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Analysis of Gauge-Equivariant Complexes and a Topological Index Theorem for Gauge-Invariant Families

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Abstract. We continue our study of gauge equivariant K-theory. We thus study the analysis of complexes endowed with the action of a family of compact Lie groups and their index in gauge equivariant K-theory. We introduce various index functions, including an axiomatic one, and show that all index functions coincide. As an application, we prove a topological index theorem for a family $D = (D_b)_{b\in B}$ of gauge-invariant elliptic operators on a $\mathcal{G}$-bundle $X \to B$, where $\mathcal{G} \to B$ is a locally trivial bundle of compact groups, with typical fiber G. More precisely, one of our main results states that $\text{a-ind}(D) = \text{t-ind}(D) \in K^0_{\mathcal{G}}(X)$, that is, the equality of the analytic index and of the topological index of the family D in the gauge-equivariant K-theory groups of X. The analytic index $\text{ind}_a(D)$ is defined using analytic properties of the family D and is essentially the difference of the kernel and cokernel $K_{\mathcal{G}}$-classes of D. The topological index is defined purely in terms of the principal symbol of D.

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1. INTRODUCTION

Analysis on singular spaces, as developed by Connes, Cordes, Melrose, Rosenberg, Skandalis, Schrohe, Schulze [12–14, 28, 41, 45, 46], and many others, leads naturally to Lie algebroids and, hence, to Lie groupoids. This was formalized in [1, 29, 36] and other papers. One of the simplest Lie groupoids is given by *a family* (*or bundle*) *of Lie groups*. In applications, one is interested mostly in the case of families of *solvable* Lie groups [2, 3, 25, 30, 34, 51]. Nevertheless, families of compact Lie groups are interesting because some of the techniques that are developed for dealing with compact groups can be used as a model in the case of solvable Lie groups as well. Moreover, families of compact Lie groups form a basic building block in the structure of *proper groupoids*, which is a very important class of groupoids. Proper groupoids have received recently a great deal of attention and several important results have been proved for this class of groupoids by Emerson and Meyer [16–19].

In this paper, we continue our study of families of *compact* Lie groups [32, 33, 49, 50] by studying gauge-equivariant complexes and the various index functions defined on them. The natural topological invariants of these complexes live in the *gauge-equivariant* K-theory defined in [32]. Among these invariants, a prominent role is played by the analytic and topological indices. Our approach is based on a careful investigation of the formal properties of index mappings, which leads us to the result stating that the analytic and topological index for these complexes coincides. This result is a generalization of the well known theorem of Atiyah and Singer [5], and is proved in the spirit of their original paper.

Let us introduce some notation to describe our results in more detail. Let $p : \mathcal{G} \to B$ be a bundle of *compact* groups. Recall that this means that each fiber $\mathcal{G}_b := p^{-1}(b)$ is a compact group and that, locally, $\mathcal{G}$ is of the form $U \times G$, where $U \subset B$ open and G a fixed compact group. Let X and B be locally compact spaces and $\pi_X : X \to B$ be a continuous mapping. In the present paper, as in [32],

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this mapping will be supposed to be a locally trivial bundle. The case of nonlocally trivial bundles will be dealt with in a later publication.

Assume that $\mathcal{G}$ acts on X. This action will always be fiber-preserving. Then we can associate to the action of $\mathcal{G}$ on X the *$\mathcal{G}$-equivariant K-theory groups* $K^i_{\mathcal{G}}(X)$ as in [32]. We shall review and slightly generalize this definition in Section 2. For X compact, the group $K^0_{\mathcal{G}}(X)$ is defined as the Grothendieck group of $\mathcal{G}$-equivariant vector bundles on X. If X is not compact, we define the groups $K^0_{\mathcal{G}}(X)$ using fiberwise one-point compactifications. We shall call these groups simply the *gauge-equivariant K-theory groups* of X when we do not want to specify $\mathcal{G}$. The reason for introducing the gauge-equivariant K-theory groups is that they are the natural range for the index of a gauge-invariant families of elliptic operators. In turn, the motivation for studying gauge-invariant families and their index is due to their connection to spectral theory and boundary value problems on noncompact manifolds. Some possible connections with Ramond-Ramond fields in String Theory were mentioned in [21, 32]. See also [7, 22, 26, 27, 31, 42]. A different approach to equivariant index constructions can be found in [43].

In this paper, in particular, we also continue our study of gauge-equivariant K-theory. Our initial motivation for this paper was to develop an equivariant topological index theorem for gauge equivariant operators in the framework of our two earlier papers on the subject [32, 33], see Theorem 5.6. In fact, the most part of the present paper was written in 2009 (and the results were presented at the conference [35]). Connections with proper groupoids [19] provide now further motivation for our study. We begin by providing two alternative definitions of the relative $K_{\mathcal{G}}$–groups, both based on complexes of vector bundles. (In this paper, all vector bundles are complex vector bundles, with the exception of the tangent bundles or where explicitly stated.) These alternative definitions, modeled on the classical case [4, 23], provide a convenient framework for the study of products, especially in the relative or noncompact cases. The products are especially useful for the proof of the Thom isomorphism in gauge-equivariant theory [33], which is one of the main ingredients for the results of this paper. A Thom isomorphism was proved later also in the framework of bivariant KK-theory [19]. Let $E \to X$ be a $\mathcal{G}$-equivariant complex vector bundle. Then the Thom isomorphism is a natural isomorphism

$$\tau_E \colon K^i_{\mathcal{G}}(X) \to K^i_{\mathcal{G}}(E). \tag{1}$$

(There is also a variant of this result for spin^c-vector bundles, but since we will not need it for the index theorem 5.6, we will not discuss it in this paper.) The Thom isomorphism allows us to define Gysin (or push-forward) mappings in K-theory. As it is well known from the classical work of Atiyah and Singer [5], the Thom isomorphism and the Gysin mappings are some of the main ingredients used for the definition and study of the topological index. In fact, we shall proceed along the lines of that paper to define the topological index for gauge-invariant families of elliptic operators. Some other approaches to Thom isomorphism in general settings of Noncommutative Geometry were the subject of [11, 20, 24, 26, 39, 47] and many other papers.

Gauge-equivariant K-theory behaves in many ways like the usual equivariant K-theory, but exhibits also some new phenomena. For example, the groups $K^0_{\mathcal{G}}(B)$ may turn out to be reduced to $K^0(B)$ when $\mathcal{G}$ has "a lot of twisting" [32, Proposition 3.6]. This is never the case in equivariant K-theory when the action of the group is trivial, but the group itself is not trivial. In [32], we addressed this problem in two ways: first, we found conditions on the bundle of groups $p : \mathcal{G} \to B$ that guarantee that $K^0_{\mathcal{G}}(X)$ is not too small (this condition is called *finite holonomy* and is recalled below), and, second, we studied a substitute of $K^0_{\mathcal{G}}(X)$ which is never too small (this substitute is $K_*(C^*(\mathcal{G}))$, the K-theory of the C^*-algebra of the bundle of compact groups $\mathcal{G}$).

In this paper, we shall again need the finite holonomy condition, so let us review it now. To define the finite holonomy condition, we introduced the *representation covering* of $\mathcal{G}$, denoted $\widehat{\mathcal{G}} \to B$. As a space, $\widehat{\mathcal{G}}$ is the union of all the representation spaces $\widehat{\mathcal{G}}_b$ of the fibers $\mathcal{G}_b$ of the bundle of compact groups $\mathcal{G}$. One measure of the twisting of the bundle $\mathcal{G}$ is the holonomy associated to the covering $\widehat{\mathcal{G}} \to B$. We say that $\mathcal{G}$ has *representation theoretic finite holonomy* if $\widehat{\mathcal{G}}$ is a union of compact-open subsets. (An equivalent conditions can be obtained in terms of the fundamental groups when B is path-connected, see Proposition 2.3 below.) Let $C^*(\mathcal{G})$ be the enveloping C^*-algebra of the bundle of compact groups $\mathcal{G}$. We have proved in [32, Theorem 5.2] that

$$K^j_{\mathcal{G}}(B) \cong K_j(C^*(\mathcal{G})), \tag{2}$$

provided that $\mathcal{G}$ has representation theoretic finite holonomy. This guarantees that $K^j_{\mathcal{G}}(B)$ is not too small. It also points out to an alternative, algebraic definition of the groups $K^i_{\mathcal{G}}(X)$.

Let us put our results into some perspective. Recently, an important paper of Emerson and Meyer with implications for our project [19] has appeared. In that paper, Emerson and Meyer developed a $\mathcal{G}$-equivariant version of the bivariant K-groups, denoted $KK^{\mathcal{G}}$ for a proper groupoid $\mathcal{G}$. Then, to any K-oriented mapping $f : X \to Y$, they had associated an element $f!_{\rm an} \in KK^{\mathcal{G}\ltimes Y}(C_0(X), C_0(Y))$ and had shown that this defined a functor. They then interpreted this result as an equivariant topological index theorem. R. Meyer has kindly informed us that some of the results, the present paper can also be obtained from this theorem in [19]. It would be quite worthwhile to complete in full detail this alternative proof. We are grateful to him and to H. Emerson for discussions on that subject at the above mentioned conference ([35]) and later. In particular, in view of the comments in the introduction to that paper, the exact relation between our gauge-invariant operators and bivariant K-theory still needs to be understood. We thus feel that the concrete constructions in this paper, done in the spirit of the original Atiyah-Singer paper, have a merit of their own. They may turn out to be useful also in cyclic homology calculations of the index and shed some new light and explain the difficult results of Emerson and Meyer.

The structure of the paper is as follows. We start with the definition of gauge-equivariant K- theory and with some basic results from [32], most of them related to the "finite holonomy condition," a condition on bundles of compact groups that we recall in Section 2. In Subsection 2.2, we describe an equivalent definition of gauge-equivariant K-theory in terms of complexes of vector bundles. This will turn out to be especially useful when studying the topological index. In Section 3, we review the Thom isomorphism in gauge-equivariant K-theory, we define and study the Gysin mappings, and we define the topological index, building on the results from [33]. In Section 4, we establish the main properties of topological index. As a consequence, in Section 5, we prove that the topological and analytical index coincide. We conclude with a discussion of the cyclic homology of the relevant groupoid algebras and with some comments on future work.

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2. PRELIMINARIES

We begin by recalling the definition of gauge-equivariant K-theory and some basic results from [32]. An important part of our discussion will be devoted to the discussion of the finite holonomy condition for a bundle of compact groups $p : \mathcal{G} \to B$, a condition introduced below.

All vector bundles considered in this paper are **complex** *vector bundles, excluding the tangent bundles to the various manifolds appearing below and if otherwise mentioned.*

2.1. Bundles of Compact Groups and Finite Holonomy Conditions

We begin with a short discussion of bundles of locally compact groups. Then we study finite holonomy conditions for bundles of compact groups. Let G be a locally compact group. We shall denote by $\hat{G}$ the set of equivalence classes of irreducible representations of G with the Jacobson topology and by $\mathrm{Aut}(G)$ we shall denote the group of continuous (group) automorphisms of G. We endow this group with the topology of uniform convergence on compact subsets. Clearly, $\mathrm{Aut}(G)$ acts on $\hat{G}$.

Definition 2.1. Let B be a locally compact space and let G be a locally compact group. A *bundle of locally compact groups* $\mathcal{G}$ *with typical fiber* G *over* B is, by definition, a fiber bundle $\mathcal{G} \to B$ with typical fiber G and structural group $\mathrm{Aut}(G)$.

We need now to introduce the representation theoretic holonomy of a bundle of Lie group with *compact fibers* $p : \mathcal{G} \to B$. Let $\mathcal{P} \to B$ be the principal $\mathrm{Aut}(G)$-bundle such that

$$\mathcal{G} \cong \mathcal{P} \times_{\mathrm{Aut}(G)} G := (\mathcal{P} \times G)/\,\mathrm{Aut}(G).$$

We fix the above notation. In particular, if $\mathcal{G}_b$ is the fiber of $\mathcal{G} \to B$ above b, then $\mathcal{G}_b \simeq G$ as groups (nonuniquely).

We assume from now on that G is compact and denote by $\widehat{\mathcal{G}}$ the (disjoint) union of the sets $\widehat{\mathcal{G}}_b$ of equivalence classes of irreducible representations of the groups $\mathcal{G}_b$. Using the natural action of $\operatorname{Aut}(G)$ on $\hat{G}$, we can naturally identify $\widehat{\mathcal{G}}$ with $\mathcal{P}\times_{\operatorname{Aut}(G)}\hat{G}$ as fiber bundles over B. Let also $\operatorname{Aut}_0(G)$ be the connected component of the identity in $\operatorname{Aut}(G)$. The group $\operatorname{Aut}_0(G)$ will act trivially on the set $\hat{G}$, because the later is discrete. Let

$$H_G := \operatorname{Aut}(G)/\operatorname{Aut}_0(G), \quad \mathcal{P}_0 := \mathcal{P}/\operatorname{Aut}_0(G), \quad \text{and} \quad \widehat{\mathcal{G}} \simeq \mathcal{P}_0 \times_{H_G} \hat{G}.$$

Above, $\widehat{\mathcal{G}}$ is defined because $\mathcal{P}_0$ is an H_G-principal bundle. The space $\widehat{\mathcal{G}}$ is called the *representation space of* $\mathcal{G}$ and the covering $\widehat{\mathcal{G}} \to B$ is called *the representation covering associated to* $\mathcal{G}$.

