, and let ${}_G\mathcal{M}=p_1^*(\mathcal{M})$ be the induced sheaf over ${}_GX$. The Hochschild–Kostant–Rosenberg theorem [10] identifies the periodic Hopf cyclic homology $HP^{\mathcal{H}}_*(A,\mathcal{M})$ with the cohomology of equivariant de Rham forms $\Omega^*({}_GX,{}_G\mathcal{M})^G$.

Let $\lambda\colon G\to\mathbb{C}$ and $m=\sum_g m_g\in\mathcal{M}$. If $p\in End(V)\otimes A$ is a λ-stable idempotent, then its Chern classes are the cohomology classes of de Rham closed forms

$$ch_n^{\lambda}(p)(m)(g,x)=\frac{1}{(2n)!}\lambda(g)m_g\mathrm{Tr}(\pi(g)p(x)dp(x)^{2n})\in\Omega^{2n}({}_GX,{}_G\mathcal{M})^G.$$

3.2. Sheaf Chern character

In this section, we define the Chern character on the sheaf of equivariant K-theories. It takes values in the Block–Getzler sheaf of equivariant cohomology (with or without coefficients). This construction is in a sense a generalization of the last example of the previous section; however, here the domain of the characteristic map is the sheaf version of the equivariant theory rather than the λ-stable K-theory.

We begin with the coefficient-free case and give a geometric definition first; we closely follow the presentation in [2]. Let $\gamma\in\Gamma(U,\mathcal{K}^0_G(X))$ be a section of the sheaf $\mathcal{K}^0_G(X)$ over an equivariant open subset $U\subseteq G$. For any $g\in U$, let $\gamma(g)\in\mathcal{K}^0_G(X)_g=K^0_{G^g}(X^g)$ be the value of the section at g (on the right-hand side, there stands the stalk of our sheaf at g). We can assume that $\gamma(g)$ is represented by an equivariant vector bundle $\xi_g\to X^g$. Moreover, for all h which are near g (in the sense of the Secs. 1.2 and 2.2), the values of the section at h are represented by restrictions to X^h of the same vector bundle.

Choose a G^g-invariant connection ∇ on ξ_g. We associate with it an *equivariant connection*, i.e., an operator $\nabla_{\mathfrak{g}^g}$ in $C^{\infty}(\mathfrak{g}^g,\,\Omega^{\bullet}(X^g,\xi_g))^{G^g}[[u]]$ (where u is a formal parameter of degree -2) given by $(\nabla_{\mathfrak{g}^g}\omega)(X)=\iota(X)(\omega(X))+u\nabla(\omega(X))$ for all $X\in\mathfrak{g}^g$ (here $\iota(X)$ stands for the contraction along the vector field operator). This operator verifies the Leibnitz rule with respect to the equivariant differential $d_{\mathfrak{g}^g}=ud+\iota$ (compare with Sec. 1.2), and we can define the equivariant curvature of $\nabla_{\mathfrak{g}}$ by setting $F_{\mathfrak{g}^g} = u^{-1}\nabla^2_{\mathfrak{g}^g}$. Then the equivariant Chern character form of ξ_g is given by $ch_G(\gamma)(g) = \mathrm{Tr}(\exp(-F_{\mathfrak{g}^g}))$. One can show in a usual way that this form is always closed and that its equivariant cohomology class does not depend on the choice of connections. Applying this construction at every point $g \in U$ (choosing a cover U by the sets of near points), we obtain a section of the Block–Getzler sheaf of equivariant cohomology (the local sections fit on the intersections since the vector bundles associated to "near" elements are isomorphic).

Let now $\gamma \in \Gamma(U, \mathcal{K}^0_G(X, E))$ be a section. We can assume that this section is equal at a point $g \in U$ to the tensor product $[\xi_g] \otimes \varphi$, where ξ_g is a G^g-equivariant vector bundle on X^g and φ is (the germ of) a section of the coefficient bundle E. At near points, the section is equal to the restriction of this element.

We consider a E_g-valued G^g-equivariant connection $\nabla^{E_g}_{\mathfrak{g}^g}$ of $\xi^g \to X^g$, which is just an operator on $C^\infty(\mathfrak{g}^g, \Omega^\bullet(X^g, \xi_g) \otimes E_g)^{G^g}[[u]]$ (recall that E_g is the space of germs of sections of the coefficient bundle in $g \in G$) that verifies the relation

$$\nabla^{E_g}_{\mathfrak{g}^g}(\alpha \wedge \omega) = d_{\mathfrak{g}^g}\alpha \wedge \omega + (-1)^{|\alpha|}\alpha \wedge \nabla^{\xi^g}_{\mathfrak{g}^g}(\omega) \tag{2}$$

for any $\omega \in C^\infty(\mathfrak{g}^g, \Omega^\bullet(X^g, \xi_g) \otimes E_g)^{G^g}[[u]]$ and any equivariant differential form α. One can obtain this sort of connection by equivariantizing the E_g-valued invariant connection on ξ^g as described above. Here the notion of V-valued G^g-invariant connection, where V is a G^g-module, is a straightforward generalization of the ordinary notion of connection, where the usual Leibnitz rule is replaced by an equation similar to (2). We pass to the equivariant curvature $\tilde{F}^{E_g}_{\mathfrak{g}^g} = u^{-1}(\nabla^{E_g}_{\mathfrak{g}^g})^2$. In this case, $F^{E_g}_{\mathfrak{g}^g}$ is a germ of operator-valued functions on $\mathfrak{g}^g$; more accurately,

$$F^{E_g}_{\mathfrak{g}^g} \in C^\infty(\mathfrak{g}^g, \Omega^\bullet(X^g, End(\xi_g)) \otimes End(E_g))^{G^g}[[u]].$$

We put $ch_{G,\,E}(\gamma)(g) = \mathrm{Tr}(\exp(-F^{E_g}_{\mathfrak{g}^g}))(\varphi)$. This is an element in the stalk of the sheaf of E-valued differential forms on G. A direct computation shows that it is closed and its class does not depend on the choices made above.

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