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ELECTROMAGNETIC METHODS

On the Well-Posedness of the Direct and Inverse Problem of Magnetostatics. Part 2

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Abstract—The main (mathematical) reason is given for the possible nonuniqueness of solution to the inverse problem of magnetostatics, which consists in reconstructing the geometrical and/or physical parameters of a magnet based on a known (measured) resultant magnetic field outside it. Examples of both unique and (essentially) ambiguous solutions to this problem are given. Some techniques for eliminating the nonuniqueness by proper arrangement of the measurement experiment are provided.

Keywords: basic equation of magnetostatics, inverse problem of magnetostatics, well-posedness of problem, magnetic nondestructive testing

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INTRODUCTION

The (Hadamard, see, for example, [1]) well-posedness of a certain mathematical problem presupposes the existence and uniqueness, as well as the stability (that is, continuous dependence in one or another metric) of its solution with respect to its initial data. In Part 1 of this paper [2], it was demonstrated that the direct problem stated in the form of the basic equation of magnetostatics [3] is well-posed. The direct problem is to determine the strength of the resultant magnetic field inside and outside an arbitrarily shaped magnet, given the following set of initial data: the shape and dimensions of the magnet, its magnetic permeability, and the magnitude and direction of an external field. An important feature of the results obtained in [2] is that in proving the stability of the field strength derived from this equation, specific estimates have been given that indicate the maximum possible error in the field strength calculated from the basic equation versus the initial error in setting the above-mentioned input data obtained as a result of measurement experiments. However, magnetic nondestructive testing is mostly concerned with the inverse problems of magnetostatics, viz. to restore the “input data”, i.e., the geometrical (the shape of the article and the shape and dimensions of defects and cavities) and/or physical (magnetic permeability) characteristics of the magnet, given an external field and a known (measured) resultant field in some finite region outside the magnet available for measurements. Various approaches to the practical solution of the inverse problem are discussed, for example, in [4–11]. Most of them are based on repeatedly solving direct problems (either by original numerical-analytical methods or using universal software packages such as ANSYS, ELCUT, or FEMM) in order to reveal regularities in the behavior of resultant magnetic fields depending on the shape and dimensions of both the magnetic article itself and internal or external defects in it. Other approaches use either the method of minimizing the functional of the deviation of the measured magnetic-field distribution from the calculated (again by repeated solution of the direct problem) fields of a “standard” defect or the construction of interpolation formulas (based on the solution of direct problems or data from actual full-scale experiments) that express the dependence of the resultant field on those or other defect parameters. In many respects, however, an important question remains about the possibility (or impossibility) in principle of unambiguous determination of the shape and dimensions of defects in a product based on measuring the reaction field outside it, that is, the question of unique solvability (included in the notion of well-posedness) of the inverse problem of magnetostatics. In this respect, of importance are the discussion papers [12–14], which informally consider various examples of single- and multiple-valued solutions to the inverse problem. Coming to the conclusion that the problem is multifaceted and complex, Pechenkov and Shcherbinin [13] infer that “for the practice of flaw detection, it is important to provide (and accumulate) examples of the geometrical shapes of bodies (with or without defects) that have a unique or ambiguous solution to the inverse problem of magnetic defectoscopy and mathematical proofs of this fact for these examples". It is this topic (but not only!) that the present article is concerned with.

POSING INVERSE PROBLEMS OF MAGNETOSTATICS

To solve the direct and inverse problems of magnetostatics, we use the so-called basic equation of magnetostatics, defined by the relation [3, p. 16]

$$\mathbf{H}(\mathbf{r}) - \nabla \mathrm{div} \int_{\Omega} \frac{\mathbf{M}(\mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' = \mathbf{H}^0(\mathbf{r}), \;\; \mathbf{r} \in \mathfrak{R}^3 \backslash (S \cup T), \tag{1}$$

and the well-known connection between vector functions $\mathbf{H}(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$ for $\mathbf{r} \in \Omega$

$$\mathbf{M}(\mathbf{r}) = \chi \cdot \mathbf{H}(\mathbf{r}), \;\; \chi := \mu - 1, \tag{2}$$