Assume now that B is a path-connected, locally simply-connected space and fix a point $b_0 \in B$. We shall denote, as usual, by $\pi_1(B, b_0)$ the fundamental group of B. Then the bundle $\mathcal{P}_0$ is classified by a morphism

$$\rho : \pi_1(B, b_0) \to H_G := \operatorname{Aut}(G)/\operatorname{Aut}_0(G), \tag{3}$$

which will be called the *holonomy of the representation covering of* $\mathcal{G}$.

For our further reasoning, we shall sometimes need the following finite holonomy condition.

Definition 2.2. We say that $\mathcal{G}$ has *representation theoretic finite holonomy* if every $\sigma \in \widehat{\mathcal{G}}$ is contained in a compact-open subset of $\widehat{\mathcal{G}}$.

In the cases we are interested in, the above condition can be reformulated as follows [32].

Proposition 2.3. *Assume that B is path-connected and locally simply-connected. Then $\mathcal{G}$ has representation theoretic finite holonomy if and only if $\pi_1(B, b_0)\sigma \subset \widehat{G}$ is a finite set for any irreducible representation σ of G.*

The case when $\mathcal{G}$ *does not* have the representation theoretic finite holonomy condition ("does not satisfy the finite holonomy condition" for short) leads to some interesting, but pathological situations [19, 32]. In particular, they lead to the appearance of bundles with nontrivial Dixmier–Douady invariants [32]. See [40, 44] for more on Dixmier–Douady invariants.

Example 2.4. For instance, let $A_1, \dots, A_k$ be *commuting* $n\times n$ matrices with integer coefficients and denote also by the same letters the corresponding automorphism of $\mathbb{Z}^n$ and $G := (S^1)^n$. Let $B := (S^1)^k$ (so both G and B are tori, possibly of different dimensions). Then the matrices $A_1, \dots, A_k$ give rise to a morphism $\pi_1(B) \simeq \mathbb{Z}^k \to \operatorname{Aut}(G)$. By choosing the matrices $A_1, \dots, A_k$ appropriately, we may arrange that the resulting G family of Lie groups will *not* have the finite holonomy condition. We may even have that $\pi_1(B) \simeq \mathbb{Z}^k \to \operatorname{Aut}(G)$ is injective.

From now on, we shall assume that $\mathcal{G}$ has representation theoretic finite holonomy, unless explicitly otherwise mentioned.

2.2. Gauge-Equivariant K-Theory

Let us now define the gauge equivariant K-theory groups of a "$\mathcal{G}$-fiber bundle" $\pi_Y : Y \to B$. All our definitions are well known if B is reduced to a point (cf. [4, 23]). First we need to fix the notation.

If $f_i : Y_i \to B$, $i = 1, 2$, are two mappings, we shall denote by

$$Y_1 \times_B Y_2 := \{(y_1, y_2) \in Y_1 \times Y_2,\ f_1(y_1) = f_2(y_2)\} \tag{4}$$

their fibered product. Let $p : \mathcal{G} \to B$ be a bundle of locally compact groups and let $\pi_Y : Y \to B$ be a continuous mapping. We shall say that $\mathcal{G}$ *acts on* Y if each group $\mathcal{G}_b$ acts continuously on $Y_b := \pi^{-1}(b)$ and the induced mapping μ defined by

$$\mathcal{G} \times_B Y := \{(g, y) \in \mathcal{G} \times Y,\ p(g) = \pi_Y(y)\} \ni (g, y) \longrightarrow \mu(g, y) := gy \in Y$$

is continuous. If $\mathcal{G}$ acts on Y, we shall say that Y is a *$\mathcal{G}$-space*. If, in addition to that, $Y \to B$ is also locally trivial, we shall say that Y is a *$\mathcal{G}$-fiber bundle*, or, simply, a *$\mathcal{G}$-bundle*. This definition is a particular case of the definition of the action of a differentiable groupoid on a space.

Let $\pi_Y : Y \to B$ be a $\mathcal{G}$-space, with $\mathcal{G}$ a bundle of compact groups over B. Recall that a vector bundle $\tilde{\pi}_E : E \to Y$ is a *$\mathcal{G}$-equivariant vector bundle* (or simply a *$\mathcal{G}$-equivariant vector bundle*) if

$$\pi_E := \pi_Y \circ \tilde{\pi}_E : E \to B$$

is a $\mathcal{G}$-space, the projection

$$\tilde{\pi}_E : E_b := \pi_E^{-1}(b) \to Y_b := \pi_Y^{-1}(b)$$

is $\mathcal{G}_b := p^{-1}(b)$ equivariant, and the induced action $E_y \to E_{gy}$ of $g \in \mathcal{G}$, between the corresponding fibers of $E \to Y$, is linear for any $y \in Y_b$, $g \in \mathcal{G}_b$, and $b \in B$.

To define gauge-equivariant K–theory, we first recall some preliminary definitions from [32]. Let $\tilde{\pi}_E : E \to Y$ be a $\mathcal{G}$-equivariant vector bundle and let $\tilde{\pi}_{E'} : E' \to Y'$ be a $\mathcal{G}'$-equivariant vector bundle, for two bundles of compact groups $\mathcal{G} \to B$ and $\mathcal{G}' \to B'$. We shall say that $(\gamma, \varphi, \eta, \psi) : (\mathcal{G}', E', Y', B') \to (\mathcal{G}, E, Y, B)$ is a *γ–equivariant morphism of vector bundles* if the following five conditions are satisfied:

(1) $\gamma\colon \mathcal{G}' \to \mathcal{G}$, $\varphi\colon E' \to E$, $\eta\colon Y' \to Y$, and $\psi\colon B' \to B$,
(2) all the resulting diagrams are commutative,
(3) $\varphi(ge) = \gamma(g)\varphi(e)$ for all $e \in E'_b$ and all $g \in \mathcal{G}'_b$,
(4) γ is a group morphism in each fiber, and
(5) f is a vector bundle morphism.

We shall say that $\phi\colon E \to E'$ *is a γ–equivariant morphism of vector bundles* if, by definition, it is part of a morphism $(\gamma, \varphi, \eta, \psi)\colon (\mathcal{G}', E', Y', B') \to (\mathcal{G}, E, Y, B)$. Note that η and ψ are determined by γ and ϕ.

Let $p\colon \mathcal{G} \to B$ be a bundle of compact groups and $\pi_Y : Y \to B$ be a $\mathcal{G}$-space. The set of isomorphism classes of $\mathcal{G}$-equivariant vector bundles $\tilde{\pi}_E\colon E \to Y$ will be denoted by $\mathcal{E}_{\mathcal{G}}(Y)$. On this set, we introduce a monoid operation, denoted "+," using the direct sum of vector bundles. This defines a monoid structure on the set $\mathcal{E}_{\mathcal{G}}(Y)$ as in the case when B consists of a point.

Definition 2.5. Suppose that $\mathcal{G} \to B$ is a bundle of compact groups acting on the $\mathcal{G}$-space $Y \to B$. Assume Y to be compact. The *$\mathcal{G}$-equivariant K-theory group* $K^0_{\mathcal{G}}(Y)$ is defined as the group completion of the monoid $\mathcal{E}_{\mathcal{G}}(Y)$.

The groups $K^0_{\mathcal{G}}(Y)$ have a natural ring structure and the functoriality properties of the usual equivariant K-theory groups extend to the gauge equivariant K-theory groups.

A $\mathcal{G}$-equivariant vector bundle $E \to Y$ on a $\mathcal{G}$-space $Y \to B$, Y compact, is called *trivial* if, by definition, there exists a $\mathcal{G}$-equivariant vector bundle $E' \to B$ such that E is isomorphic to the pull-back of E' to Y. Thus $E \simeq Y \times_B E'$. If $\mathcal{G} \to B$ has representation theoretic finite holonomy and Y is a compact $\mathcal{G}$-bundle, then every $\mathcal{G}$-equivariant vector bundle over Y can be embedded into a trivial $\mathcal{G}$-equivariant vector bundle. This embedding will necessarily be as a direct sum.

If $\mathcal{G} \to B$ does not have finite holonomy, it is possible to provide examples of $\mathcal{G}$-equivariant vector bundles that do not embed into trivial $\mathcal{G}$-equivariant vector bundles [32]. Also, a related example from [32] shows that the groups $K^0_{\mathcal{G}}(Y)$ can be fairly small if the holonomy of $\mathcal{G}$ is "large." This is seen by considering $k = 2$ in Example 2.4 and choosing an injective morphism $\mathbb{Z}^k \to \mathrm{Aut}(G)$. In this case, $K^0_{\mathcal{G}}(Y)$ and $K_0(C^*(\mathcal{G}))$ are *not* isomorphic. A very similar construction was used in [19]. A further observation is that it follows from the definitions that the tensor product of vector bundles defines a natural ring structure on $K^0_{\mathcal{G}}(Y)$.

The definition of the gauge-equivariant groups extends to noncompact $\mathcal{G}$-spaces Y as in the case of equivariant K–theory with some small modifications. Let Y be a $\mathcal{G}$-bundle. We shall denote then by $Y^+ := Y \cup B$ the compact space obtained from Y by the one-point compactification of each fiber Y_b of $\pi_Y : Y \to B$ (recall that B is compact). The need to consider the space Y^+ is the main reason for also considering bundles on B with non longitudinally smooth fibers. Then we define

$$K^0_{\mathcal{G}}(Y) := \ker\big(K^0_{\mathcal{G}}(Y^+) \to K^0_{\mathcal{G}}(B)\big).$$

Also as in the classical case, we let

$$K^n_{\mathcal{G}}(Y, Y') := K^0_{\mathcal{G}}((Y \setminus Y') \times \mathbb{R}^n)$$

for a $\mathcal{G}$-subbundle $Y' \subset Y$. Then [32] we have the following periodicity result.

Theorem 2.6. *We have natural isomorphisms*

$$K^n_{\mathcal{G}}(Y, Y') \cong K^{n-2}_{\mathcal{G}}(Y, Y').$$

The extended gauge-equivariant K-theory is then functorial with respect to open embeddings.

For the purpose of defining the Thom isomorphism, it is convenient to work with an equivalent definition of gauge-equivariant K-theory in terms of complexes of vector bundles. This will turn out to be especially useful when studying the topological index. The details and proofs can be found in [33]. Analogous results in $\mathcal{G}$-bivariant theory were obtained by [19].

Let $X \to B$ be a locally compact, paracompact $\mathcal{G}$-bundle. A *finite complex of $\mathcal{G}$–equivariant vector bundles* over X is a complex

$$(E^*, d) = \left(\cdots \xrightarrow{d_{i-1}} E^i \xrightarrow{d_i} E^{i+1} \xrightarrow{d_{i+1}} \cdots\right)$$

of $\mathcal{G}$-equivariant vector bundles over X with only finitely many E^i's different from zero. Explicitly, the E^i are $\mathcal{G}$–equivariant vector bundles, the d_i's are $\mathcal{G}$-equivariant morphisms, $d_{i+1}d_i = 0$ for every i, and $E^i = 0$ for $|i|$ large enough. We shall also use the notation $(E^*, d) = \big(E^0, \dots, E^n,\ d_i : E^i|_Y \to E^{i+1}|_Y\big)$, if $E^i = 0$ for $i < 0$ and for $i > n$. As usual, a *morphism of complexes* $f : (E^*, d) \to (F^*, \delta)$ is a sequence of morphisms $f_i \colon E^i \to F^i$ such that $f_{i+1}d_i = \delta_{i+1}f_i$, for all i. These constructions yield the category of finite complexes of $\mathcal{G}$–equivariant vector bundles. Isomorphism in this category will be denoted by $(E^*, d) \cong (F^*, \delta)$.

Definition 2.7. Let X be a compact $\mathcal{G}$-bundle and Y be a closed $\mathcal{G}$-invariant subbundle. Denote by $C^n_{\mathcal{G}}(X, Y)$ the set of (isomorphism classes of) sequences

$$(E^*, d) = \big(E^0, E^1, \dots, E^n,\ d_k : E^k|_Y \to E^{k+1}|_Y\big)$$

of $\mathcal{G}$-equivariant vector bundles over X such that $(E^k|_Y, d)$ is exact if we let $E^j = 0$ for $j < 0$ or $j > n$. We endow $C^n_{\mathcal{G}}(X, Y)$ with the semigroup structure given by the direct sums of complexes. An element in $C^n_{\mathcal{G}}(X, Y)$ is called *elementary* if it is isomorphic to a complex of the form

$$\cdots \to 0 \to E \overset{\mathrm{Id}}{\to} E \to 0 \to \cdots$$

Two complexes $(E^*, d), (F^*, \delta) \in C^n_{\mathcal{G}}(X, Y)$ are called *equivalent* if and only if there exist elementary complexes $Q^1, \dots, Q^k, P^1, \dots, P^m \in C^n_{\mathcal{G}}(X, Y)$ such that

$$E \oplus Q_1 \oplus \cdots \oplus Q_k \cong F \oplus P_1 \oplus \cdots \oplus P_m\,,$$

in which case we write $E \simeq F$. The semigroup of equivalence classes of sequences in $C^n_{\mathcal{G}}(X, Y)$ will be denoted by $L^n_{\mathcal{G}}(X, Y)$.

