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Homogenization Estimates of Operator Type for an Elliptic Equation with Quasiperiodic Coefficients

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Abstract. Error estimates for homogenization in L^2- and H^1-norms for an equation with rapidly oscillating quasiperiodic coefficients are studied.

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INTRODUCTION

Second-order differential elliptic operators of divergent and nondivergent types acting on the entire space $\mathbb{R}^d\,(d \geqslant 2)$ are considered. The coefficients of the operators are rapidly oscillating quasiperiodic functions, and ε is a small parameter determining the fast oscillation. The zeroth and the first approximations are constructed for the resolvent of the operators in the operator norms $\|\cdot\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}$ and $\|\cdot\|_{L^2(\mathbb{R}^d)\to H^1(\mathbb{R}^d)}$, respectively, with an error of the order of ε. The approximations are related to the homogenized operators which are similar to the original ones in their structure and have constant coefficients. As is usual in homogenization theory, to find these coefficients, there is a "problem on the cell," whose solvability in the quasiperiodic setting meets with difficulties because this problem is degenerate. Small denominators can occur here. For this reason, a "frequency condition" arises, which is a condition on the frequency vectors entering the formation of quasiperiodic functions. It should be noted that the frequency condition holds for the operators "in general position." This condition enables us to control the degeneration of the problem on the cell. Only a degeneration of power-law nature is admitted, and, as a result, the problem on the cell turns out to be "hypoelliptic" and is well posed in a scale of Sobolev spaces.

For nondivergent equations, all solvability problems (in the entire space, in a bounded domain, on the periodicity cell) have positive answer under an additional condition on the coefficient matrix. This additional condition is known, it is the "cone" condition or Cordes condition, in which a constraint on the scattering of eigenvalues of the coefficient matrix is imposed. In the present paper, we use the apparatus of reducing nondivergent equations with symmetric coefficient matrix to divergent equations with asymmetric coefficient matrix. Thanks to this approach, operator approximations are constructed, more or less, in the same way for divergent and nondivergent equations. An important role for nondivergent equations is played here by the so-called "acute angle inequality," which is proved here for quasiperiodic functions.

1. DIVERGENT EQUATION: ZEROTH APPROXIMATION

The resolvent equation on the entire space $\mathbb{R}^d$ is studied,

$$\begin{aligned} u_\varepsilon \in H^1(\mathbb{R}^d), \quad (\mathcal{A}_\varepsilon + 1)u_\varepsilon = f, \quad f \in L^2(\mathbb{R}^d),\\ \mathcal{A}_\varepsilon = -\operatorname{div}(a^\varepsilon(x)\nabla), \end{aligned} \tag{1.1}$$

with a rapidly oscillating coefficient matrix (as $\varepsilon \to 0$),

$$a^\varepsilon(x) = a(y)|_{y=\varepsilon^{-1}x}. \tag{1.2}$$

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Here $a(y)$, $y \in \mathbb{R}^d$, is a quasiperiodic matrix such that

$$a(y) = A(\omega^1 y, \dots, \omega^m y), \tag{1.3}$$

where $A(z_1, \dots, z_m)$ is a continuous 1-periodic (with respect to every argument $z_1, \dots, z_m$) symmetric $d\times d$-matrix, $\omega^1, \dots, \omega^m \in \mathbb{R}^d$ are the frequency vectors, and $\omega^i y = \omega^i_j y_j$ is the inner product of vectors in $\mathbb{R}^d$. The vectors $\omega^1, \dots, \omega^m$ are (rationally) independent, which means that

$$k_i \omega^i_j \neq 0, \quad j = 1, \dots, d, \quad \text{for every} \ \ 0 \neq k = (k_1, \dots, k_m) \in \mathbb{Z}^m. \tag{1.4}$$

A consequence of the independence is the equation between the mean value of an almost periodic function $a(y)$ and the mean value of the periodic function $A(z)$ [1], namely,

$$\langle a \rangle \ = \langle A \rangle, \tag{1.5}$$

$$\langle a \rangle \ = \lim_{T\to\infty} \frac{1}{(2T)^d} \int_{[-T,T]^d} a(y)\, dy, \quad \langle A \rangle = \int_{\square} A(z) dz,$$

where $\square = \square_m = [0, 1)^m$.

The matrix A in (1.3) is subjected to the ellipticity condition

$$\exists \lambda > 0: \quad \lambda |\xi|^2 \leqslant A(z)\xi \cdot \xi \leqslant \lambda^{-1} |\xi|^2 \quad \forall\, \xi \in \mathbb{R}^d. \tag{1.6}$$

In what follows, the family of frequency vectors $\omega^1, \dots, \omega^m$ is always one and the same. The expression "$b(y)$ is a quasiperiodic function" means that we have a representation of the form (1.3), i.e., $b(y) = B(\omega^1 y, \dots, \omega^m y)$, in which $B(z)$, $z \in \mathbb{R}^m$, is a continuous (or a more regular) periodic function with the periodicity cell $\square_m$. The periodic function $B(z)$ is referred to as the *support* of the quasiperiodic function $b(y)$. For example, a periodic trigonometric polynomial with integer frequencies

$$\Phi(z) = \sum_k a_k e^{i2\pi kz}, \ z \in \mathbb{R}^m, \ k \in \mathbb{Z}^m, \ i = \sqrt{-1}, \tag{1.7}$$

is the support of the quasiperiodic function

$$\varphi(y) = \sum_k a_k e^{i2\pi q_k y}, \ i = \sqrt{-1}, \ y \in \mathbb{R}^d, \ q_k = k_l \omega^l \in \mathbb{R}^d, \tag{1.8}$$

which is also a trigonometric polynomial; however, its frequencies q^j_k are not integer in general.

We are interested in the behavior of the solution of problem (1.1) as $\varepsilon \to 0$. The most general result in this direction [2] is that

$$u_\varepsilon \to u \quad \text{in } L^2(\mathbb{R}^d) \quad \text{as} \quad \varepsilon \to 0, \tag{1.9}$$

where u is the solution of the homogenized equation

$$u \in H^1(\mathbb{R}^d), \quad (\mathcal{A} + 1)u = f, \quad f \in L^2(\mathbb{R}^d), \qquad \mathcal{A} = -\operatorname{div}(a^0 \nabla). \tag{1.10}$$

The homogenized matrix a^0 is constant, symmetric, positive definite, and can be found by the rule indicated in Section 2.

The result of homogenization (1.9), (1.10) can be refined if we assume that the matrix $A(z)$ in (1.3) is sufficiently regular and if we also strengthen the independence condition (1.4) of the frequency vectors.

In what follows, to simplify the statements, we assume that $A(z)$ is a smooth periodic matrix. Let us introduce the "frequency condition" on the vectors $\omega^1, \dots, \omega^m \in \mathbb{R}^d$,

$$\exists\ c_0, \tau > 0: \quad |k_j \omega^j| > c_0^{-1} |k|^{-\tau}, \ \ \text{for every} \ \ 0 \neq k = (k_1, \dots, k_m) \in \mathbb{Z}^m. \tag{1.11}$$

This condition holds for families in general position, namely, the frequency vectors $\omega^1, \dots, \omega^m$ satisfying the inequality (1.11) for certain c_0 and τ form a set of full Lebesgue measure in $(\mathbb{R}^d)^m$. For considerations concerning this topic, see, e.g., [3].

Theorem 1.1. *Let the frequency condition hold. In this case, if u_ε, u are solutions of problems* (1.1) *and* (1.10), *respectively, then*

$$\|u_\varepsilon - u\|_{L^2(\mathbb{R}^d)} \leqslant \varepsilon C \|f\|_{L^2(\mathbb{R}^d)}, \tag{1.12}$$

where the constant C depends only on the frequency condition and on the matrix A.

The bound (1.12) is of operator form and means that

$$\|(\mathcal{A}_\varepsilon + 1)^{-1} - (\mathcal{A} + 1)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leqslant C\varepsilon. \tag{1.13}$$

By (1.9) and (1.12), the function u can be referred to as the "zeroth approximation" or an L^2-approximation to the solution u_ε. Below, we shall indicate the "first approximation" or the H^1-approximation with error estimate of the order of $O(\varepsilon)$, which is as in (1.12).

