ISSN 1061-9208, Russian Journal of Mathematical Physics, Vol. 22, No. 1, 2015, pp. 6–8. © Pleiades Publishing, Ltd., 2015.

On the Compatibility Equations in Terms of Stresses in Many-Dimensional Elastic Medium

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Received December 20, 2014

Abstract. By equating to zero all components of the Kröner incompatibility tensor of rank $2n-4$ or of the Riemann tensor dual to the Kröner tensor, $n^2(n^2-1)/12$ independent consistency equations for the stresses in an n-dimensional isotropic elastic medium are derived. The problem concerning the equivalence of the system of these equations to systems following from equating to zero either all $n(n+1)/2$ components of the Ricci tensor or only one curvature invariant is investigated. It is shown that the answer to this question depends on the dimension of the space. Three cases are singled out: $n=2$ (plane problem of elasticity theory), $n=3$ (spatial problem of elasticity theory), and $n \geqslant 4$.

DOI 10.1134/S1061920815010021

Consider the quasi-static deformation of an isotropic elastic medium belonging to Euclidean space $\mathbb{R}^n$ with a Cartesian coordinate system chosen in the space. Small deformations $\varepsilon_{ij}(\mathbf{x})$ are related in such a medium to stresses $\sigma_{ij}(\mathbf{x})$ by the inverse Hooke law

$$\varepsilon_{ij} = \frac{1}{E}\big(-\nu\Theta\delta_{ij} + (1+\nu)\sigma_{ij}\big), \quad \Theta = \sigma_{kk}, \tag{1}$$

where E stands for Young's modulus and ν for Poisson's ratio. In (1) and below, the subscripts take the values $1, 2, \dots, n$; the summation from 1 to n is assumed over the Latin indices repeated twice in any monomial.

A necessary condition for a medium to be contained in Euclidean space is the condition [1–3] that all components of the Kröner incompatibility tensor $\eta^{\{2n-4\}}(\underset{\sim}{\varepsilon}) \equiv \mathrm{Ink}\,\underset{\sim}{\varepsilon}$ of rank $2n-4$ vanish,

$$\eta_{p_1\dots p_{n-2}q_1\dots q_{n-2}}(\underset{\sim}{\varepsilon}) \equiv \epsilon_{p_1\dots p_{n-2}li}\,\epsilon_{q_1\dots q_{n-2}jk}\,\varepsilon_{ij,lk} = 0, \tag{2}$$

where $\epsilon_{p_1\dots p_{n-2}li}$ stands for the n-index Levi–Civita symbol in $\mathbb{R}^n$. A comma in the subscript stands for the partial differentiation with respect to the corresponding coordinate. The Kröner tensor is antisymmetric with respect to any pair of its $n-2$ first subscripts and of the $n-2$ other subscripts. Moreover, it is symmetric under permutations of families of $n-2$ first subscripts and $n-2$ subscripts among the last ones.

Let us multiply equation (2) by $\epsilon_{mrp_1\dots p_{n-2}}\,\epsilon_{stq_1\dots q_{n-2}}$ and then perform the summation over the $2n-4$ subscripts $p_1, \dots, p_{n-2}, q_1, \dots, q_{n-2}$,

$$\epsilon_{mrp_1\dots p_{n-2}}\,\epsilon_{p_1\dots p_{n-2}li} = \epsilon_{mrp_1\dots p_{n-2}}\,\epsilon_{lip_1\dots p_{n-2}} = (n-2)!(\delta_{ml}\delta_{ri} - \delta_{mi}\delta_{rl})$$
$$\epsilon_{stq_1\dots q_{n-2}}\,\epsilon_{q_1\dots q_{n-2}jk} = \epsilon_{stq_1\dots q_{n-2}}\,\epsilon_{jkq_1\dots q_{n-2}} = (n-2)!(\delta_{sj}\delta_{tk} - \delta_{sk}\delta_{tj}).$$