We therefore obtain natural semigroup homomorphisms

$$C^n_{\mathcal{G}}(X, Y) \to C^{n+1}_{\mathcal{G}}(X, Y) \quad \text{and} \quad C_{\mathcal{G}}(X, Y) := \bigcup_n C^n_{\mathcal{G}}(X, Y).$$

The equivalence relation $\sim$ commutes with embeddings, so the above morphisms induce morphisms $L^n_{\mathcal{G}}(X, Y) \to L^{n+1}_{\mathcal{G}}(X, Y)$. Let $L^\infty_{\mathcal{G}}(X, Y) := \varinjlim L^n_{\mathcal{G}}(X, Y)$.

Definition 2.8. Let X be a compact $\mathcal{G}$-space and $Y \subset X$ be a $\mathcal{G}$-invariant subset. An *Euler characteristic* χ_n is a natural transformation of functors $\chi_n : L^n_{\mathcal{G}}(X,Y) \to K^0_{\mathcal{G}}(X,Y)$, such that, for $Y = \emptyset$, it takes the form

$$\chi_n(E) = \sum_{i=0}^{n} (-1)^i [E^i],$$

for any sequence $E = (E^*, d) \in L^n_{\mathcal{G}}(X,Y)$.

Now, let (E,d) be a complex of $\mathcal{G}$-equivariant vector bundles over a $\mathcal{G}$-space X. A point $x \in X$ will be called a *point of acyclicity of* (E,d) if the restriction of (E,d) to x, i.e., the sequence of linear spaces

$$(E,d)_x = \left(\cdots \xrightarrow{(d_i)_x} E^i_x \xrightarrow{(d_{i+1})_x} E^{i+1}_x \xrightarrow{(d_{i+2})_x} \cdots \right)$$

is exact. The *support* of a finite complex (E,d) is defined as the complement in X of the set of its points of acyclicity and is denoted $\mathrm{supp}(E,d)$. We shall say that two complexes in $L^n_{\mathcal{G}}(X,Y)$ are *homotopic* if they are isomorphic to the restrictions to $X \times \{0\}$ and $X \times \{1\}$ of a complex defined over $X \times I$ and acyclic over $Y \times I$.

Definition 2.9. Let X be a compact $\mathcal{G}$-bundle and $Y \subset X$ be a $\mathcal{G}$-invariant subbundle. We define $E^n_{\mathcal{G}}(X,Y)$ to be the semigroup of homotopy classes of complexes of $\mathcal{G}$-equivariant vector bundles of length n over X such that their restrictions to Y are acyclic (i.e. exact).

The restriction of morphisms induces a morphism $\Phi_n : E^n_{\mathcal{G}}(X,Y) \to L^n_{\mathcal{G}}(X,Y)$.

In the case of a locally compact, paracompact $\mathcal{G}$-bundle X, we change the definitions of $L^n_{\mathcal{G}}$ and $E^n_{\mathcal{G}}$ as follows. Recall that $Y \subset X$ is closed. Then, in the definition of $L^n_{\mathcal{G}}$, the morphisms d_i have to be defined and form an exact sequence outside some compact $\mathcal{G}$-invariant subset C of $X \smallsetminus Y$. In the definition of $E^n_{\mathcal{G}}$, the complexes have to be exact outside some compact $\mathcal{G}$-invariant subset of $X \smallsetminus Y$. In particular, $L^n_{\mathcal{G}}(X,Y) = L^n_{\mathcal{G}}(X^+,Y^+)$. The following result was proved in [33].

Theorem 2.10. *One has natural isomorphisms*

$$K^0_{\mathcal{G}}(X,Y) \cong L^n_{\mathcal{G}}(X,Y) \cong E^n_{\mathcal{G}}(X,Y), \quad n \geqslant 1, \tag{5}$$

induced by χ_n *and* Φ_n *in the case of compact bundles.*

3. THE THOM ISOMORPHISM, GYSIN MAPPINGS, AND TOPOLOGICAL INDEX

We now recall the Thom isomorphism in gauge-equivariant K-theory [33]. We begin with a discussion of the compact case.

3.1. The Compact Case

Let us recall products and the Thom morphism in gauge-equivariant K-theory. Let $\pi_X : X \to B$ be a $\mathcal{G}$-space, $\tilde{\pi}_F : F \to X$ be a complex $\mathcal{G}$-vector bundle over X, and $s : X \to F$ a $\mathcal{G}$-invariant section. We shall denote by $\Lambda^i F$ the i-th exterior power of F, which is again a complex $\mathcal{G}$-equivariant vector bundle over X. As in the proof of the Thom isomorphism for ordinary vector bundles, we define the complex $\Lambda(F,s)$ of $\mathcal{G}$-equivariant vector bundles over X by

$$\Lambda(F,s) := (0 \to \Lambda^0 F \xrightarrow{\alpha^0} \Lambda^1 F \xrightarrow{\alpha^1} \cdots \xrightarrow{\alpha^{n-1}} \Lambda^n F \to 0), \tag{6}$$

where $\alpha^k(v_x) = s(x) \wedge v_x$ for $v_x \in \Lambda^k F^x$ and $n = \dim F$. It is immediate that $\alpha^{j+1}(x)\alpha^j(x) = 0$, and hence, $(\Lambda(F,s),\alpha)$ is a complex indeed.

The Künneth formula shows that the complex $\Lambda(F,s)$ is acyclic for $s(x) \neq 0$, and hence, $\mathrm{supp}(\Lambda(F,s)) := \{x \in X | s(x) = 0\}$. If this set is compact, we can associate to the complex $\Lambda(F,s)$ of Eq. (6) an element

$$[\Lambda(F,s)] \in K^0_{\mathcal{G}}(X). \tag{7}$$

Let X be a $\mathcal{G}$-bundle and $\pi_F : F \to X$ be a $\mathcal{G}$-equivariant vector bundle over X. The point of the above construction is that $\pi_F^*(F)$, the lift of F back to itself, has a canonical section $s_F(f) = (f, f)$ whose support is X. Hence, if X is a compact space, we obtain an element

$$\lambda_F := [\Lambda(\pi_F^*(F), s_F)] \in K^0_{\mathcal{G}}(F). \tag{8}$$

Recall that the tensor product of vector bundles defines a natural product $ab = a \otimes b \in K^0_{\mathcal{G}}(X)$ for any $a \in K^0_{\mathcal{G}}(B)$ and any $b \in K^0_{\mathcal{G}}(X)$, where $\pi_X : X \to B$ is a compact $\mathcal{G}$-space, as above.

Recall that all our vector bundles are assumed to be complex vector bundles, except for the ones coming from geometry (tangent bundles, their exterior powers) and where explicitly mentioned. Due to the importance that F be complex in the following definition, we shall occasionally repeat this assumption.

Definition 3.1. Let $\pi_F : F \to X$ be a (complex) $\mathcal{G}$-equivariant vector bundle. Assume the $\mathcal{G}$-bundle $X \to B$ is compact and let $\lambda_F \in K^0_{\mathcal{G}}(F)$ be the class defined in Eq. (8), then the mapping

$$\varphi^F : K^0_{\mathcal{G}}(X) \to K^0_{\mathcal{G}}(F), \qquad \varphi^F(a) = \pi_F^*(a) \otimes \lambda_F,$$

is called the *Thom morphism.*

As we shall see below, the definition of the Thom homomorphism extends to the case when X is not compact, although the Thom element itself is not defined if X is not compact.

The definition of the Thom homomorphism immediately gives the following proposition. We shall use the notation of Proposition 3.1.

Proposition 3.2. *The Thom morphism $\varphi^F : K^0_{\mathcal{G}}(X) \to K^0_{\mathcal{G}}(F)$ is an isomorphism of $K^0_{\mathcal{G}}(B)$-modules. It extends to K^1 by periodicity.*

Let $\iota : X \hookrightarrow F$ be the zero section embedding of X into F. Then ι induces a homomorphism $\iota^* : K^0_{\mathcal{G}}(F) \to K^0_{\mathcal{G}}(X)$. Then $\iota^*\varphi^F(a) = a \cdot \sum_{i=0}^n (-1)^i \Lambda^i F$.

3.2. The Noncompact Case

We consider now the case when X is locally compact, but not necessarily compact. The complex $\Lambda(\pi_F^*(F), s_F)$ has a noncompact support, and hence, it does not define an element of $K^0_{\mathcal{G}}(F)$. However, if $a = [(E, \alpha)] \in K^0_{\mathcal{G}}(X)$ is represented by the complex (E, α) of vector bundles with compact support, then we can still consider the tensor product complex

$$\big(\pi_F^*(\mathcal{E}),\, \pi_F^*(\alpha)\big) \otimes \Lambda(\pi_F^* F, s_F).$$

From the Künneth formula for the homology of a tensor product, we obtain that the support of a tensor product complex is the intersection of the supports of the two complexes. In particular, we obtain

$$\begin{aligned} \operatorname{supp}\{(\pi_F^* E, \pi_F^*\alpha) \otimes \Lambda(\pi_F^* F, s_F)\} &\subset \operatorname{supp}(\pi_F^* E, \pi_F^*\alpha) \cap \operatorname{supp}\Lambda(\pi_F^* F, s_F) \\ &\subset \operatorname{supp}(\pi_F^* E, \pi_F^*\alpha) \cap X = \operatorname{supp}(E, \alpha). \end{aligned} \tag{9}$$

Thus, the complex $(\pi_F^*\mathcal{E}, \pi_E^*\alpha) \otimes \Lambda(\pi_F^* F, s_F)$ has compact support and, hence, defines an element in $K^0_{\mathcal{G}}(F)$. Of course, the reason for this is that the Thom element is more naturally an element of a bivariant K-theory group.

Proposition 3.3. *The homomorphism of $K^0_{\mathcal{G}}(B)$-modules*

$$\varphi^F : K^0_{\mathcal{G}}(X) \to K^0_{\mathcal{G}}(F), \quad \varphi^F(a) = [(\pi_F^*\mathcal{E}, \pi_F^*\alpha) \otimes \Lambda(\pi_F^* F, s_F)], \tag{10}$$

defined in Eq. (9) extends the Thom morphism to the case of not necessarily compact X. The Thom morphism φ^F satisfies $i^*\varphi^F(a) = a \cdot \sum_{i=0}^n (-1)^i \Lambda^i F$.

We are now ready to formulate the Thom isomorphism in the setting of gauge-equivariant vector bundles.

Theorem 3.4 [33, Theorem 4.5]. *Let $X \to B$ be a $\mathcal{G}$-bundle and $F \to X$ a complex $\mathcal{G}$- equivariant vector bundle, then $\varphi^F : K^i_{\mathcal{G}}(X) \to K^i_{\mathcal{G}}(F)$ is an isomorphism.*

We now discuss a few constructions related to the Thom isomorphism, which will be necessary for the definition of the topological index. The most important one is the Gysin mapping. For several of the constructions below, the setting of $\mathcal{G}$-spaces and even $\mathcal{G}$-bundles is too general, and we shall have to consider *longitudinally smooth* $\mathcal{G}$-fiber bundles $\pi_X : X \to B$. The main reason why we need longitudinally smooth bundles to define the Gysin mapping is the same as in the definition of the Gysin mapping for embeddings of smooth manifolds. We shall denote by $T_{\mathrm{vert}}X$ the *vertical* tangent bundle to the fibers of $X \to B$. All tangent bundles below will be *vertical* tangent bundles.

Let X and Y be longitudinally smooth $\mathcal{G}$-fiber bundles, $i : X \to Y$ be an equivariant fiberwise embedding, and $p_T : T_{\mathrm{vert}}X \to X$ be the vertical tangent bundle to X. Assume Y is equipped with a $\mathcal{G}$-invariant Riemannian metric and let $p_N : N_{\mathrm{vert}} \to X$ be the fiberwise normal bundle to the image of i.

Let us choose a function $\varepsilon : X \to (0,\infty)$ such that the mapping $\rho : N_{\mathrm{vert}} \to N_{\mathrm{vert}}$, defined by $\phi(\xi) = \frac{\xi}{1+|\xi|}$ is $\mathcal{G}$-equivariant and defines a $\mathcal{G}$-diffeomorphism $\Phi : N_{\mathrm{vert}} \to W$ onto a bundle of open tubular neighborhoods $W \supset X$ in Y.

Let $(N \oplus N)_{\mathrm{vert}} := N_{\mathrm{vert}} \oplus N_{\mathrm{vert}}$. The embedding $i : X \to Y$ can be written as a composition of two fiberwise embeddings $i_1 : X \to W$ and $i_2 : W \to Y$. Passing to differentials, we obtain

$$T_{\mathrm{vert}}X \xrightarrow{di_1} T_{\mathrm{vert}}W \xrightarrow{di_2} T_{\mathrm{vert}}Y \quad \text{and} \quad d\Phi : T_{\mathrm{vert}}N \to T_{\mathrm{vert}}W,$$

where we use the simplified notation $T_{\mathrm{vert}}N = T_{\mathrm{vert}}N_{\mathrm{vert}}$.