2. DIVERGENT EQUATION: FIRST APPROXIMATION

Introduce the problem of finding a quasiperiodic function $n^j(y)$, $y \in \mathbb{R}^d$, such that

$$\operatorname{div}_y a(y)(\nabla_y n^j(y) + e_j) = 0, \quad \langle n^j \rangle = 0, \quad j = 1, \dots, d, \tag{2.1}$$

where $e_1, \dots, e_d$ stands for the canonical basis in $\mathbb{R}^d$.

Lemma 2.1. *If the frequency condition* (1.11) *holds, then there are functions $\{n^j(y)\}_{j=1}^d$ that are quasiperiodic together with all their derivatives and satisfy equation* (2.1).

Lemma 2.1 is a consequence of a more general lemma, Lemma 3.1.

Having solutions of problem (2.1), one can define the homogenized matrix a^0 in equation (1.10) by the equation

$$a^0 e_j = \langle a(y)(\nabla_y n^j(y) + e_j)\rangle, \quad j = 1, \dots, d, \tag{2.2}$$

in terms of the mean value of an almost periodic function.

The first approximation to the solution of the equation (1.1) is the function

$$v_\varepsilon(x) = u(x) + \varepsilon n^j(y) \frac{\partial u(x)}{\partial x_j}, \qquad y = \varepsilon^{-1} x, \tag{2.3}$$

which is obtained by adding a corrector to the zeroth approximation, which enables us to achieve the closeness of the approximation in the H^1-norm. Obviously, $v_\varepsilon(x)$ belongs to $H^1(\mathbb{R}^d)$, because $u \in H^2(\mathbb{R}^d)$ due to the elliptic bound for the averaged equation, and $n^j(y)$ and $\nabla n^j(y)$ are bounded functions by Lemma 2.1.

Theorem 2.1. *Let the frequency condition hold. Then the following bound holds*:

$$\|u_\varepsilon - v_\varepsilon\|_{H^1(\mathbb{R}^d)} \leqslant \varepsilon C \|f\|_{L^2(\mathbb{R}^d)}, \tag{2.4}$$

where the constant C is of the same type as that in (1.12).

Proof. From (2.3), using simple manipulations, we obtain

$$\begin{aligned}
\nabla v_\varepsilon(x) &= \nabla u(x) + \nabla n^j(y) \frac{\partial u(x)}{\partial x_j} + \varepsilon \nabla \frac{\partial u(x)}{\partial x_j} n^j(y), \quad y = \varepsilon^{-1} x, \\
R_\varepsilon(x) &\equiv a_\varepsilon(x) \nabla v_\varepsilon(x) - a^0 \nabla u(x) \\
&= [a(y)(\nabla_y n^j(y) + e_j) - a^0 e_j] \frac{\partial u(x)}{\partial x_j} + \varepsilon a(y) \nabla \frac{\partial u(x)}{\partial x_j} n^j(y) \\
&= g^j(y) \frac{\partial u(x)}{\partial x_j} + r_1^\varepsilon(x), \qquad r_1^\varepsilon(x) = \varepsilon a(y) \nabla \frac{\partial u(x)}{\partial x_j} n^j(y), \qquad y = \varepsilon^{-1} x,
\end{aligned} \tag{2.5}$$

where, by (2.2) and (2.1),

$$g^j(y) = a(y)(\nabla_y n^j(y) + e_j) - \langle a(\cdot)(\nabla_y n^j(\cdot) + e_j)\rangle$$

is a smooth solenoidal quasiperiodic vector with zero mean, $\operatorname{div}_y g^j(y) = 0$, $\langle g^j(\cdot)\rangle = 0$. By Lemma 3.2 presented below, we have the representation

$$g^j(y) = \operatorname{div}_y\, p^j(y), \tag{2.6}$$

in which $p^j(y)$ is a smooth quasiperiodic skew-symmetric matrix. Therefore,

$$\begin{gathered} R_\varepsilon(x) = \operatorname{div}\Big(\varepsilon p^j\Big(\frac{x}{\varepsilon}\Big)\frac{\partial u(x)}{\partial x_j}\Big) + r_1^\varepsilon(x) + r_2^\varepsilon(x), \quad \operatorname{div} = \operatorname{div}_x \\ r_2^\varepsilon(x) = -\varepsilon p^j\Big(\frac{x}{\varepsilon}\Big)\nabla\frac{\partial u(x)}{\partial x_j}, \end{gathered} \tag{2.7}$$

$$\operatorname{div}\, R_\varepsilon(x) = \operatorname{div}\,(r_1^\varepsilon(x) + r_2^\varepsilon(x)) \tag{2.8}$$

by the skew symmetry of the matrix $p^j(y)$. Hence,

$$\begin{aligned} -\operatorname{div}[a_\varepsilon(x)\nabla(v_\varepsilon(x) - u_\varepsilon(x))] + (v_\varepsilon(x) - u_\varepsilon(x)) &= -\operatorname{div}[a_\varepsilon(x)\nabla v_\varepsilon(x)] + v_\varepsilon(x) - f \\ &= -\operatorname{div}[a_\varepsilon(x)\nabla v_\varepsilon(x)] + v_\varepsilon(x) + \operatorname{div}\, a^0\nabla u(x) - u(x) \\ &= -\operatorname{div}(a_\varepsilon(x)\nabla v_\varepsilon(x) - a^0\nabla u(x)) + (v_\varepsilon(x) - u(x)) = -\operatorname{div} R_\varepsilon(x) + r_0^\varepsilon(x), \\ r_0^\varepsilon(x) = \varepsilon n^j(\frac{x}{\varepsilon})\frac{\partial u(x)}{\partial x_j}. \end{aligned} \tag{2.9}$$

The solution of the equation

$$z_\varepsilon \in H^1(\mathbb{R}^d), \qquad -\operatorname{div}(a_\varepsilon(x)\nabla z_\varepsilon) + z_\varepsilon = \operatorname{div}\, F + F_0, \qquad F_0 \in L^2(\mathbb{R}^d), \qquad F \in L^2(\mathbb{R}^d)^d,$$

satisfies the energy estimate

$$\int_{\mathbb{R}^d}(|z_\varepsilon|^2 + |\nabla z_\varepsilon|^2)dx \leqslant c\int_{\mathbb{R}^d}(|F_0|^2 + |F|^2)dx, \quad c = \operatorname{const}(\lambda),$$

according to which we derive from (2.9) that

$$\int_{\mathbb{R}^d}(|v_\varepsilon - u_\varepsilon|^2 + |\nabla(v_\varepsilon - u_\varepsilon)|^2)dx \leqslant c\int_{\mathbb{R}^d}(|r_0^\varepsilon|^2 + |r_1^\varepsilon|^2 + |r_2^\varepsilon|^2)dx, \qquad c = \operatorname{const}(\lambda),$$

where the expression for $R_\varepsilon(x)$ in terms of $r_i^\varepsilon(x)$ is used, see (2.8). Every term $r_i^\varepsilon(x)$ contains ε as a factor (see (2.5), (2.7), and (2.9)) and admits the bound

$$\|r_i^\varepsilon\|_{L^2(\mathbb{R}^d)} \leqslant \varepsilon\|u\|_{H^2(\mathbb{R}^d)}\|b\|_{L^\infty(\mathbb{R}^d)} \leqslant \varepsilon C\|f\|_{L^2(\mathbb{R}^d)},$$

where, for the function b, we use n^j, an^j, and p^j, which are obviously bounded pointwise. This completes the proof of the theorem.

The bound (1.12) obviously follows from (2.4).

3. PROBLEMS ON THE CELL

In this section, we prove Lemma 2.1 and the representation (2.6), which were used in the proof of the bound (2.4). Assume that the frequency condition (1.11) holds; we do not mention this assumption below.