where Ω is the (simply connected or not) domain, with a boundary S, occupied by the investigated magnet; $\mathbf{H}^0(\mathbf{r})$ is the strength of an external field created by currents concentrated within a certain closed domain T; $\mathbf{H}(\mathbf{r})$ is the intensity of the resultant field; $\mathbf{M}(\mathbf{r})$ is a magnetization arising in the magnet; μ is its magnetic permeability; and χ is its magnetic susceptibility. Depending on the situation, the permeability (and, accordingly, susceptibility) may be constant (μ = const) or depend on the coordinates [$\mu = \mu(\mathbf{r})$] or the intensity or magnetization [$\mu = \mu(\mathbf{r}, \mathbf{H}(\mathbf{r}))$ or $\mu = \mu(\mathbf{r}, \mathbf{M}(\mathbf{r}))$]. In what follows, for convenience, the intensity of the resultant field $\mathbf{H}(\mathbf{r})$ inside (when $\mathbf{r} \in \Omega$) and outside [when $\mathbf{r} \in \Omega_1 := \mathfrak{R}^3 \backslash (\Omega \cup S \cup T)$] the magnet will be denoted, respectively, as $\mathbf{H}^i(\mathbf{r})$ and $\mathbf{H}^e(\mathbf{r})$.

Relations (1) and (2) are fully equivalent to the system of Maxwell's equations for the case of magnetostatics; however, they have a number of undoubted merits, discussed, for example, in [15, 16], that greatly facilitate both the study and derivation per se of analytical formulas or numerical algorithms for solving the direct and inverse problems of magnetostatics.

The following equation for determining the magnetization $\mathbf{M}(\mathbf{r})$ is derived from relations (1) and (2) to study and solve direct problems:

$$\chi^{-1} \cdot \mathbf{M}(\mathbf{r}) + (A\mathbf{M})(\mathbf{r}) = \mathbf{H}^0(\mathbf{r}), \;\; \mathbf{r} \in \Omega, \tag{3}$$

where the operator

$$(A\mathbf{M})(\mathbf{r}) := -\nabla \mathrm{div} \int_{\Omega} \frac{\mathbf{M}(\mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}'. \tag{4}$$

Equation (3) and operator A are considered in a real Hilbert space $\mathbf{L}_2(\Omega)$ of vector-valued functions square integrable over domain Ω, with the scalar product defined by the formula $(\mathbf{u}, \mathbf{v}) := \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, d\mathbf{r}$.

The statement (in the form of the basic equation of magnetostatics) of the direct problem of magnetostatics and the study of its well-posedness are discussed in detail in [2]. In the present paper, the well-posedness is investigated for the *inverse* problem of magnetostatics stated in the same form, viz., to restore the "input data", i.e., the geometrical (magnet shape Ω) and/or physical (magnet permeability μ) characteristics given an external field $\mathbf{H}^0(\mathbf{r})$ and a known (measured) resultant field $\mathbf{H}(\mathbf{r})$ in some finite domain Ω' outside the magnet accessible for measurement. Hereinafter, the shape of the magnet is understood to be not only its geometrical affiliation but also its relevant dimensions and position in space (localization). Depending on what characteristics are subject to reconstruction, we will distinguish between the following three types of inverse problems:

problem A—given a known shape of the magnet domain Ω, find the distribution of magnetic permeability μ in it;

problem B—given a known *model* of the distribution of magnetic permeability μ in a magnet, determine its shape Ω (thereby determining the shape of possible defects in the form of a cavity);

problem C—determine both the shape Ω of a magnet and the distribution of magnetic permeability μ in it.

We note that whereas the problem of the well-posedness of the direct problem of magnetostatics has been positively solved for a wide class of practically important problems, the issues of the well-posedness of the above inverse problems are much more sophisticated, and the relevant theory is still in its infancy. In the present paper, we will touch on only one of the three well-posedness conditions mentioned in the introduction, namely, the uniqueness of the solution to the above-formulated inverse problems, a question that is important in magnetic nondestructive testing, since for cases of the positive answer to this question, there exists at least the possibility in principle for unambiguously reconstructing the distribution of magnetic permeability in an article and/or detecting various types of flaws in it and describing their shapes and locations.

WEYL DECOMPOSITION AND ITS CONNECTION WITH THE OPERATOR IN THE BASIC EQUATION OF MAGNETOSTATICS

When studying and practically solving the direct and inverse problems of magnetostatics, the so-called Weyl decomposition of the space $\mathbf{L}_2(\Omega)$ into the sum of three orthogonal subspaces plays an important role [17]

$$\mathbf{L}_2(\Omega) = \mathbf{U}(\Omega) \oplus \mathbf{G}(\Omega) \oplus \mathbf{J}(\Omega), \tag{5}$$