Lemma 3.5. (cf. [23, p. 112]) *The manifold $T_{\mathrm{vert}}N$ can be identified with $p_T^*(N \oplus N)_{\mathrm{vert}}$ using a $\mathcal{G}$-equivariant diffeomorphism ψ that makes the following diagram commutative*:

$$\begin{array}{ccccc} p_T^*(N \oplus N)_{\mathrm{vert}} & & \xleftarrow{\quad\psi\quad} & & T_{\mathrm{vert}}N \\ \downarrow & & & & \downarrow \\ T_{\mathrm{vert}}X & & & & N_{\mathrm{vert}} \\ & \searrow^{p_T} & & {}^{p_N}\swarrow & \\ & & X, & & \end{array}$$

where $p_T : T_{\mathrm{vert}}X \to X$ is the canonical projection.

With the help of the relation $i \cdot (n_1, n_2) = (-n_2, n_1)$, we can equip

$$p_T^*(N \oplus N)_{\mathrm{vert}} = p_T^*(N_{\mathrm{vert}}) \oplus p_T^*(N_{\mathrm{vert}})$$

with the structure of a complex manifold. Then we can consider the Thom homomorphism

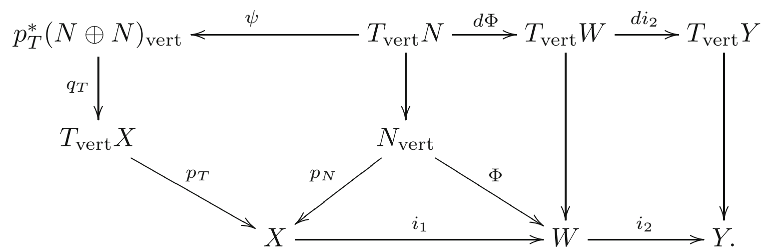
$$\varphi : K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \to K^0_{\mathcal{G}}(p_T^*(N \oplus N)_{\mathrm{vert}}).$$

Since $T_{\mathrm{vert}}W$ is an open $\mathcal{G}$-stable subset of $T_{\mathrm{vert}}Y$ and $di_2 : T_{\mathrm{vert}}W \to T_{\mathrm{vert}}Y$ is a fiberwise embedding, by using the direct image morphism, we obtain a homomorphism $(di_2)_* : K^0_{\mathcal{G}}(T_{\mathrm{vert}}W) \to K^0_{\mathcal{G}}(T_{\mathrm{vert}}Y)$.

Definition 3.6. Let $i : X \to Y$ be an equivariant embedding of $\mathcal{G}$-bundles. The *Gysin homomorphism* is the mapping

$$i_! : K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \to K^0_{\mathcal{G}}(T_{\mathrm{vert}}Y), \qquad i_! = (di_2)_* \circ (d\Phi^{-1})^* \circ \psi^* \circ \varphi.$$

We thus see that the Gysin homomorphism is obtained by passage to K-groups in the upper part of the diagram



A different choice of a metric or neighborhood W induces a homotopic mapping and (by (3) of Theorem 3.7 below) the same Gysin homomorphism $i_!$. Recall from the following result from [33], where $p_T : T_{\mathrm{vert}}X \to X$ is the canonical projection.

Theorem 3.7. [Properties of Gysin homomorphism] *Let $i : X \to Y$ be a $\mathcal{G}$-embedding.*

(1) *$i_!$ is a homomorphism of $K^0_{\mathcal{G}}(B)$-modules.*
(2) *Let $i : X \to Y$ and $j : Y \to Z$ be two fiberwise $\mathcal{G}$-embeddings, then $(j \circ i)_! = j_! \circ i_!$.*
(3) *Let fiberwise embeddings $i_1 : X \to Y$ and $i_2 : X \to Y$ be $\mathcal{G}$-homotopic in the class of embeddings. Then $(i_1)_! = (i_2)_!$.*
(4) *Let $i : X \to Y$ be a fiberwise $\mathcal{G}$-diffeomorphism and $di\colon T_{\mathrm{vert}}X \to T_{\mathrm{vert}}Y$ be the differential of i. Then $i_!^{-1} = (di)^*$.*
(5) *A fiberwise embedding $i : X \to Y$ can be represented as a compositions of embeddings X in N_{vert} (as the zero section $s_0\colon X \to N$) and $N_{\mathrm{vert}} \to Y$ by $i_2 \circ \Phi\colon N_{\mathrm{vert}} \to Y$. Then $i_! = (i_2 \circ \Phi)_!(s_0)_!$.*
(6) *Consider the complex bundle $p_T^*(N_{\mathrm{vert}} \otimes \mathbb{C})$ over $T_{\mathrm{vert}}X$. Form the complex $\Lambda(p_T^*(N_{\mathrm{vert}} \otimes \mathbb{C}), 0)$:*

$$0 \to \Lambda^0(p_T^*(N_{\mathrm{vert}} \otimes \mathbb{C})) \xrightarrow{0} \cdots \xrightarrow{0} \Lambda^k(p_T^*(N_{\mathrm{vert}} \otimes \mathbb{C})) \to 0$$

with noncompact support. If $a \in K^0_{\mathcal{G}}(T_{\mathrm{vert}}X)$, then the complex

$$a \otimes \Lambda(p_T^*(N_{\mathrm{vert}} \otimes \mathbb{C}), 0)$$

has compact support and defines an element of $K^0_{\mathcal{G}}(T_{\mathrm{vert}}X)$. Then

$$(di)^* i_!(a) = a \cdot \Lambda(p_T^*(N_{\mathrm{vert}} \otimes \mathbb{C}), 0),$$

where di is the differential of the embedding i.
(7) *$i_!(x(di)^*y) = i_!(x) \cdot y$, where $x \in K^0_{\mathcal{G}}(T_{\mathrm{vert}}X)$ and $y \in K^0_{\mathcal{G}}(T_{\mathrm{vert}}Y)$.*

We shall need also the following properties of the Gysin mapping. If $X = B$, the trivial longitudinally smooth $\mathcal{G}$-bundle, we shall identify $T_{\mathrm{vert}}X = B$ and $T_{\mathrm{vert}}\mathcal{V} = \mathcal{V} \otimes \mathbb{C}$ for a real bundle $\mathcal{V} \to B$.

Theorem 3.8. [33, Theorem 5.4] *Suppose that $\mathcal{V} \to B$ is a $\mathcal{G}$-equivariant real vector bundle and that $X = B$. Then the mapping*

$$i_! : K^0_{\mathcal{G}}(B) = K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \to K^0_{\mathcal{G}}(T_{\mathrm{vert}}\mathcal{V}) = K^0_{\mathcal{G}}(\mathcal{V} \otimes \mathbb{C})$$

coincides with the Thom homomorphism $\varphi^{\mathcal{V} \otimes \mathbb{C}}$.

Now we recall our "fibered Mostow–Palais theorem" that will be useful in defining the index. A generalization of this result to proper groupoids can be found in [19].

Theorem 3.9. [33, Theorem 6.1] *Let $\pi_X\colon X \to B$ be a compact $\mathcal{G}$-fiber bundle. Then there exists a real $\mathcal{G}$-equivariant vector bundle $\mathcal{V} \to B$ and a fiberwise smooth $\mathcal{G}$-embedding $X \to \mathcal{V}$. After averaging one can assume that the action of $\mathcal{G}$ on $\mathcal{V}$ is orthogonal.*

Let us now turn to the definition of the topological index. Let $X \to B$ be a compact, longitudinally smooth $\mathcal{G}$-bundle. From Theorem 3.9, it follows that there exists a $\mathcal{G}$-equivariant *real* vector bundle $\mathcal{V} \to B$ and a fiberwise smooth $\mathcal{G}$-equivariant embedding $i : X \to \mathcal{V}$. We can assume that $\mathcal{V}$ is endowed with an orthogonal metric and that $\mathcal{G}$ preserves this metric. Thus, the Gysin homomorphism

$$i_!\colon K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \to K^0_{\mathcal{G}}(T_{\mathrm{vert}}\mathcal{V}) = K^0_{\mathcal{G}}(\mathcal{V} \otimes \mathbb{C})$$

is defined. Since $T_{\mathrm{vert}}\mathcal{V} = \mathcal{V} \otimes \mathbb{C}$ is a complex vector bundle, we have the following Thom isomorphism:

$$\varphi\colon K^0_{\mathcal{G}}(B) \xrightarrow{\sim} K^0_{\mathcal{G}}(T_{\mathrm{vert}}\mathcal{V}).$$

Definition 3.10. The *topological index* is, by definition, the morphism

$$\text{t-ind}^X_{\mathcal{G}} : K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \to K^0_{\mathcal{G}}(B), \qquad \text{t-ind}^X_{\mathcal{G}} := \varphi^{-1} \circ i_!.$$

The topological index satisfies the following properties.

Theorem 3.11 [33, Theorem 6.3]. *Let $X \to B$ be a longitudinally smooth bundle and*

$$\text{t-ind}^X_{\mathcal{G}}\colon\ K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \to K^0_{\mathcal{G}}(B)$$

be its associated topological index. Then

(1) $\text{t-ind}^X_{\mathcal{G}}$ *does not depend on the choice of the $\mathcal{G}$-equivariant vector bundle $\mathcal{V}$ and on the $\mathcal{G}$-equivariant embedding $i\colon X \to \mathcal{V}$.*
(2) $\text{t-ind}^X_{\mathcal{G}}$ *is a $K^0_{\mathcal{G}}(B)$-homomorphism.*
(3) *If $X = B$, then the mapping*

$$\text{t-ind}^X_{\mathcal{G}}\colon\ K^0_{\mathcal{G}}(B) = K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \to K^0_{\mathcal{G}}(B)$$

coincides with $\mathrm{Id}_{K^0_{\mathcal{G}}(B)}$.
(4) *Suppose X and Y are compact longitudinally smooth $\mathcal{G}$-bundles, $i : X \to Y$ is a fiberwise $\mathcal{G}$-embedding. Then the diagram*

$$\begin{array}{ccc} K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) & \xrightarrow{\quad i_! \quad} & K^0_{\mathcal{G}}(T_{\mathrm{vert}}Y) \\ & {\scriptstyle \text{t-ind}^X_{\mathcal{G}}} \searrow \quad \swarrow {\scriptstyle \text{t-ind}^Y_{\mathcal{G}}} & \\ & K^0_{\mathcal{G}}(B). & \end{array}$$

commutes.

4. AN AXIOMATIC APPROACH

Condition 4.1. From now on, we shall assume some additional smoothness properties of the spaces and actions involved. Namely, we assume B to be a *smooth* (compact) manifold and $X \to B$ to be a *smooth* bundle. We also assume that all vector bundles involved are smooth. Let us trivialize over an open subset $U \subset B$, so that $X|_U = X_0 \times U$ and $\mathcal{G}|_U = G \times U \subset G \times \mathbb{R}^n$ (we consider U as an open neighborhood of zero in $\mathbb{R}^n$). We then assume that the induced action of $G \times U$ on X_0 is smooth.

Below, by a *morphism of bundles of groups* we shall mean a morphism of bundles which is a fiberwise homomorphism of groups. The morphism ψ^* is naturally induced by the group-bundle morphism $\psi : \mathcal{H} \to \mathcal{G}$. We now introduce the important concept of an “index function,” extending the definition in [5].

Definition 4.2. An *index function* is a family of $K^0_{\mathcal{G}}(B)$-homomorphisms $\{\mathrm{ind}^X_{\mathcal{G}}\}$

$$\mathrm{ind}^X_{\mathcal{G}} : K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \to K^0_{\mathcal{G}}(B),$$

where $\mathcal{G}$ runs over the set of bundles of compact Lie groups and X runs over compact longitudinally smooth $\mathcal{G}$-bundles. This family is required to satisfy the following two conditions.

(1) If $f : X \to Y$ is a $\mathcal{G}$-diffeomorphism, then the diagram

$$\begin{array}{ccc} K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) & \xrightarrow{(df^{-1})^*} & K^0_{\mathcal{G}}(T_{\mathrm{vert}}Y) \\ & {\scriptstyle \mathrm{ind}^X_{\mathcal{G}}} \searrow \qquad \swarrow {\scriptstyle \mathrm{ind}^Y_{\mathcal{G}}} & \\ & K^0_{\mathcal{G}}(B) & \end{array}$$

is commutative.

(2) If $\psi\colon \mathcal{H} \to \mathcal{G}$ is a morphism of bundles of groups over B, then the diagram

$$\begin{array}{ccc} K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) & \xrightarrow{\psi^*} & K_{\mathcal{H}}(T_{\mathrm{vert}}X) \\ {\scriptstyle \mathrm{ind}^X_{\mathcal{G}}}\downarrow & & \downarrow{\scriptstyle \mathrm{ind}^X_{\mathcal{H}}} \\ K^0_{\mathcal{G}}(B) & \xrightarrow{\psi^*} & K^0_{\mathcal{H}}(B) \end{array}$$

is commutative.

We have the following.