3.1. Let $a(y)$ be a quasiperiodic matrix in (1.3) and let $f(y)$ be a given quasiperiodic vector function. The problem of finding a quasiperiodic function $n(y)$ such that

$$\operatorname{div}_y a(y)\nabla_y n(y) = \operatorname{div}_y f(y), \qquad \langle n\rangle = 0, \tag{3.1}$$

is reduced to solving the periodic problem on the cell $\square = [0,1)^m$,

$$\operatorname{div}_y A(z)\nabla_y N(z) = \operatorname{div}_y F(z), \quad \langle N\rangle = 0, \tag{3.2}$$

if one passes from quasiperiodic functions to their periodic supports. Problem (3.2) is not so obvious as can appear at first glance, because it is degenerate, since it contains a differentiation with respect to the argument y related to the variables z_j by the equations $z_j = \omega^j y$, $j = 1, \dots, m$. To study this problem, we introduce the spaces that we need.

Let $\mathrm{Trig}(\mathbb{R}^m)$ be the set of trigonometric polynomials of the form (1.7). For every $s \in \mathbb{R}$, denote by H_s the completion of the set $\mathrm{Trig}(\mathbb{R}^m)$ with respect to the Sobolev norm

$$\|U\|_s^2 = \sum_k |a_k|^2(k^2+1)^s, \qquad U(z) = \sum_k a_k e^{i2\pi kz}.$$

By the Parseval formula, the spaces H_s, $s = 1, 2, \dots$, coincide with the Sobolev spaces $H^s(\square)$ of periodic function square integrable over the cell $\square$ together with all their derivatives up to the order s.

The operator $\Lambda_p U = \sum_k a_k(k^2+1)^{p/2}e^{i2\pi kz}$ defines an isomorphism $\Lambda_p\colon H_{s+p} \to H_s$ for every $s, p \in \mathbb{R}$. For an even $p > 0$, we have $\Lambda_p = (-\Delta_z + 1)^{p/2}$, where Δ_z stands for the Laplace operator with respect to the variable z.

Let $C^l(\square)$ be the space of all l times continuously differentiable periodic functions with the periodicity cell $\square$.

Lemma 3.1. *If $F \in H_s$ for sufficiently large $s > s(m,\tau)$, then equation* (3.2) *has a solution $N \in C^2(\square)$. If $F \in C^\infty(\square)$, then $N \in C^\infty(\square)$.*

Proof. We follow the scheme suggested in [4] to study a similar problem.

Let us first prove *a priori* estimates for a solution of equation (3.2) and then use them to prove the existence of a solution. Consider the corresponding integral identity on specially chosen test functions,

$$\begin{gathered}(A(z)\nabla_y N(z), \nabla_y \Phi(z))_{L^2(\square)} = (F(z), \nabla_y \Phi(z))_{L^2(\square)}, \\ \Phi = \Lambda_s N, \quad s/2 > 0 \text{ is even.}\end{gathered} \tag{3.3}$$

Let us use the fact that the operator Λ_s is self-adjoint and the equations

$$\Lambda_s = \Lambda_{s/2}\Lambda_{s/2}, \quad \Lambda_{s/2}(A\nabla_y N) = A\nabla_y \Lambda_{s/2} N + \cdots,$$

where the dots stand for the summands containing the derivatives of the function N with respect to z of order less than s. Then, using (3.3), one can readily imply the inequality

$$\lambda\|\nabla_y N\|_s^2 \leqslant c(s)(\|\nabla_y N\|_0^2 + \|F\|_s^2). \tag{3.4}$$

Moreover, formula (3.3) for $s = 0$ gives the energy inequality $\|\nabla_y N\|_0^2 \leqslant c(\lambda)\|F\|_0^2$. Finally, (3.4) implies the bound

$$\|\nabla_y N\|_s \leqslant c(\lambda)\|F\|_s, \quad s/2 > 0 \text{ is even.} \tag{3.5}$$

By the frequency condition and the equation $\langle N\rangle = 0$, we have

$$\|N\|_{s-\tau} \leqslant c\|\nabla_y N\|_s,$$

and therefore, (3.5) implies the bound

$$\|N\|_{s-\tau} + \|\nabla_y N\|_s \leqslant c\|F\|_s. \tag{3.6}$$

If we now solve equation (3.2) by the Galerkin method, then the bound (3.6), which certainly holds for the Galerkin approximation (in the form of a trigonometric polynomial), enables us to justify the passage to the limit and to obtain a periodic solution of the equation in the space $H_{s-\tau}$ under the condition $F \in H_s$. Since $H_{s-\tau} \subset C^2(\Box)$ for a sufficiently large s, we obtain the desired solution $N \in C^2(\Box)$. This completes the proof of the lemma.

Lemma 2.1 is an obvious corollary to Lemma 3.1.

3.2. When deriving the bound (2.4), we used the following special representation of solenoidal vectors.

Lemma 3.2. *Let a smooth quasiperiodic vector $g(y)$, $y \in \mathbb{R}^d$, be solenoidal and have zero mean value,* div $g = 0$, $\langle g\rangle = 0$. *Then the representation $g =$ div p holds, where the smooth quasiperiodic matrix $p = \{p^{lj}\}$ is skew-symmetric, $p^{lj} = -p^{jl}$. Moreover, the following bound holds for periodic supports P and G of these functions*:

$$\|P\|_s \leqslant c_0\|G\|_{s+\tau}, \tag{3.7}$$

where c_0 and τ are the constants in the frequency condition.

Proof. Represent the vector g using a series of the form (1.8), taking its properties into account, namely,

$$g(y) = \sum_{k\neq 0} a_k e^{i2\pi q_k y}, \quad a_k, q_k \in \mathbb{R}^d, \quad a_k \perp q_k, \quad q_k \neq 0, \tag{3.8}$$

and using the orthogonality property $a_k \perp q_k$ is explained in the remark below. The desired matrix is

$$p(y) = \sum_{k\neq 0} p_k e^{i2\pi q_k y}, \qquad p_k^{lj} = \frac{1}{2\pi i}\frac{a_k^l q_k^j - a_k^j q_k^l}{|q_k|^2}. \tag{3.9}$$

Indeed, for any chosen $k \neq 0$, we have $a_k e^{i2\pi q_k y} = \operatorname{div}(p_k e^{i2\pi q_k y})$, because

$$\sum_j \frac{\partial}{\partial y_j}(p_k^{lj} e^{i2\pi q_k y}) = \sum_j \frac{a_k^l q_k^j q_k^j - a_k^j q_k^l q_k^j}{|q_k|^2} e^{i2\pi q_k y} = \sum_j \frac{a_k^l q_k^j q_k^j}{|q_k|^2} e^{i2\pi q_k y} = a_k^l e^{i2\pi q_k y}.$$

The bound (3.7) follows from the frequency condition, from the definition of the norm $\|\cdot\|_s$, and from precise formulas for the representation (3.9). For example, by (1.11), we have

$$\frac{q_k^j}{|q_k|^2} \leqslant \frac{1}{|q_k|} < c_0|k|^\tau, \quad j = 1, \dots, d.$$

This proves the lemma.

Remark. Let a quasiperiodic vector $g(y)$ have a smooth periodic support $G(z)$, and let, moreover, $\operatorname{div}_y g(y) = 0$, i.e., let g be solenoidal. Then the following equation holds:

$$\langle g(\cdot)\cdot\nabla_y\varphi(\cdot)\rangle = 0 \tag{3.10}$$

for every trigonometric polynomial φ of the form (1.8). Indeed,

$$0=\langle \varphi(\cdot)(\operatorname{div}_y g)(\cdot)\rangle=\int_\square \Phi(z)(\operatorname{div}_y G)(z)\,dz=-\int_\square G(z)\cdot\nabla_y\Phi(z)\,dz=-\langle g(\cdot)\cdot(\nabla_y\varphi)(\cdot)\rangle,$$

which proves (3.10). In this chain of equations, the second and fourth equations follow from a property similar to (1.5) and the third equation is obtained by integration by parts in the integral over the cell $\square$ and using the periodicity of the functions G and Φ, because $\partial_{y_j}=\omega_j^k\partial_{z_k}$.