where $\mathbf{U}(\Omega)$ is the closure [in the metric of space $\mathbf{L}_2(\Omega)$] of the gradients of harmonic functions smooth in $\bar{\Omega}$; $\mathbf{G}(\Omega)$ is the closure of the gradients of functions that are smooth in $\bar{\Omega}$ and vanish at the boundary S of domain Ω; and $\mathbf{J}(\Omega)$ is the closure of solenoidal vector-valued functions that are smooth in $\bar{\Omega}$ and have zero normal component at the boundary S. The Weyl expansion (5) turns out to be closely related to Eq. (3) and the operator (4) appearing in it, namely, the subspace $\mathbf{J}(\Omega)$ coincides with the kernel of operator A (the set of functions turned into the zero function by the operator A), $\mathbf{G}(\Omega)$ is a proper subspace of operator A corresponding to eigenvalue 1, and the subspace $\mathbf{U}(\Omega)$ is invariant under operator A, that is, $A\mathbf{M} \subset \mathbf{U}(\Omega)$ for all $\mathbf{M} \in \mathbf{U}(\Omega)$. The proof of these assertions can be found, for example, in [18] or [16]. Moreover, in the monograph [16], the Weyl expansion (5) was obtained precisely in the way of investigating the properties of operator A. According to this decomposition, any vector $\mathbf{M} \in \mathbf{L}_2(\Omega)$ is unique represented as the sum of three orthogonal [in the sense of the scalar product in $\mathbf{L}_2(\Omega)$] vectors

$$\mathbf{M} = \mathbf{M}_U + \mathbf{M}_G + \mathbf{M}_J \tag{6}$$

that belong to $\mathbf{U}(\Omega)$, $\mathbf{G}(\Omega)$, and $\mathbf{J}(\Omega)$, respectively, and are the orthogonal projections of the vector $\mathbf{M}$ on these subspaces. Note that if the vector $\mathbf{M}$ is sufficiently smooth in Ω, its projections $\mathbf{M}_U$, $\mathbf{M}_G$, and $\mathbf{M}_J$ have the same degree of smoothness [17].

REASONS FOR POSSIBLE NONUNIQUENESS OF SOLUTION TO INVERSE PROBLEMS OF MAGNETOSTATICS

Let us turn to the question of the uniqueness of solution to inverse problem A, which consists in reconstructing the magnetic permeability μ based on a known (measured) resultant field $\mathbf{H}^e(\mathbf{r})$ in some region Ω' accessible for measurement outside the magnet. The following approach to solving this problem seems most natural [19]. Assuming that $\mathbf{r} \in \Omega'$ in Eq. (1), we arrive at the following integro-differential equation for determining magnetization $\mathbf{M}(\mathbf{r})$:

$$A\mathbf{M}(\mathbf{r}) = \mathbf{H}^0(\mathbf{r}) - \mathbf{H}^e(\mathbf{r}), \ \ \mathbf{r} \in \Omega', \tag{7}$$

with the right-hand side of it known and the operator A, of the form defined in Eq. (4), treated as an operator from $\mathbf{L}_2(\Omega)$ into $\mathbf{L}_2(\Omega')$. After finding the distribution of the magnetization vector $\mathbf{M}(\mathbf{r})$ from this equation, we assume that $\mathbf{r} \in \Omega$ in (1) and by direct calculation find the field $\mathbf{H}^i(\mathbf{r})$ inside Ω as

$$\mathbf{H}^i(\mathbf{r}) = \mathbf{H}^0(\mathbf{r}) + \nabla \mathrm{div} \int_{\Omega} \frac{\mathbf{M}(\mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}', \ \ \mathbf{r} \in \Omega. \tag{8}$$

Finally, knowing $\mathbf{M}(\mathbf{r})$ and $\mathbf{H}^i(\mathbf{r})$, from relation (2) we find the magnetic permeability μ.

When deciding on the uniqueness of determining the magnetic permeability μ with such an approach, the question of the uniqueness of solution to Eq. (7) arises in a natural way. Let us prove, however, that such uniqueness does not take place.

First of all, we derive a more convenient expression for operator A in Eq. (4) when $\mathbf{r} \notin \bar{\Omega}$ (in particular, for $\mathbf{r} \in \Omega'$). Let $\mathbf{M}(\mathbf{r})$ be a function sufficiently smooth in Ω, with all of its projections in (6) being therefore smooth, too. Substituting the expansion in Eq. (6) for operator A in Eq. (4) and using the corresponding integral formulas, we can obtain the following expression for this operator for $\mathbf{r} \in \mathfrak{R}^3 \backslash \bar{\Omega}$ (in particular, $\mathbf{r} \in \Omega'$) after a series of natural transformations:

$$(A\mathbf{M})(\mathbf{r}) = \nabla \int_S (M_U)_n \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}', \quad \mathbf{r} \in \mathfrak{R}^3 \backslash \bar{\Omega}, \tag{9}$$

where $(M_U)_n := \mathbf{M}_U \cdot \mathbf{n}$ is the normal component (more precisely, its limiting value from within Ω) on the surface S of projection $\mathbf{M}_U$, where $\mathbf{n}$ is the vector of the unit outer normal to S.