Proposition 4.3. *The topological index* t-$\mathrm{ind}^X_{\mathcal{G}}$ *is an index function.*

Proof. Indeed, we need to check the two conditions defining an index function. To prove (1), let us suppose that we have a mapping $i : Y \hookrightarrow \mathcal{V}$ and let $j := i \circ f : X \hookrightarrow \mathcal{V}$. By (2) of Theorem 3.7 (on the properties of the Gysin mapping), the following diagram is commutative.

$$\begin{array}{ccccc} K^0_{\mathcal{G}}(T_{\mathrm{vert}}Y) & & & & \\ \uparrow & \searrow{\scriptstyle i_!} & & & \\ {\scriptstyle f_!} & & K^0_{\mathcal{G}}(T_{\mathrm{vert}}\mathcal{V}) & \xrightarrow{\varphi^{-1}} & K^0_{\mathcal{G}}(B). \\ | & \nearrow{\scriptstyle j_!} & & & \\ K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) & & & & \end{array}$$

By (4) of the same theorem, we have $f_! = (df^{-1})^*$ in our case, and then we complete the proof of (1) by using this definition of t-ind.

Property (2) immediately follows from the definitions if, on $\mathcal{V}$, we consider the action of $\mathcal{H}$ induced by ψ.

Let us consider the following two axioms for an index function $\{\mathrm{ind}^X_{\mathcal{G}}\}$ family of mappings $\mathrm{ind}^X_{\mathcal{G}} : K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \to K^0_{\mathcal{G}}(B)$, defined for all compact, longitudinally smooth $\mathcal{G}$-bundles $X \to B$, and satisfying the two conditions above.

Axiom A1. *If* $X = B$, *then* $\mathrm{ind}^X_{\mathcal{G}} : K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \to K^0_{\mathcal{G}}(B)$ *coincides with* $\mathrm{Id}_{K^0_{\mathcal{G}}(B)}$.

Axiom A2. *Suppose $i : X \to Y$ is a fiberwise $\mathcal{G}$-embedding. Then the diagram*

$$\begin{array}{ccc} K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) & \xrightarrow{\quad i_! \quad} & K^0_{\mathcal{G}}(T_{\mathrm{vert}}Y) \\ & {\scriptstyle \mathrm{ind}^X_{\mathcal{G}}} \searrow \qquad \swarrow {\scriptstyle \mathrm{ind}^Y_{\mathcal{G}}} & \\ & K^0_{\mathcal{G}}(B) & \end{array}$$

is commutative.

We have the following corollary to Theorem 3.11.

Corollary 4.4. *The topological index* $\text{t-ind}^X_{\mathcal{G}}$ *satisfies Axioms* A1 *and* A2.

We can now prove the following theorem.

Theorem 4.5. *Let* $\mathrm{ind}^X_{\mathcal{G}}$ *be an index function satisfying Axioms* A1 *and* A2. *Then* $\mathrm{ind}^X_{\mathcal{G}} = \text{t-ind}^X_{\mathcal{G}}$.

Proof. Consider a $\mathcal{G}$-embedding $i : X \to \mathcal{V}$ of a longitudinally smooth $\mathcal{G}$-bundle X in a real vector $\mathcal{G}$-bundle $\mathcal{V} \to B$. The fiberwise one-point compactification $\mathcal{V}^+$ (i.e., a sphere bundle) is a $\mathcal{G}$-bundle with the canonical $\mathcal{G}$-inclusion $\varepsilon^+ : \mathcal{V} \to \mathcal{V}^+$. Put $i^+ := \varepsilon^+ \circ i : X \to \mathcal{V}^+$. If $P = B \subset \mathcal{V}$ and $j : P \to \mathcal{V}$ is the inclusion, then we obtain the diagram

$$\begin{array}{ccccc} & {\scriptstyle i_!} \swarrow & K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) & \searrow {\scriptstyle \mathrm{ind}^X_{\mathcal{G}}} & \\ & & \downarrow {\scriptstyle i^+_!} & & \\ K^0_{\mathcal{G}}(T_{\mathrm{vert}}\mathcal{V}) & \xrightarrow{(\varepsilon^+)_!} & K^0_{\mathcal{G}}(T_{\mathrm{vert}}\mathcal{V}^+) & \xrightarrow{\mathrm{ind}^{\mathcal{V}^+}_{\mathcal{G}}} & K^0_{\mathcal{G}}(B) \\ & & \uparrow {\scriptstyle j^+_!} & & \\ & {\scriptstyle j_!} \nwarrow & K^0_{\mathcal{G}}(T_{\mathrm{vert}}P) = K^0_{\mathcal{G}}(B), & \nearrow {\scriptstyle \mathrm{ind}^P_{\mathcal{G}}} & \end{array}$$

where $j^+ = \varepsilon^+ \circ j : P \to \mathcal{V}^+$. By (2) of Theorem 3.7 (respectively, by Axiom A2), the left (respectively, right) triangles commute. By Axiom A1, $\mathrm{ind}^P_{\mathcal{G}}$ is the identity mapping. Since $j_! : K^0_{\mathcal{G}}(B) \to K^0_{\mathcal{G}}(T\mathcal{V}) = K^0_{\mathcal{G}}(\mathcal{V} \otimes \mathbb{C})$ coincides with the Thom homomorphism, one has

$$\begin{aligned} \mathrm{ind}^X_{\mathcal{G}} &= \mathrm{ind}^{\mathcal{V}^+}_{\mathcal{G}} \circ i^+_! = \mathrm{ind}^{\mathcal{V}^+}_{\mathcal{G}} \circ (\varepsilon^+)_! \circ i_! = \mathrm{ind}^{\mathcal{V}^+}_{\mathcal{G}} \circ (j^+)_! \circ j_!^{-1} \circ i_! \\ &= \mathrm{ind}^P_{\mathcal{G}} \circ j_!^{-1} \circ i_! = j_!^{-1} \circ i_! = \text{t-ind}^X_{\mathcal{G}} . \end{aligned}$$

The theorem is proved.

We would like to replace Axiom A2 by new axioms. We start from the following formulation.

Axiom B1 (excision). *Let U be a (noncompact) longitudinally smooth $\mathcal{G}$-bundle and*

$$j_1 : U \to X', \qquad j_2 : U \to X''$$

be fiberwise $\mathcal{G}$-embeddings of U onto open subsets of the compact longitudinally smooth $\mathcal{G}$-bundles X' and X''. Then the diagram

$$\begin{array}{ccccc} & {\scriptstyle (dj_1)_*} \nearrow & K^0_{\mathcal{G}}(T_{\mathrm{vert}}X') & \searrow {\scriptstyle \mathrm{ind}^{X'}_{\mathcal{G}}} & \\ K^0_{\mathcal{G}}(T_{\mathrm{vert}}U) & & & & K^0_{\mathcal{G}}(B) \\ & {\scriptstyle (dj_2)_*} \searrow & K^0_{\mathcal{G}}(T_{\mathrm{vert}}X'') & \nearrow {\scriptstyle \mathrm{ind}^{X''}_{\mathcal{G}}} & \end{array}$$

is commutative.

Suppose at least one of the embeddings j_1 or j_2 is defined. Then by Axiom B1, the index mapping

$$\mathrm{ind}^U_{\mathcal{G}}\colon K^0_{\mathcal{G}}(T_{\mathrm{vert}}U) \to K^0_{\mathcal{G}}(B)$$

is well defined.

Let Y be a smooth, compact manifold and H be a compact Lie group acting on Y. Let D be an elliptic (pseudo)differential operator acting between suitable sections of two vector bundles on Y. Then the kernel and cokernel of D are finite dimensional, complex representations of H, and thus define elements in $R(H)$, the representation ring of H. We shall denote by $\mathbb{C}\text{-}\mathrm{ind}^Y_H(D) := [\ker(D)] - [\mathrm{coker}(D)] \in R(H)$ the classical H-equivariant index of D. We thus obtain a well defined morphism

$$\mathbb{C}\text{-}\mathrm{ind}^Y_H\colon K^0_H(TY) \to R(H).$$

We have the following statement (see [5]). Suppose $j\colon * \to \mathbb{R}^n$ is the embedding of the origin, hence, $j_!\colon R(O(n)) \to K^0_{O(n)}(T\mathbb{R}^n)$. Then

$$\mathbb{C}\text{-}\mathrm{ind}^{\mathbb{R}^n}_{O(n)}\, j_!(1) = 1. \tag{11}$$

Let $\pi\colon P \to X$ be a compact longitudinally smooth principal bundle for a compact bundle of Lie groups $\mathcal{H} \to B$, i.e., we have a (right) free action of $\mathcal{H}$ on P and $X = P/\mathcal{H}$. Suppose we have a left action of the bundle $\mathcal{G} \to B$ on P and these two actions commute. Let F be a compact longitudinally smooth left $(\mathcal{G} \times \mathcal{H})$-bundle, where we write $\mathcal{G} \times \mathcal{H}$ instead of $\mathcal{G} \times_B \mathcal{H}$ for a more compact notation. We can form the associated bundle $\pi_1 : Y = P \times_{\mathcal{H}} F \to X$ with the natural action of $\mathcal{G}$. Consider the tangent bundle along the fibers of π_1 (which is automatically "vertical"). Let us denote it by $T_F Y$. Then $T_F Y$ is a $\mathcal{G}$-invariant real subbundle of TY_{vert} and $T_F Y = P \times_{\mathcal{H}} TF_{\mathrm{vert}}$. Using the metric, it is possible to decompose TY_{vert} into a direct sum $T_{\mathrm{vert}}Y = T_F Y \oplus \pi_1^*(T_{\mathrm{vert}}X)$. Therefore, the multiplication

$$\begin{array}{ccc} K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}}(T_F Y) & & \\ \downarrow & & \\ K^0_{\mathcal{G}}(\pi_1^* T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}}(T_F Y) & \to & K^0_{\mathcal{G}}(T_{\mathrm{vert}}Y) \end{array}$$

is well defined. There exists a mapping

$$K^0_{\mathcal{G}\times\mathcal{H}}(T_{\mathrm{vert}}F) \to K^0_{\mathcal{G}\times\mathcal{H}}(P \times T_{\mathrm{vert}}F) \cong K^0_{\mathcal{G}}(P \times_{\mathcal{H}} T_{\mathrm{vert}}F) = K^0_{\mathcal{G}}(T_F Y).$$

Hence, we can define a mapping

$$\gamma : K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}\times\mathcal{H}}(T_{\mathrm{vert}}F) \to K^0_{\mathcal{G}}(T_{\mathrm{vert}}Y).$$

Denote $\gamma(a \otimes b)$ by $a \cdot b$.

If $\mathcal{V} \to B$ is a complex vector $(\mathcal{G} \times \mathcal{H})$-bundle, then $P \times_{\mathcal{H}} \mathcal{V}$ is a complex vector $\mathcal{G}$-bundle over X. We obtain the following ring homomorphism, which is a homomorphism of $K^0_{\mathcal{G}}(B)$-modules:

$$\mu_P\colon K^0_{\mathcal{G}\times\mathcal{H}}(B) \to K^0_{\mathcal{G}}(X), \qquad [\mathcal{V}] \mapsto [P \times_{\mathcal{H}} \mathcal{V}].$$

Since $K^0_{\mathcal{G}}(T_{\mathrm{vert}}X)$ has a $K^0_{\mathcal{G}}(X)$-module structure, we can formulate the following axiom.

Axiom B2. *If* $a \in K^0_{\mathcal{G}}(T_{\mathrm{vert}}X)$, $b \in K^0_{\mathcal{G}\times\mathcal{H}}(T_{\mathrm{vert}}F)$, *then*

$$\mathrm{ind}^Y_{\mathcal{G}}(a \cdot b) = \mathrm{ind}^X_{\mathcal{G}}(a \cdot \mu_P(\mathrm{ind}^F_{\mathcal{G}\times\mathcal{H}}(b))),$$

i.e., the diagram

$$\begin{array}{ccc} K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}\times\mathcal{H}}(T_{\mathrm{vert}}F) & \xrightarrow{1 \otimes \mathrm{ind}^F_{\mathcal{G}\times\mathcal{H}}} & K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}\times\mathcal{H}}(B) \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle 1\otimes\mu_P} \\ K^0_{\mathcal{G}}(T_{\mathrm{vert}}Y) & & K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}}(X) \\ \downarrow{\scriptstyle \mathrm{ind}^Y_{\mathcal{G}}} & & \downarrow \\ K^0_{\mathcal{G}}(B) & \xleftarrow{\mathrm{ind}^X_{\mathcal{G}}} & K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \end{array}$$

is commutative

The following two statements were briefly mentioned in [33]; we recall them for the convenience of the reader.