Setting $\varphi(y)=e^{-i2\pi q_j y}$ in (3.10) for a chosen $0\neq j\in\mathbb{Z}^m$, we derive the orthogonality condition for the coefficients of the Fourier series in (3.8) if we take into account the equality $\nabla e^{-i2\pi q_j y}=-i2\pi q_j e^{-i2\pi q_j y}$ and

$$\langle e^{i2\pi(q_j-q_k)y}\rangle=\int_\square e^{i2\pi(j-k)z}\,dz\neq 0,$$

only if $k{=}j$. Here again we use the coincidence of the mean values (see (1.5)).

4. CASE m⩽d

Above we have imposed no conditions on the dimensions d and m of the quasiperiodic and periodic variables y and z, respectively (see the definition of the matrix (1.3)). We claim now that, in the case $m\leqslant d$, the frequency condition (1.11) can be replaced by the simpler condition of linear independence of the frequency vectors. It suffices to show that the assertions of Lemmas 3.1 and 3.2 remain valid.

Let us begin with Lemma 3.1. The periodic problem of finding a function $N\in H^1(\square)=H_1$ such that

$$(A(z)\nabla_y N(z),\nabla_y\Phi(z))_{L^2(\square)}=(F(z),\nabla_y\Phi(z))_{L^2(\square)}\quad\forall\ \Phi{\in}H_1 \tag{4.1}$$

is solvable for every $F\in L^2(\square)$, because, on the left-hand side of the identity (4.1), after passing to the gradients with respect to the variable z, a nondegenerate form arises, namely,

$$(\tilde A(z)\nabla_z N(z),\nabla_z\Phi(z))_{L^2(\square)},\quad \tilde A>0.$$

The matrix $\tilde A=\{\tilde A_{sj}\}$ with the entries $\tilde A_{sj}=A_{lk}\omega_l^s\omega_k^j$ is the Gram matrix of the vectors $\omega^1,\dots,\omega^m$, and it is positive definite by the ellipticity condition (1.6). Moreover, the right-hand side of the identity (4.1) is a linear form in $\nabla_z\Phi(z)$ (we can take into account that $\partial_{y_j}=\omega_j^k\partial_{z_k}$). It can readily be seen that the assertion of Lemma 3.1 hold if the right-hand side of F in (4.1) is sufficiently regular.

Consider Lemma 3.2. Let us construct the periodic support P of the desired matrix p from the periodic support $G=(G_1,\dots,G_d)$ of the given quasiperiodic vector g.

Consider the problem of finding a periodic vector function $B=(B_1,\dots,B_d)$ satisfying the relations

$$-\Delta_y B(z)=G(z),\qquad \langle B\rangle=0, \tag{4.2}$$

where the equation is understood in the case of integral identity for every component,

$$B_j\in H^1(\square),\quad (\nabla_y B_j(z),\nabla_y\Phi(z))_{L^2(\square)}=(G_j(z),\Phi(z))_{L^2(\square)}\quad\forall\ \Phi\in H^1(\square). \tag{4.3}$$

Problem (4.2) has a unique solution, because $\langle G\rangle=\langle g\rangle=0$ and the left-hand side of the identity (4.3), after the passage to the gradients with respect to the variable z, defines the form $(\hat A\nabla_z B_j(z),\nabla_z\Phi(z))_{L^2(\square)}$. Here $\hat A=\{\hat A_{sj}\}_{s,j=1}^m$, and $\hat A_{sj}=\omega^s\cdot\omega^j$ is the inner product of the vectors $\omega^s,\omega^j\in\mathbb{R}^d$, and hence, $\hat A>0$ by a property of the Gram matrix. The solution of problem (4.2) satisfies the elliptic bound

$$\|B\|_{H^2(\square}\leqslant c\,\|G\|_{L^2(\square)},\quad c=\operatorname{const}(\omega^1,\dots,\omega^m). \tag{4.4}$$

Moreover, by (4.2) and by the solenoidal property of the vector g, the function $\tilde{B} = \operatorname{div}_y B$ satisfies the relations

$$-\Delta_y \tilde{B}(z) = \operatorname{div}_y G(z) = 0, \qquad \langle \tilde{B} \rangle = 0,$$

and thus, $\tilde{B} = \operatorname{div}_y B = 0$.

By setting $P^{sj} = \partial B_j / \partial y_s - \partial B_s / \partial y_j$, we obtain the desired matrix $P = \{P^{sj}\}_{s,j=1}^d$. Indeed, by (4.4), the bound $\|P\|_{H^1(\square)} \leqslant c \|G\|_{L^2(\square)}$ holds, which can be regarded as an analog of the bound (3.7) for $s = 1$. Moreover,

$$\frac{\partial P^{sj}}{\partial y_j} = \frac{\partial}{\partial y_j}\Big(\frac{\partial B_j}{\partial y_s} - \frac{\partial B_s}{\partial y_j}\Big) = \frac{\partial}{\partial y_s}(\operatorname{div}_y B) - \Delta_y B_s = -\Delta_y B_s = G_s,$$

i.e., the representation $\operatorname{div}_y P = G$ holds. For a smooth function G, differentiating the equation (4.2), we first derive the smoothness of the function B and then also the smoothness of the function P.

Example. Let $m = d = 2$ and let a matrix $a(y)$, $y \in \mathbb{R}^2$, be obtained from a matrix $A(z)$, $z \in \mathbb{R}^2$, which is periodic with respect to the variables z_1, z_2 (the periodicity cell is the unit square $[0,1)^2$), by the substitution $a(y) = A(\omega^1 y, \omega^2 y)$, where ω^1 and ω^2 are noncollinear vectors on the plane. Depending on the choice of the vectors ω^1 and ω^2, two versions are possible: either $a(y)$ turns out to be periodic (possibly with another periodicity cell) or $a(y)$ is not periodic. For example, if $\omega^1 = (\sqrt{2}, 0)$ and $\omega^2 = (0, \sqrt{3})$, then $a(y_1, y_2) = A(\sqrt{2} y_1, \sqrt{3} y_2)$ is periodic with the cell $[0, 1/\sqrt{2}) \times [0, 1/\sqrt{3})$. If $\omega^1 = (\sqrt{2}, 1)$ and $\omega^2 = (1, \sqrt{3})$, then $a(y_1, y_2) = A(\sqrt{2} y_1 + y_2, y_1 + \sqrt{3} y_2)$ is aperiodic. In any case, as is shown by the above analysis, auxiliary "problems on the cell" are reduced to nondegenerate periodic problems. Thus, in this section we have identified the "easy" case of quasiperiodic coefficients, which differs little from the case of periodic coefficients, although does not belong formally to the periodic case.

5. NONDIVERGENT EQUATION

Consider the nondivergent equation

$$u_\varepsilon \in H^1(\mathbb{R}^d), \quad (\mathcal{A}_\varepsilon + 1) u_\varepsilon = f, \quad f \in L^2(\mathbb{R}^d), \qquad \mathcal{A}_\varepsilon = -a_{ij}^\varepsilon(x) \frac{\partial^2}{\partial x_i \partial x_j}, \tag{5.1}$$

where $a^\varepsilon(x) = \{a_{ij}^\varepsilon(x)\}$ is a quasiperiodic matrix defined by relations (1.2)–(1.6). The elliptic theory ensures the existence of the resolvent $(\mathcal{A}_\varepsilon + s)^{-1}$ if $s > 0$ is sufficiently large. In our case, as will be seen below, there is a nondegenerate weight $p_\varepsilon(x) = p(x/\varepsilon)$ such that $0 < \nu \leqslant p(\cdot) \leqslant 1/\nu$ and the coercive bound holds in the weighted L^2-space,

$$(\mathcal{A}_\varepsilon u, u)_{L^2(\mathbb{R}^d, p_\varepsilon dx)} \geqslant \lambda \|\nabla u\|^2_{L^2(\mathbb{R}^d, p_\varepsilon dx)}.$$

Hence, to show the solvability of equation (5.1), one can apply the Lax–Milgram lemma by equipping $H^1(\mathbb{R}^d)$ with the weighted norm (equivalent to the standard norm) given by the equation $(\|\nabla u\|^2_{L^2(\mathbb{R}^d, p_\varepsilon dx)} + \|u\|^2_{L^2(\mathbb{R}^d, p_\varepsilon dx)})^{1/2}$.