We obtain

$$2R_{rmst}(\underset{\sim}{\varepsilon}) \equiv \varepsilon_{rs,mt} + \varepsilon_{mt,rs} - \varepsilon_{ms,rt} - \varepsilon_{rt,ms} = 0, \tag{3}$$

where $R^{\{4\}}(\underset{\sim}{\varepsilon})$ stands for the linearized Riemann tensor constructed on $\underset{\sim}{\varepsilon}$ regarded as the metric tensor of the space; the linearized Riemann tensor is more convenient for manipulations because its rank is equal to four for any n. The numbers of independent (or, in the terminology of [4], *essential*) components of the tensors $R^{\{4\}}$ and $\eta^{\{2n-4\}}$ are equal because these tensors are dual to each other, i.e., are obtained from each other using only contractions with the Levi–Civita symbols. Taking into account the C_n^4 Ricci identities

$$R_{rmst}(\underset{\sim}{\varepsilon}) + R_{rstm}(\underset{\sim}{\varepsilon}) + R_{rtms}(\underset{\sim}{\varepsilon}) = 0, \tag{4}$$

which become nontrivial for $n \geqslant 4$, we see that these numbers are equal to $n^2(n^2-1)/12 = A_n$. First-order differential relations known in differential geometry as Bianchi identities do not make the number of independent components of $R^{\{4\}}$ (and hence, the number of independent strain compatibility equations, or Saint-Venant n-dimensional identities (3)) less than A_n [5].

Substituting relations (1) into (3), we obtain the desired stress consistency equations in the multidimensional elastic medium

$$\begin{aligned} 2R_{rmst}\big(\underset{\sim}{\varepsilon}(\underset{\sim}{\sigma})\big) =& \sigma_{rs,mt} + \sigma_{mt,rs} - \sigma_{ms,rt} - \sigma_{rt,ms} \\ &+ \frac{\nu}{1+\nu}(\Theta_{,rt}\delta_{ms} + \Theta_{,ms}\delta_{rt} - \Theta_{,mt}\delta_{rs} - \Theta_{,rs}\delta_{mt}) = 0, \end{aligned} \tag{5}$$

among which there are also A_n independent equations.

Let us further form the traces of the tensor $R^{\{4\}}$, namely, the symmetric Ricci tensor $\underset{\sim}{R}$ with $n(n+1)/2 = a_n$ independent components and the curvature invariant R,

$$R_{ms} = R_{rmsr}, \quad R = R_{mm} = R_{rmmr}. \tag{6}$$

Equating to zero all components $R_{ms}\big(\underset{\sim}{\varepsilon}(\underset{\sim}{\sigma})\big)$ leads to a_n consequences of the general compatibility equations (5),

$$\Delta\sigma_{ms} + \frac{1+(3-n)\nu}{1+\nu}\,\Theta_{,ms} - \frac{\nu}{1+\nu}\,\Delta\Theta\,\delta_{ms} - \sigma_{mr,rs} - \sigma_{sr,rm} = 0 \tag{7}$$

and equating to zero the invariant R leads to a relationship for scalars,

$$\Delta\Theta = \frac{1+\nu}{1+(2-n)\nu}\,\sigma_{mr,mr}. \tag{8}$$

Taking (8) into account, we can represent equations (7) as follows:

$$\Delta\sigma_{ms} + \frac{1+(3-n)\nu}{1+\nu}\,\Theta_{,ms} - \frac{\nu}{1+(2-n)\nu}\,\sigma_{rt,tr}\,\delta_{ms} - \sigma_{mr,rs} - \sigma_{sr,rm} = 0. \tag{9}$$