Theorem 4.6. *Let $\pi : P \to X$ be a principal right $\mathcal{H}$-bundle with a left action of $\mathcal{G}$ commuting with $\mathcal{H}$. Suppose F is a longitudinally smooth $(\mathcal{G}\times\mathcal{H})$-bundle. Let us denote by Y the space $P\times_{\mathcal{H}} F$. Let $j : X' \to X$ and $k : F' \to F$ be fiberwise $\mathcal{G}$- and $(\mathcal{G}\times\mathcal{H})$-embeddings, respectively. Let $\pi' : P' \to X'$ be the principal $\mathcal{H}$-bundle induced by j on X'. Assume that $Y' := P' \times_{\mathcal{H}} F'$. The embeddings j and k induce $\mathcal{G}$-embedding $j * k : Y' \to Y$. Then the diagram*

$$\begin{array}{ccc}
K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes_{K^0_{\mathcal{G}}(B)} K^0_{\mathcal{G}\times\mathcal{H}}(T_{\mathrm{vert}}F) & \xrightarrow{\gamma} & K^0_{\mathcal{G}}(T_{\mathrm{vert}}Y) \\
\uparrow{\scriptstyle j_!\otimes k_!} & & \uparrow{\scriptstyle (j*k)_!} \\
K^0_{\mathcal{G}}(T_{\mathrm{vert}}X') \otimes_{K^0_{\mathcal{G}}(B)} K^0_{\mathcal{G}\times\mathcal{H}}(T_{\mathrm{vert}}F') & \xrightarrow{\gamma} & K^0_{\mathcal{G}}(T_{\mathrm{vert}}Y')
\end{array}$$

is commutative.

Let us remark that, in the statement of this theorem, there is no compactness assumption on X, X', F, and F', since there is no compactness assumption in the definition of the Gysin homomorphism. This is not so in the definition of the topological index, where we start with a compact $\mathcal{G}$-bundle $X \to B$.

Proof. Let us use the definition of γ:

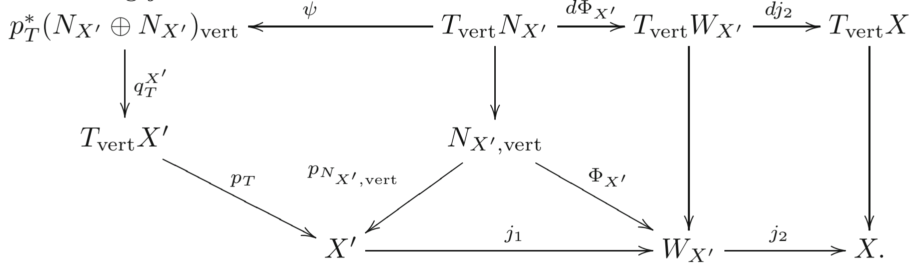
$$\begin{array}{ccccc}
K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}\times\mathcal{H}}(T_{\mathrm{vert}}F) & \longrightarrow & K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}\times\mathcal{H}}(P\times T_{\mathrm{vert}}F) & \xrightarrow{\cong} & \\
\uparrow{\scriptstyle j_!\otimes k_!} & \boxed{1} & \uparrow{\scriptstyle \varepsilon} & \boxed{2} & \\
K^0_{\mathcal{G}}(T_{\mathrm{vert}}X') \otimes K^0_{\mathcal{G}\times\mathcal{H}}(T_{\mathrm{vert}}F') & \longrightarrow & K^0_{\mathcal{G}}(T_{\mathrm{vert}}X') \otimes K^0_{\mathcal{G}\times\mathcal{H}}(P'\times T_{\mathrm{vert}}F') & \xrightarrow{\cong} &
\end{array}$$

$$\begin{array}{ccc}
\cong K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}}(P\times_{\mathcal{H}} T_{\mathrm{vert}}F) \to & & K^0_{\mathcal{G}}(\pi_1^* T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}}(P\times_{\mathcal{H}} T_{\mathrm{vert}}F) \to \\
\uparrow{\scriptstyle \beta} & \boxed{3} & \uparrow{\scriptstyle \alpha} \\
\cong K^0_{\mathcal{G}}(TX') \otimes K^0_{\mathcal{G}}(P'\times_{\mathcal{H}} T_{\mathrm{vert}}F') \to & & K^0_{\mathcal{G}}((\pi_1')^* T_{\mathrm{vert}}X') \otimes K^0_{\mathcal{G}}(P'\times_{\mathcal{H}} T_{\mathrm{vert}}F') \to
\end{array}$$

$$\begin{array}{c}
\to K^0_{\mathcal{G}}((\pi_1^* T_{\mathrm{vert}}X) \times (P\times_{\mathcal{H}} T_{\mathrm{vert}}F)) \to \\
\boxed{4} \\
\to K^0_{\mathcal{G}}((\pi_1')^* T_{\mathrm{vert}}X' \times (P'\times_{\mathcal{H}} T_{\mathrm{vert}}F')) \to
\end{array} \tag{12}$$

$$\begin{array}{ccc}
\to K^0_{\mathcal{G}}(\pi_1^* T_{\mathrm{vert}}X \oplus (P\times_{\mathcal{H}} T_{\mathrm{vert}}F)) & = & K^0_{\mathcal{G}}(T_{\mathrm{vert}}Y) \\
\boxed{4} & & \uparrow (j*k)_! \\
\to K^0_{\mathcal{G}}((\pi_1')^* T_{\mathrm{vert}}X' \oplus (P'\times_{\mathcal{H}} T_{\mathrm{vert}}F')) & = & K^0_{\mathcal{G}}(TY'_{\mathrm{vert}}),
\end{array}$$

where the projections $\pi_1 : Y = P\times_{\mathcal{H}} F \to X$ and $\pi_1' : Y' = P\times_{\mathcal{H}} F' \to X'$ are defined as above. Here we use the isomorphism $K^0_{\mathcal{G}\times\mathcal{H}}(P\times W) \cong K^0_{\mathcal{G}}(P\times_{\mathcal{H}} W)$ for a free $\mathcal{H}$-bundle P (see [33, Theorem 2.6]). Let us recall the following diagram that was used for the definition of the Gysin homomorphism of an embedding $j : X' \to X$:



From the similar diagrams for $k_!$ and $(j*k)_!$ and the explicit form of these mappings, it follows that the square $\boxed{4}$ in (12) is commutative if and only if α has the following form:

$$\alpha(\sigma\otimes\rho) = (\pi_1^*)\Big\{(dj_2)_*\,(d\Phi_{X'}^{-1})^*\,\psi_{X'}^*\Big\}\circ\varphi^S(\sigma)$$
$$\otimes\,(\pi^* j_2\times_{\mathcal H} dk_2)_*\Big((\pi^*\Phi_{X'}\times_{\mathcal H} d\Phi_{F'})^{-1}\Big)^*\,(1\times_{\mathcal H}\psi_{F'})^*\,\varphi^R(\rho),$$

where S and T are bundles of the form

$$S:\quad \begin{matrix}\pi_1^*\Big((p_T^{X'})^*\{N_{X'}\oplus N_{X'}\}\Big)\\ \downarrow (\pi_1')^* q_T^{X'}\\ (\pi_1')^*\,(T_{\mathrm{vert}}X'),\end{matrix}\qquad R:\quad \begin{matrix}\pi^* N_{X'}\times_{\mathcal H}(p_T^{F'})^*\,(N_{F'}\oplus N_{F'})\\ \downarrow (\pi')^*(p_{N_{X'}})\times_{\mathcal H} q_T^{F'}\\ \pi^*\,X'\times_{\mathcal H} T_{\mathrm{vert}}F' = P'\times_{\mathcal H} T_{\mathrm{vert}}F'.\end{matrix}$$

Hence, the square $\boxed{3}$ in (12) is commutative if and only if the homomorphism β has the form

$$\beta(\tau\otimes\rho) = j_!(\tau)\otimes(\pi^* j_2\times_{\mathcal H} dk_2)_*\Big((\pi^*\Phi_{X'}\times_{\mathcal H} d\Phi_{F'})^{-1}\Big)^*\,(1\times_{\mathcal H}\psi_{F'})^*\,\varphi^R(\rho),$$
$$\tau\in K^0_{\mathcal G}(TX'),\qquad \rho\in K^0_{\mathcal G}(P'\times_{\mathcal H} TF').$$

In turn, the square $\boxed{2}$ in (12) is commutative if and only if the homomorphism ε has the form

$$\varepsilon(\tau\otimes\delta) = j_!(\tau)\otimes(\pi^* j_2\times_{\mathcal H} dk_2)_*\Big((\pi^*\Phi_{X'}\times_{\mathcal H} d\Phi_{F'})^{-1}\Big)^*\,(1\times_{\mathcal H}\psi_{F'})^*\,\varphi^{\widetilde R}_{\mathbb C}(\delta),$$
$$\tau\in K^0_{\mathcal G}(TX'),\qquad \delta\in K^0_{G\times\mathcal H}(P'\times TF'),$$

where $\widetilde R$ is the following bundle:

$$\widetilde R:\quad \begin{matrix}\pi^* N_{X'}\times(p_T^{F'})^*\,(N_{F'}\oplus N_{F'})\\ \downarrow (\pi')^*(p_N)\times q_T^{F'}\\ P'\times TF'.\end{matrix}$$

Suppose $\delta = [\underline{\mathbb C}]\widehat\otimes\omega$, where $[\underline{\mathbb C}]\in K^0_{\mathcal G\times\mathcal H}(P')$, $\underline{\mathbb C}$ is the one-dimensional trivial bundle and $\omega\in K^0_{\mathcal G\times\mathcal H}(TF')$. Then

$$\varepsilon(\tau\otimes\delta) = j_!(\tau)\otimes\Big\{\pi^*(j_2)_*(\Phi_{X'}^{-1})^*[\underline{\mathbb C}]\widehat\otimes k_!(\omega)\Big\} = j_!(\tau)\otimes\Big\{[\underline{\mathbb C}]\widehat\otimes k_!(\omega)\Big\}.$$

Since the mapping $K^0_{\mathcal G\times\mathcal H}(TF)\to K^0_{\mathcal G\times\mathcal H}(P\times TF)$ (as well as the lower line in (12)) has the form $\omega\mapsto[\underline{\mathbb C}]\widehat\otimes\omega$, we have proved the commutativity of $\boxed{1}$ in (12).

From this theorem, we obtain the following corollary.

Corollary 4.7. *Let M be a compact smooth H-manifold, let $\mathcal H = B\times H$, and let P be a principal longitudinally smooth $\mathcal H$-bundle over X carrying also an action of $\mathcal G$ commuting with the action of $\mathcal H$. Also, let $X\to B$ be a compact longitudinally smooth $\mathcal G$-bundle. Let $Y := P\times_H M\to X$ be associated longitudinally smooth $\mathcal G$-bundle. Taking $F = B\times M$, we define $T_MY := T_FY$. Then T_MY is a $\mathcal G$-invariant real subbundle of $T_{\mathrm{vert}}Y$ and $T_MY = P\times_H TM$. Let $j : X'\to X$ be a fiberwise $\mathcal G$-equivariant embedding and let $k : M'\to M$ be an H-embedding. Denote by $\pi' : P'\to X'$ the principal $\mathcal H$-bundle induced by j on X' and assume that $Y' := P'\times_H M'$. The embeddings j and k induce $\mathcal G$-embedding $j*k : Y'\to Y$. Then the diagram*

$$\begin{array}{ccc} K^0_{\mathcal G}(T_{\mathrm{vert}}X)\otimes K^0_H(TM) & \xrightarrow{\ \gamma\ } & K^0_{\mathcal G}(T_{\mathrm{vert}}Y)\\ \uparrow{\scriptstyle j_!\otimes k_!} & & \uparrow{\scriptstyle (j*k)_!}\\ K^0_{\mathcal G}(T_{\mathrm{vert}}X')\otimes K^0_H(TM') & \xrightarrow{\ \gamma\ } & K^0_{\mathcal G}(T_{\mathrm{vert}}Y') \end{array}$$

is commutative.

Let us assume that, in Axiom B2, the class $b \in K^0_{\mathcal{G}\times\mathcal{H}}(B)$ is actually in $K^0_{\mathcal{G}}(B)$, namely,

$$\operatorname{ind}^F_{\mathcal{G}\times\mathcal{H}}(b) \in K^0_{\mathcal{G}}(B) \subset K^0_{\mathcal{G}\times\mathcal{H}}(B).$$

To be precise, the mapping $K^0_{\mathcal{G}}(B) \to K^0_{\mathcal{G}\times\mathcal{H}}(B)$ is induced at the level of semigroups by sending a $\mathcal{G}$-vector bundle E with an action $g : e \mapsto ge$ ($e \in E_b$, $g \in \mathcal{G}_b$) to the same bundle with the action $(g,h) : e \mapsto ge$ ($h \in \mathcal{H}_b$). The existence of a left inverse mapping (restriction of action) implies injectivity.

We now consider the two following weak forms of Axiom B2.

Axiom B2′. *If* $\operatorname{ind}^F_{\mathcal{G}\times\mathcal{H}}(b) \in K^0_{\mathcal{G}}(B) \subset K^0_{\mathcal{G}\times\mathcal{H}}(B)$, *then*

$$\operatorname{ind}^Y_{\mathcal{G}}(a \cdot b) = \operatorname{ind}^X_{\mathcal{G}}(a) \cdot \operatorname{ind}^F_{\mathcal{G}\times\mathcal{H}}(b).$$

Assume in B2 $X = P$, $\mathcal{H} = B$. We can then formulate the following axiom.