The crucial point in our considerations is the existence of a smooth quasiperiodic positive function $p(y)$ (see Lemmas 5.1 and 5.2) satisfying the equation

$$\operatorname{div}_y \operatorname{div}_y (a(y) p(y)) = 0 \tag{5.2}$$

and also the conditions of normalization and nondegeneracy,

$$\langle p \rangle = 1, \qquad 0 < \nu \leqslant p(\cdot) \leqslant 1/\nu. \tag{5.3}$$

It follows from (5.2) that the vector

$$g(y) = \operatorname{div}_y(a(y)p(y)) \tag{5.4}$$

is solenoidal and, by Lemma 3.2, the representation

$$g(y) = \operatorname{div}_y s(y) \tag{5.5}$$

holds, where $s(y)$ is a smooth quasiperiodic skew-symmetric matrix with zero mean value.

The following transformations are related to the function $p(y)$ and the vector $g(y)$:

$$p(y)a_{ij}(y)\frac{\partial^2}{\partial y_i \partial y_j} = \frac{\partial}{\partial y_i}\Big(p(y)a_{ij}(y)\frac{\partial}{\partial y_j}\Big) - g_j(y)\frac{\partial}{\partial y_j},$$

$$pa_{ij}\frac{\partial^2}{\partial y_i \partial y_j} = \frac{\partial}{\partial y_i}\Big(pa_{ij}\frac{\partial}{\partial y_j}\Big) - \Big(\frac{\partial}{\partial y_i}s_{ij}\Big)\frac{\partial}{\partial y_j} = \frac{\partial}{\partial y_i}\Big[(pa_{ij} - s_{ij}))\frac{\partial}{\partial y_j}\Big] = \frac{\partial}{\partial y_i}\Big(\tilde{a}_{ij}\frac{\partial}{\partial y_j}\Big), \tag{5.6}$$

$$\tilde{a}_{ij}(y) = p(y)a_{ij}(y) - s_{ij}(y). \tag{5.7}$$

We obtain a divergent operator with asymmetric matrix $\tilde{a}(y)$. This matrix is solenoidal, since

$$\operatorname{div}_y \tilde{a}(y) = \operatorname{div}_y(pa(y)) - \operatorname{div}_y s(y) \overset{(5.4)}{=} g(y) - \operatorname{div}_y s(y) \overset{(5.5)}{=} 0. \tag{5.8}$$

As is well known (see [2, Russian p. 24]), in this case, the homogenized matrix is evaluated by the rule $a^0 = \langle \tilde{a} \rangle$, and therefore,

$$a^0 = \langle pa \rangle. \tag{5.9}$$

By setting

$$p_\varepsilon(x) = p(x/\varepsilon), \qquad \tilde{a}^\varepsilon(x) = \tilde{a}(x/\varepsilon), \tag{5.10}$$

we obtain (by (5.6)) the representation

$$p_\varepsilon(x)\mathcal{A}_\varepsilon = -\frac{\partial}{\partial x_i}\tilde{a}^\varepsilon_{ij}(x)\frac{\partial}{\partial x_j} \tag{5.11}$$

with asymmetric matrix $\tilde{a}^\varepsilon(x)$, and equation (5.1) can be represented in an equivalent form

$$u_\varepsilon \in H^1(\mathbb{R}^d), \quad -\operatorname{div}(\tilde{a}^\varepsilon \nabla u_\varepsilon) + p_\varepsilon u_\varepsilon = p_\varepsilon f. \tag{5.12}$$

In [2], the following homogenization result for the equation (5.1) (or (5.12)) was proved:

$$u_\varepsilon \rightharpoonup u \quad \text{in } H^1(\mathbb{R}^d), \tag{5.13}$$

where u is a solution of equation (1.10) with the matrix a^0 of the form (5.9). Let us return to the crucial object of our considerations, namely, to equation (5.2).

Lemma 5.1. *Let the matrix $A(z)$ in* (1.3) *be close to the identity matrix. Then there is a quasiperiodic positive solution $p(y)$ of equation* (5.2)*. The solution can be regarded as an arbitrarily smooth function.*

This assertion is customarily referred to as the "lemma on the ground state." In the statement of the lemma, one can take any constant positive definite matrix instead of the identity matrix. One can see from the proof of the lemma, which uses perturbation theory (see Section 6), what is the precise meaning of the assertion that the matrix $A(z)$ is close to a constant matrix. The lemma remains valid under a more general assumption.

Lemma 5.2. *Let a matrix $A(z)$ be subjected to the "cone" condition or the Cordes condition (for details, see Section 6). Then the statement of Lemma* 5.1 *holds.*

As is known [5], the Cordes condition follows from the ellipticity condition (1.6) in dimension $d = 2$ (see Section 7); this fails to hold in the general case, and the Cordes condition must be imposed in addition to (1.6). The "cone" condition gives a constraint on the scattering of the eigenvalues of the matrix $A(z)$. The solvability of nondivergent equations without conditions of this type is not established.

Thanks to the lemma on the ground state, the matrix a^0 in (5.9) is elliptic, and thus, the statement of the homogenized problem with this matrix is well posed. Moreover, also using Lemma 3.2, one can reformulate the original problem (5.1) in the form of equation (5.12).

The result of homogenization (5.13) stated above can be refined in the following sense. The solution u_ε can be approximated by the solution of the homogenized problem in the H^1-norm, and with a controlled bound for the approximation of order ε, if one takes a somewhat corrected equation as compared with (1.10) for the averaged equation, namely,

$$v_\varepsilon \in H^1(\mathbb{R}^d), \quad -\operatorname{div}\, a^0\nabla v_\varepsilon + v_\varepsilon = p_\varepsilon f, \quad f \in L^2(\mathbb{R}^d), \tag{5.14}$$

where the matrix a^0 and the weights p_ε are defined in (5.9) and (5.10), respectively.

Theorem 5.1. *Let the assumptions of Lemma* 5.1 *or of Lemma* 5.2 *hold, together with the frequency condition* (1.11). *Then the difference between the solutions of equations* (5.1) *and* (5.14) *satisfies the bound*

$$\|u_\varepsilon - v_\varepsilon\|_{H^1(\mathbb{R}^d)} \leqslant \varepsilon C\|f\|_{L^2(\mathbb{R}^d)}, \tag{5.15}$$

where the constant C depends on the matrix $A(z)$ and on the frequency condition.

Proof. We claim that, for the solution of equation (5.12), the zeroth approximation v_ε is also the first approximation. To simplify our notation, here and below, we write $v_\varepsilon = v$. Transform the difference of the flows, namely, the difference between the approximate flow and the averaged flow,

$$\begin{aligned} R_\varepsilon(x) &= \tilde{a}^\varepsilon(x)\nabla v(x) - a^0\nabla v(x) \overset{(5.7),(5.9)}{=\!=\!=} (pa(y) - s(y) - \langle pa - s\rangle)\nabla v(x) \\ &= h_j(y)\frac{\partial v(x)}{\partial x_j} = (\operatorname{div}_y \varsigma_j(y))\frac{\partial v(x)}{\partial x_j}, \quad y = \frac{x}{\varepsilon}, \end{aligned}$$

where $h_j(y) = (pa(y) - s(y) - \langle pa - s\rangle)e_j$, $j = 1, \dots, d$, is a solenoidal vector (see (5.8)) with mean value $\langle h_j\rangle = 0$. For h_j, we use the representation of the skew-symmetric matrix ς_j given by Lemma 3.2. Therefore,

$$R_\varepsilon(x) = \operatorname{div}\left(\varepsilon\varsigma_j\Big(\frac{x}{\varepsilon}\Big)\frac{\partial v(x)}{\partial x_j}\right) - \varepsilon\varsigma_j\Big(\frac{x}{\varepsilon}\Big)\nabla\frac{\partial v(x)}{\partial x_j}, \qquad -\operatorname{div} R_\varepsilon(x) = \varepsilon\operatorname{div}\left(\varsigma_j\Big(\frac{x}{\varepsilon}\Big)\nabla\frac{\partial v(x)}{\partial x_j}\right) \tag{5.16}$$

due to the skew symmetry of the matrix ς_j. Therefore,

$$\begin{aligned} &-\operatorname{div}[\tilde{a}^\varepsilon\nabla(v - u_\varepsilon)] + p_\varepsilon(v - u_\varepsilon) \overset{(5.12)}{=} -\operatorname{div}(\tilde{a}^\varepsilon\nabla v) + p_\varepsilon v - p_\varepsilon f \\ &\overset{(5.14)}{=} -\operatorname{div}(\tilde{a}^\varepsilon\nabla v) + p_\varepsilon v + \operatorname{div}\, a^0\nabla v - v = -\operatorname{div}(\tilde{a}^\varepsilon\nabla v - a^0\nabla v) + (p_\varepsilon - 1)v = -\operatorname{div}\, R_\varepsilon + (p_\varepsilon - 1)v, \end{aligned} \tag{5.17}$$

where, by Lemma 5.3 presented below, we have

$$\|(p_\varepsilon - 1)v\|_{H^{-1}(\mathbb{R}^d)} \leqslant \varepsilon C\|v\|_{H^1(\mathbb{R}^d)}. \tag{5.18}$$