Traditionally (although this is not necessary), in consistency equations reflecting the differential-geometric structure of the dependence of $\underset{\sim}{\sigma}$ on the coordinates, one takes into account the fact that $\underset{\sim}{\sigma}$ is a physical object that satisfies the postulates of continuum mechanics and, in particular, the n equations of equilibrium with given volume forces $\mathbf{X}(\mathbf{x})$,

$$\sigma_{ij,j} = -X_i. \tag{10}$$

Their presence enables us to simplify the form of (9),

$$\Delta\sigma_{ms} + \frac{1+(3-n)\nu}{1+\nu}\,\Theta_{,ms} + \frac{\nu}{1+(2-n)\nu}\,\operatorname{div}\mathbf{X}\,\delta_{ms} + X_{m,s} + X_{s,m} = 0. \tag{11}$$

For $n = 3$, equations (11) obviously coincide with the classical Beltrami–Michell equations of elasticity theory.

Let us study the following problem: Is the system of a_n equations (9) (or (7)) equivalent to the system of A_n equations (5) and for what n are these systems equivalent?

Case $n = 2$. We have $A_2 = 1$ and $a_2 = 3$. In the planar problem, there is only one independent compatibility equation, which can be taken in the form (8),

$$\Delta\Theta = (1+\nu)\sigma_{mr,mr}. \tag{12}$$

It can readily be seen that (12) implies the triples of equations (7) and (9), into which one must substitute $n = 2$, i.e., each of the systems (5), (7), and (9) is equivalent to a single equation, namely, (12).

Case $n = 3$. The relation $A_3 = a_3 = 6$ enables one to replace the six equations (5) by a system consisting of six arbitrary independent linear combinations of these equations, i.e., for example, by systems (9) or (7) obtained by contractions (5). Each of these systems is equivalent to (5), and the compatibility equations in stresses in $\mathbb{R}^3$ can be represented in the form (9) or (7), after substituting $n = 3$ in (9) and (7), respectively.[1]

Case $n \geqslant 4$. Beginning with $n = 4$, we have the inequality $A_n > a_n$, and therefore, no system of a_n linear combinations A_n of equations (5), including systems (7) and (9), cannot be equivalent to system (5) itself. System (7) is weaker than (5).

As a visible counterexample, we take the field of stresses

$$\sigma_{ms}(\mathbf{x}) = cx_3x_4(\delta_{1m}\delta_{2s} + \delta_{1s}\delta_{2m}), \qquad c = \text{const}, \tag{13}$$

$$\sigma_{ms,rt} = c(\delta_{3r}\delta_{4t} + \delta_{3t}\delta_{4r})(\delta_{1m}\delta_{2s} + \delta_{1s}\delta_{2m}), \tag{14}$$

and substitute this into systems (7) and (5). All summands in (7) are contractions of expressions (14) with respect to some pair of indices, and thus vanish, whereas the nontrivial condition $c = 0$ occurs in (5) for $m = 1$, $s = 2$, $r = 3$, and $t = 4$. For $c \neq 0$, the stress field (13) is incompatible.

Thus, with respect to the a_n components of the symmetric tensor $\underset{\sim}{\sigma}(\mathbf{x})$, the statement of the second boundary value problem of the isotropic elasticity theory in stresses [6, 7] in an n-dimensional domain V (with boundary ∂V which has a unique outward pointing unit normal $\mathbf{n}$ at every point of ∂V) is in the solution of A_n equations (5) and n equations (10) in V satisfying n boundary conditions $\sigma_{ij}n_j = P_i^\circ(\mathbf{x})$ on ∂V, where $\mathbf{P}^\circ(\mathbf{x})$ stands for the vector of surface loads given on ∂V. The replacement of A_n equations (5) by a_n generalized Beltrami–Michell equations (11) is nonequivalent for $n \geqslant 4$ and weakens the system in V.

ACKNOWLEDGMENTS

The research was supported by RFBR (grants nos. 14-01-00317a and 15-01-00848a) .

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[1] Certainly, after this, one can take into account also the equilibrium relations (10), which leads to the Beltrami–Michell equations (11).