Axiom B2″. *If X and F are longitudinally smooth $\mathcal{G}$-bundles, then*

$$\operatorname{ind}^{X\times F}_{\mathcal{G}}(a \cdot b) = \operatorname{ind}^X_{\mathcal{G}}(a) \cdot \operatorname{ind}^F_{\mathcal{G}}(b).$$

Since μ_P and $\operatorname{ind}^X_{\mathcal{G}}$ are $K^0_{\mathcal{G}}(B)$-homomorphisms, then Axioms B2′ and B2″ are consequences of Axiom B2.

Theorem 4.8. *Suppose that an index function* $\operatorname{ind}^X_{\mathcal{G}}$ *satisfies Axioms* A1, B1, B2′, *then*

$$\operatorname{ind}^X_{\mathcal{G}} = \text{t-ind}^X_{\mathcal{G}}.$$

Proof. First, we extend Axiom B2′ to the noncompact case under some restrictions in the following way. Suppose that, in Axiom B2′, F is equal to an open $(\mathcal{G}\times\mathcal{H})$-subbundle of the compact longitudinally smooth bundle $\widetilde{F}$. Let $j : F \hookrightarrow \widetilde{F}$. Then

$$\operatorname{ind}^Y_{\mathcal{G}}(a\cdot b) = \operatorname{ind}^{\widetilde{Y}}_{\mathcal{G}}(dJ_*)(a\cdot b) = \operatorname{ind}^{\widetilde{Y}}_{\mathcal{G}}(a\cdot((dj)_* b)) = \operatorname{ind}^X_{\mathcal{G}}(a)\cdot\operatorname{ind}^{\widetilde{F}}_{\mathcal{G}\times\mathcal{H}}((dj)_* b) = \operatorname{ind}^X_{\mathcal{G}}(a)\cdot\operatorname{ind}^F_{\mathcal{G}\times\mathcal{H}}(b), \quad (13)$$

where J is the embedding

$$Y = P \times_{\mathcal{H}} F \overset{\operatorname{Id}\times_{\mathcal{H}} j}{\hookrightarrow} P \times_{\mathcal{H}} \widetilde{F} = \widetilde{Y}.$$

Indeed, let us consider the diagram

$$\begin{array}{ccc}
K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}\times\mathcal{H}}(T_{\mathrm{vert}}F) & \longrightarrow & K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}\times\mathcal{H}}(P \times T_{\mathrm{vert}}F) \cong \\
\big\downarrow 1\otimes(dj)_* & & \big\downarrow 1\otimes(\operatorname{Id}\times dj)_* \\
K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}\times\mathcal{H}}(T_{\mathrm{vert}}\widetilde{F}) & \longrightarrow & K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}\times\mathcal{H}}(P \times T_{\mathrm{vert}}\widetilde{F}) \cong
\end{array}$$

$$\begin{array}{ccc}
\cong K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}}(P \times_{\mathcal{H}} T_{\mathrm{vert}}F) & \Longrightarrow & K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}}(T_F Y) \longrightarrow \\
\big\downarrow 1\otimes(\operatorname{Id}\times_{\mathcal{H}} dj)_* & & \big\downarrow 1\otimes\alpha_* \\
\cong K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}}(P \times_{\mathcal{H}} T_{\mathrm{vert}}\widetilde{F}) & \Longrightarrow & K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}}(T_{\widetilde{F}} Y) \longrightarrow
\end{array}$$

$$\begin{array}{ccc}
\longrightarrow K^0_{\mathcal{G}}(\pi_1^* T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}}(T_F Y) & \longrightarrow & K^0_{\mathcal{G}}(T_{\mathrm{vert}}Y) \\
\big\downarrow 1\otimes\alpha_* & & \big\downarrow (dJ)_* \\
\longrightarrow K^0_{\mathcal{G}}(\pi_1^* T_{\mathrm{vert}}X) \otimes K^0_{\mathcal{G}}(T_{\widetilde{F}} Y) & \longrightarrow & K^0_{\mathcal{G}}(T_{\mathrm{vert}}\widetilde{Y}).
\end{array}$$

This diagram is commutative. In fact, we have

$$\begin{array}{ccc} T_{\mathrm{vert}}Y & = & T_F Y \oplus \pi_1^*(T_{\mathrm{vert}}X) \\ \downarrow{\scriptstyle dJ} & & \downarrow{\scriptstyle \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}} \\ T_{\mathrm{vert}}\widetilde{Y} & = & T_{\widetilde{F}} Y \oplus \pi_1^*(T_{\mathrm{vert}}X), \end{array}$$

and $\alpha = \mathrm{Id} \times_{\mathcal{H}} dj$ under the identification $T_F Y = P \times_{\mathcal{H}} TF$. We have proved the second equality in (13), the remaining are obvious.

Let us now take, in particular,

$$F = \mathbb{R}^n \times B, \quad \widetilde{F} = (\mathbb{R}^n)^+ \times B = S^n \times B, \quad \mathcal{H} = O(n) \times B, \quad b = \phi_!(1), \quad 1 = [\underline{\mathbb{C}}],$$

where $\phi : \vec{0} \times B \hookrightarrow \mathbb{R}^n \times B$ is the natural embedding. Then P is a principal $O(n) \times B$-bundle over X, the bundle of groups $\mathcal{G}$ acts on P commuting with $O(n) \times B$. Suppose $\mathcal{G}$ acts on $\mathbb{R}^n \times B$ in a trivial way. We form the associated real $\mathcal{G}$-bundle

$$Y := P \times_{O(n)} \mathbb{R}^n = P \times_{O(n)\times B} (\mathbb{R}^n \times B) \to X.$$

Let us denote by

$$i : X \to Y, \qquad i = 1_X * \phi,$$

the embedding of X as of the zero section. Assume that, in Theorem 4.6, we have $F = \mathbb{R}^n \times B$, $X' = X$, $F' = B$. Then we obtain the commutative diagram

$$\begin{array}{ccc} K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes_{K^0_{\mathcal{G}}(B)} K^0_{\mathcal{G}\times(O(n)\times B)}(T_{\mathrm{vert}}(\mathbb{R}^n \times B)) & \xrightarrow{\gamma} & K^0_{\mathcal{G}}(T_{\mathrm{vert}}Y) \\ \uparrow{\scriptstyle (1_X)_! \otimes \phi_!} & & \uparrow{\scriptstyle (1_X * \phi)_! = i_!} \\ K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) \otimes_{K^0_{\mathcal{G}}(B)} K^0_{\mathcal{G}\times(O(n)\times B)}(B) & \xrightarrow{\gamma} & K^0_{\mathcal{G}}(T_{\mathrm{vert}}X). \end{array}$$

Since $\gamma\,(a \otimes 1) = a$ on the bottom line, we have

$$i_!(a) = \gamma\left(((1_X)_! \otimes \phi_!)\,(a \otimes 1)\right) = \gamma\,(a \otimes \phi_!(1)) = a \cdot \phi_!(1) = a \cdot b.$$

By Equation (11),

$$\mathrm{ind}^{\mathbb{R}^n}_{\mathcal{G}\times O(n)}\ \phi_!(1) = 1,$$

where $\mathcal{G}$ acts on $\mathbb{R}^n \times B$ in a trivial way. Now by equality in Equation (13),

$$\mathrm{ind}^X_{\mathcal{G}}(a) = \mathrm{ind}^X_{\mathcal{G}}(a \cdot 1) = \mathrm{ind}^X_{\mathcal{G}}(a \cdot \mu_P(1)) = \mathrm{ind}^X_{\mathcal{G}}(a \cdot \mu_P(\mathrm{ind}^{\mathbb{R}^n}_{\mathcal{G}\times(O(n)\times B)}(b))) = \mathrm{ind}^Y_{\mathcal{G}}(a \cdot b) = \mathrm{ind}^Y_{\mathcal{G}}\, i_!(a). \tag{14}$$

Let $k\colon X \to Z$ be a fiberwise embedding of X in a compact longitudinally smooth $\mathcal{G}$-bundle Z with the fiberwise normal bundle N and a fiberwise $\mathcal{G}$-invariant tubular neighborhood $\Phi\colon N \to W$. By the definition of the Gysin homomorphism, $k_! = (di_2 \circ d\Phi)_* i_!$, where $di_2\colon T_{\mathrm{vert}}W \to T_{\mathrm{vert}}Z$ is an embedding of vertical tangent bundle and $i\colon X \to N$ is the fiberwise embedding of X as of the zero section in the normal bundle. In the diagram

$$\begin{array}{ccccc} K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) & \xrightarrow{i_!} & K^0_{\mathcal{G}}(T_{\mathrm{vert}}N) & \xrightarrow{(di_2\circ d\Phi)_*} & K^0_{\mathcal{G}}(T_{\mathrm{vert}}Z) \\ & {\scriptstyle \mathrm{ind}^X_{\mathcal{G}}}\searrow & \downarrow{\scriptstyle \mathrm{ind}^N_{\mathcal{G}}} & \swarrow{\scriptstyle \mathrm{ind}^Z_{\mathcal{G}}} & \\ & & K^0_{\mathcal{G}}(B) & & \end{array}$$

the left triangle is commutative by (14). Indeed, we can take P equal to the principal $O(n)$-bundle of normal vertical orthonormal frames and $Y = N$. The mapping $i_2 \cdot \Phi$ is an open embedding. Hence, by Axiom B1, the right triangle is commutative too. Therefore, $\operatorname{ind}_{\mathcal{G}}^X = \operatorname{ind}_{\mathcal{G}}^Z \circ k_!$. Hence, Axiom A2 is satisfied. To complete the proof, it remains to apply Theorem 4.5.

Let us remark that we have used only a very particular case of Axiom B2$'$, namely, the following one.

Axiom B2^0. Let P be the principal $O(n) \times B$-bundle of normal (vertical) orthonormal frames of the embedding $k\colon X \to Z$, i.e., the bundle of frames of N. Suppose $\mathcal{G}$ acts on $\mathbb{R}^n \times B$ in a trivial way. The associated real $\mathcal{G}$-bundle

$$Y := P \times_{O(n)} \mathbb{R}^n = P \times_{O(n)\times B} (\mathbb{R}^n \times B) \to X$$

is just N. Let

$$i\colon X \to Y, \qquad i = 1_X * \phi,$$

be the embedding of X as of the zero section. Then the diagram

$$\begin{array}{ccc} K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) & \xrightarrow{\;i_!\;} & K^0_{\mathcal{G}}(T_{\mathrm{vert}}Y) \\ & {\scriptstyle \operatorname{ind}_{\mathcal{G}}^X} \searrow \quad \swarrow {\scriptstyle \operatorname{ind}_{\mathcal{G}}^Y} & \\ & K^0_{\mathcal{G}}(B) & \end{array}$$

commutes.

In the formulation of the next theorem, we require the excision axiom, hence, one can use instead of Axiom B2^0 its reformulation for the fiber-wise compactification, demanding commutativity of the following diagram:

$$\begin{array}{ccc} K^0_{\mathcal{G}}(T_{\mathrm{vert}}X) & \xrightarrow{\;i_!\;} & K^0_{\mathcal{G}}(T_{\mathrm{vert}}Y^\bullet)\ , \\ & {\scriptstyle \operatorname{ind}_{\mathcal{G}}^X} \searrow \quad \swarrow {\scriptstyle \operatorname{ind}_{\mathcal{G}}^{Y^\bullet}} & \\ & K^0_{\mathcal{G}}(B) & \end{array}$$

where $Y^\bullet = P \times_{O(n)\times B} (S^n \times B)$. So we have.

Theorem 4.9. *Suppose that an index function* $\operatorname{ind}_{\mathcal{G}}^X$ *satisfies Axioms* A1, B1, B2^0, *then*

$$\operatorname{ind}_{\mathcal{G}}^X = \text{t-}\operatorname{ind}_{\mathcal{G}}^X\ .$$

5. PROOF OF THE INDEX THEOREM

First of all, let us notice that the analytical index is an index function. Indeed, it has the property (1) of Definition 4.2, since a $\mathcal{G}$-diffeomorphism takes KER to KER and COK to COK. The property (2) of 4.2 for the analytical index means that, in the presence of ψ, the bundles KER and COK can be considered as $\mathcal{G}$-bundles and $\mathcal{H}$-bundles in a coherent way. Thus the analytic index function a-ind also satisfies the property (2) of the definition of an index function.

Lemma 5.1. *The analytical index* a-ind *satisfies Axiom* A1.

Proof. An elliptic family of operators over the trivial bundle $X = B \to B$ is a $\mathcal{G}$-morphism $P : \mathcal{V} \to \mathcal{W}$ of equivariant vector bundles and $[\sigma(P)] = [\mathcal{V}] - [\mathcal{W}] = \text{a-ind}\, P \in K^0_{\mathcal{G}}(B)$.

Before going further, let us note that if a Fredholm $\mathcal{G}$-family is fiberwise surjective, then its kernel forms a $\mathcal{G}$-vector bundle over B, i.e., KER = ker and COK = coker = 0, see [32] for the definitions of KER and COK.