For the equation

$$-\operatorname{div}(\tilde{a}^\varepsilon\nabla z) + z = \operatorname{div}\, F + F_0,$$

we have the energy bound

$$\|\nabla z\|_{L^2(\mathbb{R}^d)} + \|z\|_{L^2(\mathbb{R}^d)} \leqslant c(\|F\|_{L^2(\mathbb{R}^d)} + \|F_0\|_{H^{-1}(\mathbb{R}^d)}).$$

As applied to (5.17), this bound gives the desired inequality (5.15) if we take into account formulas (5.16) and (5.18), the sufficient smoothness of ς_j, and the elliptic bound $\|v\|_{H^2(\mathbb{R}^d)} \leqslant C\|f\|_{L^2(\mathbb{R}^d)}$. This completes the proof of the theorem.

Lemma 5.3. *Let the frequency condition* (1.11) *hold, let* $b(y)$ *be a sufficiently smooth quasiperiodic function with zero mean, and let* $b_\varepsilon(x) = b(x/\varepsilon)$*. Then*

$$\|b_\varepsilon v\|_{H^{-1}(\mathbb{R}^d)} \leqslant \varepsilon C \|v\|_{H^1(\mathbb{R}^d)}. \tag{5.19}$$

Proof. We shall first obtain the representation

$$b(y) = \operatorname{div}_y\, h(y) \tag{5.20}$$

in which $h(y)$ is a bounded quasiperiodic function. One must take $h(y) = \nabla_y w(y)$, where $w(y)$ is a solution of the equation $\Delta_y w(y) = b(y)$. The solvability of this equation in the class of sufficiently smooth quasiperiodic functions is ensured by the frequency condition. Indeed, for periodic supports, we have an equation on the cell $\Box$, namely,

$$\Delta_y W(z) = B(z), \qquad \langle B\rangle = 0.$$

The solution W is written out by an explicit formula using a Fourier series, which implies a bound in the scale of H_s spaces (see Section 3), $\|W\|_{H_{s-2\tau}} \leqslant c\|B\|_{H_s}$. For sufficiently large s, we have $H_{s-2\tau} \subset C^2(\Box)$, and thus, the representation (5.20) is justified. Using this representation, one can readily derive inequality (5.19) immediately from the definition of the H^{-1}-norm. This completes the proof of the lemma.

6. PROOF OF THE LEMMA ON THE GROUND STATE

6.1. Lemma 5.1 is proved in [4]. For the sake of completeness of our presentation, we reproduce this proof.

Consider the space $X = L^2(\Box)^{d^2}$ on the m-dimensional cell $\Box$ and the subset S of X formed by the elements of the form $\{\frac{\partial^2}{\partial y_l \partial y_j}\Phi\}$, where Φ ranges over $\mathrm{Trig}(\mathbb{R}^m_z)$. Since $z_j = \omega^j y$, it follows that $\partial/\partial y_j = \omega^l_j \partial/\partial z_l$. Denote by W^2_2 the closure of S in X and by W^{-2}_2 the dual space of W^2_2 with respect to the inner product in X. Introduce an operator Δ_y by the equation

$$\Delta_y V = \sum_{i=1}^d V_{ii}, \qquad V = \{V_{ij}\} \in W^2_2.$$

On the set S, which is dense in W^2_2, the operator Δ_y is the ordinary Laplace differential operator. The following bound holds:

$$\|\Delta V\|_{L^2(\Box)} \geqslant c_1 \|V\|_{W^2_2}. \tag{6.1}$$

If $V \in \mathrm{Trig}(\mathbb{R}^m)$, then

$$\frac{\partial}{\partial y_j} V = i2\pi \sum a_k q^j_k e^{i2\pi kz}, \qquad \frac{\partial^2}{\partial y_l \partial y_j} V = (i2\pi)^2 \sum a_k q^l_k q^j_k e^{i2\pi kz},$$

where the components of the vectors $q_k = k_i \omega^i$ are used, and so inequality (6.1) follows from the Parseval formula and the elementary inequality $2\alpha\beta \leqslant \alpha^2 + \beta^2$, $\alpha, \beta \in \mathbb{R}$.

It follows from (6.1) that the image of the adjoint operator $\Delta^*_y : L^2(\Box) \to W^{-2}_2$ coincides with W^{-2}_2. The kernel of the operator Δ^*_y is formed by the constants only, because every trigonometric polynomial Ψ with zero mean is representable in the form $\Psi = \Delta_y \Phi$. It should be noted that the frequency vectors $\omega^1, \dots, \omega^m$ are rationally independent, and thus, $q_k = k_i \omega^i \neq 0$. This implies that, for every $F \in W^{-2}_2$, there is a unique solution of the equation

$$\Delta^*_y U = F, \qquad U \in L^2(\Box), \qquad \langle U\rangle = 0,$$

where $\|U\|_{L^2(\square)} \leqslant c_0\|F\|_{W_2^{-2}}$. Denote the solution operator $W_2^{-2} \to L^2(\square)$ by $\mathcal{R}$.

Now let us introduce the family of operators $A_s\colon W_2^2 \to L^2(\square)$. For an element $V = \{V_{ij}\} \in W_2^2$, set

$$A_s V = \Lambda_{-s} a_{ij} \Lambda_s \{V_{ij}\}, \quad s/2 \geqslant 0 \quad \text{is an integer},$$

where the operator Λ_s is defined in Section 3. Assume that the matrix a_{ij} is close to the identity matrix δ_{ij}. Then $a_{ij} = \delta_{ij} + \chi_{ij}$ and

$$A_s V = \Delta_y V + \Lambda_{-s}\chi_{ij}\Lambda_s\{V_{ij}\} = \Delta_y V + \mathcal{K}_s V,$$

where the operator $\mathcal{K}_s$ is sufficiently small if the $C^s(\square)$-norms of the functions χ_{ij} are sufficiently small. By perturbation theory, the equation $A_s^* V = 0$ (i.e., $(\Delta_y^* + \mathcal{K}_s^*)V = 0$) has a solution, and it can be obtained as follows. Write

$$V^{(N)} = \sum_{i=0}^{N} V_i, \quad V_0 = 1, \quad V_i = -\mathcal{R}\mathcal{K}_s^* V_{i-1} \quad (i > 0).$$

Then $V^{(N)}$ is a Cauchy sequence in $L^2(\square)$, and its limit is the solution V. It is clear that V is close to $V_0 = 1$, and thus, is positive.

Thus, we have found a function V such that

$$\int_\square A_s^* V \Psi dz = 0 \quad \Longleftrightarrow \int_\square V A_s \Psi dz = 0, \qquad \int_\square V\Lambda_{-s}(\Delta_y \Phi + \chi_{ij}\frac{\partial^2}{\partial y_l \partial y_j}\Phi) dz = 0,$$
$$\|V - 1\|_{L^2(\square)} \leqslant \delta,$$

where $\delta > 0$ is sufficiently small and $\Psi = \Lambda_{-s}\Phi$, Φ is a trigonometric polynomial. Then $P = \Lambda_{-s}V$ satisfies the identity

$$\int_\square P\Big(\Delta_y \Phi + \chi_{ij}\frac{\partial^2}{\partial y_l \partial y_j}\Phi\Big) dz = 0,$$

i.e., P is a solution of the equation $\operatorname{div}\operatorname{div}(aP) = 0$, and it is sufficiently close to 1 in the $C^2(\square)$-norm. Increasing s, one can achieve an arbitrarily high smoothness of the function P. The properties of P thus obtained ensure the statement of Lemma 5.1.

We stress that, in Lemma 5.1, the frequency condition (1.11) is not required. The solvability problem for equation (5.2) has a positive answer which follow from smallness considerations, because equation (5.2) differs only slightly from the solvable equation with the operator Δ_y^*.

6.2. We now turn to Lemma 5.2.

Let us first formulate the "cone" condition introduced by Cordes [6], which a condition on the matrix $A(z)$,

$$\exists\, \delta > 0: \quad (d-1)\Big(1 + \frac{d(d-2)}{d^2-1}\Big) \sum_{i<j=2}^{d} (\lambda_i(z) - \lambda_j(z))^2 \leqslant (1-\delta)(\operatorname{Tr} A(z))^2, \tag{6.2}$$

where $\lambda_1(z), \dots, \lambda_d(z)$ are the eigenvalues of the matrix $A(z)$ and $\operatorname{Tr} A(z) = \sum_{i=1}^d \lambda_i(z)$ stands for the trace of the matrix $A(z)$. An important consequence of condition (6.2), which was proved in [5], is the "acute angle inequality," which, in our case, becomes

$$\tilde\lambda \int_{\mathbb{R}^d} |\Delta\, u|^2\, dx \leqslant -\int_{\mathbb{R}^d} \mathcal{A}_\varepsilon u \Delta\, u\, dx, \quad u \in C_0^\infty(\mathbb{R}^d). \tag{6.3}$$

The constant $\tilde\lambda > 0$ depends only on δ in (6.2) and on the ellipticity constant λ in (1.6).

We need the following version of the "acute angle inequality."

Lemma 6.1. *Let condition* (6.2) *hold, and let* $\varphi(y)$ *be a smooth quasiperiodic function. Then the support of this function* $\Phi(z)$ *satisfies the inequality*

$$\tilde{\lambda}\int_{\square}|\Delta_y\Phi|^2\,dz \leqslant -\int_{\square}\mathcal{A}_y\Phi\Delta_y\Phi\,dz, \tag{6.4}$$

where $\mathcal{A}_y = -A_{ij}(z)\partial^2/\partial y_i\partial y_j$ *and the constant* $\tilde{\lambda} > 0$ *is just the same as in* (6.3).

Proof. Write out inequality (6.3) for $u(x) = \alpha(x)\varphi(x/\varepsilon)$, where $\alpha \in C_0^\infty(\mathbb{R}^d)$. Since

$$\Delta u(x) = \varepsilon^{-2}\alpha(x)\Delta_y\varphi(y) + O(\varepsilon^{-1}), \quad y = x/\varepsilon,$$
$$-\mathcal{A}_\varepsilon u(x) = \varepsilon^{-2}\alpha(x)a_{ij}(y)\frac{\partial^2}{\partial y_i\partial y_j}\varphi(y) + O(\varepsilon^{-1}), \quad y = x/\varepsilon,$$

we obtain

$$\tilde{\lambda}\int_{\mathbb{R}^d}|\alpha(x)\Delta_y\varphi(y)|^2\,dx \leqslant \int_{\mathbb{R}^d}\alpha(x)a_{ij}(y)\frac{\partial^2}{\partial y_i\partial y_j}\varphi(y)\alpha(x)\Delta_y\varphi(y)\,dx + O(\varepsilon), \quad y = x/\varepsilon. \tag{6.5}$$

Using the properties of the mean value $\langle\psi(\cdot)\rangle$ of the quasiperiodic function $\psi(y) = \Psi(z)|_{z_i=\omega^i y}$, i.e., the weak L^1_{loc}-convergence $\psi(x/\varepsilon) \rightharpoonup \langle\psi(\cdot)\rangle$ and the coincidence of means $\langle\psi(\cdot)\rangle = \langle\Psi(\cdot)\rangle$, we find the limits as $\varepsilon \to 0$ of the integrals in (6.5); these limits are equal to

$$\int_{\mathbb{R}^d}|\alpha(x)|^2\langle|\Delta_y\Phi(\cdot)|^2\rangle\,dx, \quad \int_{\mathbb{R}^d}|\alpha(x)|^2\langle A_{ij}(\cdot)\frac{\partial^2}{\partial y_i\partial y_j}\Phi(\cdot)\Delta_y\Phi(\cdot)\rangle\,dx.$$

Hence, passing to the limit in (6.5), we see that

$$\tilde{\lambda}\int_{\square}|\Delta_y\Phi(z)|^2\,dz \leqslant \int_{\square}A_{ij}(z)\frac{\partial^2}{\partial y_i\partial y_j}\Phi(z)\Delta_y\Phi(z)\,dz,$$

as was to be proved. This completes the proof of the lemma.

Having the inequality (6.4), consider the operator $\mathcal{A}_y\colon W_2^2 \to L^2(\square)$, where the space W_2^2 is defined in Subsection 6.1. The range of this operator is closed due to the bound (which follows from (6.1)) $c_1\|u\|_{W_2^2} \leqslant \|\mathcal{A}_y u\|_{L^2(\square)} \leqslant c_2\|u\|_{W_2^2}$. Introduce the space $\tilde{L} = \{f \in L^2(\square) : \langle f\rangle = 0\}$ and the orthogonal projection $\mathfrak{P}\colon L^2(\square) \to \tilde{L}$. It can readily be seen that $\Delta_y\colon W_2^2 \to \tilde{L}$ is an isomorphism, and we can define the operator $\mathfrak{P}\mathcal{A}_y\Delta_y^{-1}\colon \tilde{L} \to \tilde{L}$. Formula (6.4) implies the coercive property

$$-\langle\mathfrak{P}\mathcal{A}_y\Delta_y^{-1}f\,f\rangle = -\langle\mathfrak{P}\mathcal{A}_y u\,\Delta_y u\rangle = -\langle\mathcal{A}_y u\,\Delta_y u\rangle \geqslant c\|\Delta_y u\|^2_{L^2(\square)} = c\|f\|^2_{L^2(\square)},$$

where $u = \Delta_y^{-1}f$. In this case, by the Lax–Milgram lemma, $\mathfrak{P}\mathcal{A}_y\colon W_2^2 \to \tilde{L}$ is an isomorphism, and the range of the operator $\mathcal{A}_y\colon W_2^2 \to L^2(\square)$ can have codimension equal to zero or one. The case of zero codimension is excluded, because the problem $\mathcal{A}_y u = 1$, $u \in W_2^2$, is not solvable, as is shown by inequality (6.4). Thus, the dimension of the range of the operator $\mathcal{A}_y\colon W_2^2 \to L^2(\square)$ is equal to one. In this case, the adjoint operator has a one-dimensional kernel, i.e., equation (5.2) has precisely one solution p normalized by the condition $\langle p\rangle = 1$. It is positive, which follows from the fact that, otherwise, there would be an $f > 0$ such that $\langle fp\rangle = 0$ and the equation $\mathcal{A}_y u = f$ would be solvable. This contradicts the maximum principle. Lemma 5.2 is proved.

We stress that, in Lemma 5.2, we do not assume the frequency condition (1.11), and the solvability problem has a positive answer due to the "acute angle inequality" in the form (6.4), which is ensured by the Cordes condition (6.2).

7. SOME REMARKS

On the method. Diverse aspects of homogenizing were studied earlier for equations with quasiperiodic coefficients in [7–9, 4]. In particular, bounds for the homogenization errors were treated in [7]. However, these bounds were not in operator form and were obtained using the maximum principle under excessive regularity conditions on the right-hand side f of the equation. The majorant in the error bounds depended on the Sobolev norms of the function f or on the C^k-norms of f. In this case, it is impossible to represent the error bounds as bounds for the difference between the resolvents of the original and the homogenized operator in the operator $L^2 \to L^2$ norm, as in (1.13).