Lemma 5.2. *Let H_b^0 and H_b^1 be $\mathcal{G}$-bundles of Hilbert spaces together with an equivariant Fredholm family $D_b : H_b^0 \to H_b^1$, $b \in B$. Let $L \to B$ be a finite-dimensional $\mathcal{G}$-bundle and $T : L \to H$ a morphism of $\mathcal{G}$-bundles such that $D_b + T_b : H_b \oplus L_b \to H_b$ is surjective for all b (that is, $B + T$ is fiberwise surjective). Then*

$$\text{a-ind}(D) = \ker(D+T) - [L].$$

Proof. Denote by Q the family $D + T$ considered as a family of fiberwise mappings from $H \oplus L \to H \oplus L$. Then Q is a fiberwise compact perturbation of $D \oplus \mathrm{Id}$. Hence, $\text{a-ind}(Q) = \text{a-ind}(D)$. Also $\text{a-ind}(Q) = \ker(D + T) - [L]$.

Also we need the following statement.

Lemma 5.3. *Let $(D_b)_{b\in B}$ be an elliptic family with invariant principal symbol family $(\sigma_b)_{b\in B}$. Then the fiberwise average $\mathrm{Av}_{\mathcal{G}} D$ is an invariant elliptic family with the same principal symbol.*

Proof. All facts are known from single operator theory, except for the continuity of $\mathrm{Av}_{\mathcal{G}} D$. However, since this is a local question, we can assume that $X|_U = X_0 \times U$, $\mathcal{G}|_U = G \times U$, D_b depends continuously on $b \in U \subset B$, and the action of $G \times U$ on X_0 is smooth (see Condition 4.1). Then, by [5, item 5.5], the action of $\mathcal{G}$ is continuous on families of operators, and hence we can integrate over G to project onto the fixed-point set.

Theorem 5.4. *The index* a-ind *satisfies Axiom* B1.

Proof. We shall use the notation introduced in the statement of Axiom B1. Thus, suppose that $a \in K^0_{\mathcal{G}}(T_{\mathrm{vert}} U)$, that

$$\begin{array}{ccc} j_1 : U & \longrightarrow & X' \\ & \searrow \quad \swarrow & \\ & B & \end{array} \quad \text{and} \quad \begin{array}{ccc} j_2 : U & \longrightarrow & X'' \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

are fiberwise $\mathcal{G}$-embeddings, and that $\pi : T_{\mathrm{vert}} U \to U$ is the natural projection. Let the sequence

$$0 \to \pi^* E \xrightarrow{\rho} \pi^* F \to 0$$

of $\mathcal{G}$-bundles be exact for $x \in U \setminus L$, $|\xi| > c$ (point and (co)vector), where $E \to U$ and $F \to U$ are longitudinally smooth $\mathcal{G}$-bundles and L is some $\mathcal{G}$-invariant compact subbundle of U. Suppose

$$\alpha : E|_{U\setminus L} \cong (U \setminus L) \times_B N, \qquad \beta : F|_{U\setminus L} \cong (U \setminus L) \times_B N,$$
$$\text{and} \quad \rho = (\pi^*\beta)^{-1} (\pi^*\alpha),$$

where $N \to B$ is a vector $\mathcal{G}$-bundle. More precisely, one can assume that, for a $\mathcal{G}$-invariant metric, L is a bundle of open balls of continuously changing radius over B. Take a representative of the homotopy class of ρ being a symbol of order zero. Then it is possible to assume the corresponding symbols $\sigma_1 \in \mathrm{Smbl}_0(X', E_1, F_1)$, $\sigma_2 \in \mathrm{Smbl}_0(X'', E_2, F_2)$ (we use the notation Smbl_k, Int_k and CZ_k for family symbols, Fourier integral operators and Calderon-Zygmund operators (cf. [37])) be as follows. Suppose

$$E_1 = E \cup_{j_1\alpha} (X' \setminus j_1 L) \times_B N, \quad E_2 = E \cup_{j_2\alpha} (X' \setminus j_2 L) \times_B N.$$

Let the similar equalities hold for F_1 and F_2, and $\sigma_1 = \rho \cup_{j_1} \mathrm{Id}$, $\sigma_2 = \rho \cup_{j_2} \mathrm{Id}$. Let us pass to the construction of families of operators $\widetilde{D}_1$ and $\widetilde{D}_2$, which represent these families of symbols in $\mathrm{Int}_0(X'; E_1, F_1)$ and $\mathrm{Int}_0(X''; E_2, F_2)$, respectively. Let us take a trivializing cover, a partition of unity and smoothing functions on U. Pull them back on $j_1 U$ and $j_2 U$, and then complete these collections of open sets (to obtain covers) by some open sets not intersecting with $j_1 L$ and $j_2 L$, respectively. By our symbols and with the help of this data, let us construct in the usual way (noninvariant) families of operators D_1, $D_2 \in \mathrm{CZ}_0$, and then (keeping in mind Lemma 5.3)

$$\widetilde{D}_1 = \mathrm{Av}_{\mathcal{G}} D_1 \in \mathrm{Int}_0(X'), \quad \widetilde{D}_2 = \mathrm{Av}_{\mathcal{G}} D_2 \in \mathrm{Int}_0(X'').$$

It is necessary to verify the equality

$$\text{a-ind}\,\widetilde{D}_1 = \text{a-ind}\,\widetilde{D}_2 \in K^0_{\mathcal{G}}(B).$$

Since L is invariant, the averaging over this set is the same for both operators. Since the operators have the order zero, we compute the index in continuous families of L_2 (square integrable) spaces. For these spaces,

$$L_2(X', E_1) \cong L_2(j_1 L, E_1|_{j_1 L}) \oplus L_2(X' \setminus j_1 L, E_1|_{X' \setminus j_1 L})$$

and

$$\widetilde{D}_1 : L_2(X' \setminus j_1 L, E_1|_{X' \setminus j_1 L}) \cong L_2(X' \setminus j_1 L, E_1|_{X' \setminus j_1 L})$$

(this is the identity operator). These decompositions are continuous (in L_2-norms) in $b \in B$. Similar relations hold for $\widetilde{D}_2$. On the second summand of the decomposition of L_2, we have the commutative diagram

$$\begin{array}{ccc} \Gamma(E_1|_{j_1 L}) & \xrightarrow{\widetilde{D}_1} & \Gamma(F_1|_{j_1 L}) \\ {\scriptstyle (j_2 j_1^{-1})}\downarrow\cong & & \cong\downarrow{\scriptstyle (j_2 j_1^{-1})} \\ \Gamma(E_2|_{j_2 L}) & \xrightarrow{\widetilde{D}_2} & \Gamma(F_2|_{j_2 L}). \end{array}$$

This diagram of $\mathcal{G}$-mappings demonstrates the coincidence of indices, because KER and COK for $\widetilde{D}_1$ can serve as KER and COK for $\widetilde{D}_2$.

See also [8, 10].

Theorem 5.5. *The analytical index* a-ind *satisfies Axiom* B2^0.

Proof. Denote by $\mathcal{B}$ an $O(n)$-equivariant elliptic operator of order 1 over S^n, $\mathcal{B}\colon \Gamma^\infty(S^n, F^0) \to \Gamma^\infty(S^n, F^1)$, such that

1. its symbol $\sigma(\mathcal{B})\colon \pi_S^* F^0 \to \pi_S^* F^1$ represents the class $\phi_!(1)$, where we denote by ϕ the injection $0 \hookrightarrow \mathbb{R}^n$ as well as its lift to B and $\pi_S\colon TS^n \to S^n$ is the natural projection;
2. $\ker \mathcal{B}^* = 0$ and $\ker \mathcal{B}$ is a one dimensional trivial $O(n)$-module.

The existence of such $\widetilde{\mathcal{B}}$ follows, e.g., from [6, Lemma 4.1]. Let an $a \in K^0_{\mathcal{G}}(T_{\text{vert}} X)$ be presented by a symbol $s : \pi_X^* E^0 \to \pi_X^* E^1$ of order one. As it was explained in the proof of Theorem 4.8, we have $i_!(a) = a \cdot \phi_!(1)$. Thus, by (1) above, $i_!(a)$ has a representative

$$S = \begin{pmatrix} s \otimes_B \mathrm{Id} & -\mathrm{Id} \otimes_B \sigma(\mathcal{B})^* \\ \mathrm{Id} \otimes_B \sigma(\mathcal{B}) & s^* \otimes_B \mathrm{Id} \end{pmatrix} : \pi_{Y^\bullet}^*((E^0 \otimes_B F^0) \oplus (E^1 \otimes_B F^1)) \to \pi_{Y^\bullet}^*((E^1 \otimes_B F^0) \oplus (E^0 \otimes_B F^1)) \tag{15}$$

because γ or $\cdot$ is locally the tensor multiplication of 2-complexes reduced over B. Now we need to verify that $\text{a-ind}^X_{\mathcal{G}}(s) = \text{a-ind}^{Y^\bullet}_{\mathcal{G}}(S)$.

Starting from a family A with the symbol s and operator B, using local lifts and averaging, we construct in a standard way (see, e.g., [6] and [48, p. 173]) an elliptic family

$$D = \begin{pmatrix} \widetilde{\mathcal{A}} & -\widetilde{\mathcal{B}}^* \\ \widetilde{\mathcal{B}} & \widetilde{\mathcal{A}}^* \end{pmatrix}$$

with the symbol S. The main difference with the standard argument is the new way of averaging over $\mathcal{G}$. It is explained in Lemma 5.3 that the averaging over $\mathcal{G}$ respects continuity of families, the other properties are known from the single operator equivariant theory [5].

It remains to verify that a-ind(A) = a-ind(D). Indeed, let $h_0 \in \Gamma^\infty(S^n, F^0)$ be a generator of the one dimensional $O(n)$-module $\ker \mathcal{B}$. Define for any $\mathcal{G}$-equivariant bundle $\widetilde{E}$ over X the mapping

$$f : H^s(X, \widetilde{E}) \to H^{s-1}(Y^\bullet, \widetilde{E} \otimes F^0), \qquad f(u) = u \otimes h_0.$$

Since h_0 is $O(n)$-fixed, f is a well-defined injective $\mathcal{G}$-vector bundle homomorphism. Let L be a finite-dimensional $\mathcal{G}$-vector bundle over B and $T : L \to H^{s-1}(X, \widetilde{E}^1)$ be a $\mathcal{G}$-vector bundle homomorphism such that

$$Q_b : H^s(X, \widetilde{E}_0)_b \oplus L_b \to H^{s-1}(X, \widetilde{E}^1)_b, \qquad Q_b(u, v) = \mathcal{A}_b(u) + T(v),$$

is surjective for any b. For L one can take (a bundle representing) COK$(\mathcal{A})$ and for T one can take the natural inclusion.

Consider a mapping

$$R_b : H^s(Y^\bullet, \widetilde{E}^0 \otimes_B F^0 \oplus \widetilde{E}^1 \otimes_B F^1)_b \oplus L_b \to H^{s-1}(Y^\bullet, \widetilde{E}^1 \otimes_B F^0 \oplus \widetilde{E}^0 \otimes_B F^1)_b$$

defined by the formula

$$R_b(u, v) = D_b(u) + (f \circ T(v) \oplus 0).$$

A standard argument (see, e.g., [48, pp. 174–175]) shows that R_b is surjective for any b and $\ker R = \ker Q$. Thus, by Lemma 5.2,

$$\text{a-ind}(\mathcal{A}) = \ker(Q) - [L] = \ker(R) - [L] = \text{a-ind}(D)$$

and we are done.

We can now prove the following topological index theorem for gauge-equivariant operators.

Theorem 5.6. *The index functions* a-ind *and* t-ind *coincide. More precisely, suppose that $\mathcal{G}$ satisfies the finite holonomy condition, that $Y \to B$ is a longitudinally smooth bundle and that P is a gauge-equivariant family of pseudodifferential operators on Y. Then*

$$\text{a-ind}(P) = \text{t-ind}(P)\,.$$

Proof. From the results of this section (Lemma 5.1 and Theorems 5.4 and 5.5), it follows that we can apply Theorem 4.9 to conclude that a-ind = t-ind.

We conclude with a brief discussion of the homology of the groupoid algebras, in view of its connections to index theory [12].

Remark 5.7. Let us denote for any Lie group G by $I(G) := C^\infty(G)^G$, the space of smooth *class* functions on G. We shall use this only for compact G. Let $\mathcal{G} \to B$ be a longitudinally smooth bundle of Lie groups. Then $I(\mathcal{G}_b)$, $b \in B$, is a naturally flat bundle over B. It follows then from the Künneth formula in Hochschild and cyclic homology and using also localization with respect to the maximal ideals of $C^\infty(B)$ that the Hochschild homology groups of $C^\infty(\mathcal{G})$ are isomorphic to the space of forms on B with values in the sheaf defined by $I(\mathcal{G}_b)$. Similarly, the periodic cyclic homology groups of $C^\infty(\mathcal{G})$ are isomorphic to the cohomology groups of B with coefficients in the the sheaf $I(\mathcal{G}_b)$. It would be interesting to establish a cohomological index theorem in cyclic homology, but this seems hard even in the case of a single operator without any group action, in spite of the many recent advances on the subject. See [38] and the references therein.

As mentioned in Introduction, operators invariant with respect to groups appear in analysis on singular spaces, see [1–3, 9, 29, 36, 51], for example. It would be quite important to extend the results of this paper to operators invariant with respect to bundles of *solvable* Lie groups [15, 25, 30, 34, 49, 50].

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