The operator estimate (1.13) is proved by the classical "method of first approximation" in the modification originating from [10], where divergent uniformly elliptic equations with periodic measurable coefficients were studied. In the present paper, we are forced to assume the sufficient regularity of the coefficients of the equation (the measurability is certainly insufficient), which is related to the "problem on the cell" in the class of quasiperiodic functions. In the case of divergent uniformly elliptic equations with periodic coefficients (we suppose that these coefficients are measurable), the corresponding "problem on the cell" is an elliptic periodic problem on $\square$, and this problem has a solution in the Sobolev $W^{1,2}$-space of periodic functions (belonging to $L^2(\square)$ together with all its derivatives), which is sufficient for the method (see [10]). Moreover, in the scalar case, this periodic solution turns out to be bounded by the maximum principle (see [11, Th. B.2 in Chap. II]).

If the coefficients of a uniformly elliptic equation are quasiperiodic, then the "problem on the cell" in the class of quasiperiodic functions on $\mathbb{R}^d$ is reduced, by the coincidence of the mean values (see (1.5)), to a "hypoelliptic" problem in the class of periodic functions on $\mathbb{R}^m$. For $m > d$, this periodic problem is degenerate, and there are difficulties concerning its solvability. Moreover, one must ensure the existence of solution in the Sobolev space H^k on the cell $\square_m$ (for the definition, see Section 3) for sufficiently large k and ensure that the solution is at least continuous and correctly define the quasiperiodic function. For these reasons, we need the sufficient smoothness of the coefficients of the equation under consideration.

On the reduction technique. When considering nondivergent equations, we have used the trick of reducing these equations to divergent equations. The reduction technique was developed in [4, 9]. Using this approach, one can obtain (from our results) interesting consequences for equations which do not belong formally to the case under consideration and can be reduced to this case according to [4, 9]. Consider, for example, the stationary Schrödinger equation

$$\begin{gathered} u_\varepsilon \in H^1(\mathbb{R}^d), \qquad (\mathcal{A}_\varepsilon + 1)u_\varepsilon = f, \quad f \in L^2(\mathbb{R}^d), \\ \mathcal{A}_\varepsilon = -\Delta + \varepsilon^{-2}\varphi_\varepsilon(x), \end{gathered} \tag{7.1}$$

with oscillating quasiperiodic potential $\varphi_\varepsilon(x) = \varphi(y)$, $y = \varepsilon^{-1}x$, where, as usual, $\varphi(y) = \Phi(\omega^1 y, \dots, \omega^m y)$ and $\Phi(z_1, \dots, z_m)$ is a continuous function 1-periodic with respect to every argument. Suppose that $\lambda_0 = 0$ is the lowest eigenvalue of the operator $-\Delta_y + \Phi(z)$ on the unit cube $[0,1)^m$ with periodic boundary conditions. The corresponding eigenfunction $P(z)$ may be chosen posititive with normalizing condition $\langle P^2 \rangle = 1$. It means that there exists a positive quasiperiodic function $p(y)$ such that

$$-\Delta_y p(y) + \varphi(y)p(y) = 0, \qquad \langle p^2 \rangle = 1. \tag{7.2}$$

The following reduction formula holds:

$$p(-\Delta_y + \varphi)pv = -\frac{\partial}{\partial y_i} p^2 \frac{\partial}{\partial y_i} v,$$

thanks to which the operator $\mathcal{A}_\varepsilon$ from (7.1) is positive, thereby, the resolvent $(\mathcal{A}_\varepsilon + 1)^{-1}$ really exists. Setting $\rho_\varepsilon(x) = p^2(x/\varepsilon)$, $p_\varepsilon(x) = p(x/\varepsilon)$, and $\tilde{u}_\varepsilon = u_\varepsilon(x)/p(x/\varepsilon)$, after multiplying (7.1) by $p(x/\varepsilon)$, we obtain

$$\tilde{u}_\varepsilon \in H^1(\mathbb{R}^d), \quad -\frac{\partial}{\partial x_i}\rho_\varepsilon \frac{\partial}{\partial x_i}\tilde{u}_\varepsilon + \rho_\varepsilon \tilde{u}_\varepsilon = p_\varepsilon f. \tag{7.3}$$

Let us find the homogenized matrix a^0 for an isotropic matrix $a^\varepsilon(x) = \{\rho_\varepsilon(x)\delta_{ij}\}$, as is described in Section 2, and introduce the operator $\mathcal{A} = -\operatorname{div}(a^0\nabla)$. For the homogenized problem, we take problem (5.14). The difference between the solutions of problems (7.3) and (5.14) satisfies the bound

$$\|\tilde{u}_\varepsilon - v_\varepsilon\|_{L^2(\mathbb{R}^d)} \leqslant \varepsilon C\|f\|_{L^2(\mathbb{R}^d)},$$

where the constant C depends on the potential $\Phi(z)$ and on the frequency condition. This bound can be represented in operator form as

$$\Big\|p_\varepsilon\Big(-\frac{\partial}{\partial x_i}\rho_\varepsilon\frac{\partial}{\partial x_i} + \rho_\varepsilon\Big)^{-1}p_\varepsilon - p_\varepsilon(\mathcal{A}+1)^{-1}p_\varepsilon\Big\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leqslant \varepsilon C.$$

By the reduction formula, the first bordered resolvent can be replaced by the resolvent of the operator $\mathcal{A}_\varepsilon = -\Delta + \varepsilon^{-2}\varphi_\varepsilon(x)$, namely,

$$\|(\mathcal{A}_\varepsilon+1)^{-1} - p_\varepsilon(\mathcal{A}+1)^{-1}p_\varepsilon\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leqslant \varepsilon C.$$

The somewhat nonstandard normalization condition in (7.2) enables one to use Lemma 5.3 when deriving this bound.

Suppose that instead of (7.2) the ground state equation is of the form $(-\Delta_y + \varphi(y) - \lambda_0)p(y) = 0$, where $\lambda_0 > 0$. In this case the problem (7.1) is again well posed and due to the reduction formula it may be transformed into the equation $-div\rho_\varepsilon\nabla\tilde{u}_\varepsilon + (\varepsilon^{-2}+1)\rho_\varepsilon\tilde{u}_\varepsilon = p_\varepsilon f$. Hence, one can readily conclude that $\|\tilde{u}_\varepsilon\|_{L^2} = O(\varepsilon^2)$ instead of above estimates.

On the Cordes condition. *In the planar case* ($d = 2$), (6.2) *follows from* (1.6). Indeed, the Cordes condition becomes $(\lambda_1 - \lambda_2)^2 \leqslant (1-\delta)(\lambda_1+\lambda_2)^2$, and it certainly holds with $\delta = 2\lambda^2/(1+\lambda^2) < 1$. Here λ is the ellipticity constant, and we may always assume that $\lambda < 1$. By (1.6), the point (λ_1, λ_2) belongs to the square $Q = \{t \in \mathbb{R}^2 : \lambda \leqslant t_i \leqslant 1/\lambda, i = 1, 2\}$, which is contained in the acute angle between the lines $\gamma_\pm$ with the equations $t_2 = \lambda^{\pm 2}t_1$. In the new orthogonal system of coordinates $\tilde{t}_1 = t_2 + t_1$, $\tilde{t}_2 = t_2 - t_1$, the straight lines $\gamma_\pm$ have the angular coefficient $k_\pm = \pm(1-\lambda^2)/(1+\lambda^2)$. Therefore, for every point of the square Q, we have $|\tilde{t}_2/\tilde{t}_1| \leqslant \frac{1-\lambda^2}{1+\lambda^2}$. Then $(\lambda_1-\lambda_2)^2 \leqslant (\lambda_1+\lambda_2)^2(\frac{1-\lambda^2}{1+\lambda^2})^2 < (1 - 2\frac{\lambda^2}{1+\lambda^2})(\lambda_1+\lambda_2)^2$, as was to be proved.

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