Let us prove that the kernel of operator A [i.e., the set of vectors that this operator translates into the zero vector, denoted by ker A] acting from $\mathbf{L}_2(\Omega)$ into $\mathbf{L}_2(\Omega')$ according to formula (4) coincides with the subspace $\mathbf{G}(\Omega) \oplus \mathbf{J}(\Omega)$

$$\ker A = \mathbf{G}(\Omega) \oplus \mathbf{J}(\Omega). \tag{10}$$

Indeed, if $\mathbf{M} \in \mathbf{G}(\Omega) \oplus \mathbf{J}(\Omega)$, then it follows from the single-valuedness of the expansion in Eq. (6) that $\mathbf{M}_U = \mathbf{0}$, and therefore from (9) we obtain $A\mathbf{M} = \mathbf{0}$, that is, $\mathbf{M} \in \ker A$. Conversely, let $\mathbf{M} \in \ker A$ and let us then prove that $\mathbf{M}_U = \mathbf{0}$, which will complete the proof of relation (10). If $(A\mathbf{M})(\mathbf{r}) = \mathbf{0}$ for $\mathbf{r} \in \Omega'$, then it follows from Eq. (9) that the simple-layer potential, appearing under the gradient sign in Eq. (9),

$$V(\mathbf{r}) := \int_S (M_U)_n \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \tag{11}$$

is a constant C for all $\mathbf{r} \in \Omega'$. Since the simple-layer potential is harmonic in the domain $\mathfrak{R}^3 \backslash \bar{\Omega}$ and $\Omega' \subset \mathfrak{R}^3 \backslash \bar{\Omega}$, it follows from the uniqueness theorem for harmonic functions that this potential is the constant C in the entire unbounded domain $\mathfrak{R}^3 \backslash \bar{\Omega}$. However, since the simple-layer potential in Eq. (11) tends to zero as $|\mathbf{r}| \to \infty$ [20, p. 208], i.e., $C = 0$, we have $V(\mathbf{r}) = 0$ for all $\mathbf{r} \in \mathfrak{R}^3 \backslash \bar{\Omega}$. Considering the continuity of the simple-layer potential in the entire space $\mathfrak{R}^3$, we conclude that this potential vanishes at the surface S of domain Ω, too. The function $V(\mathbf{r})$, harmonic in domain Ω, thus vanishes at the boundary S of this domain, and, therefore, it follows from the uniqueness theorem for the solution of the internal Dirichlet problem for the Laplace equation [20, p. 274] that $V(\mathbf{r}) = 0$ for all $\mathbf{r} \in \Omega$. We thus obtain that the simple-layer potential $V(\mathbf{r}) \equiv 0$ in $\mathfrak{R}^3$. In this case, the limiting (at S) values of the normal derivatives $(\partial V/\partial n)_i$ and $(\partial V/\partial n)_e$ also turn zero both from inside and outside of Ω. Since the jump in the normal derivative of the simple-layer potential $V(\mathbf{r})$ upon passage through the surface S is described by the formula [20, p. 267]

$$(\partial V/\partial n)_i - (\partial V/\partial n)_e = (M_U)_n,$$

we obtain that $(M_U)_n \equiv 0$ at S. Since $\mathbf{M}_U \in \mathbf{U}(\Omega)$, $\mathbf{M}_U$ is a solenoidal function, and therefore the fact that $(M_U)_n \equiv 0$ at S entails $\mathbf{M}_U \in \mathbf{J}(\Omega)$. However, since the zero vector is the only intersection of the subspaces $\mathbf{U}(\Omega)$ and $\mathbf{J}(\Omega)$, we have $\mathbf{M}_U = \mathbf{0}$, which completes the proof of (10).

It follows immediately from (10) that Eq. (7) has an infinite set of solutions. Next, let us determine their structure. First of all, we prove that the projection $\mathbf{M}_U$ of any solution $\mathbf{M}$ of Eq. (7) onto the subspace $\mathbf{U}(\Omega)$ is the same, and therefore uniquely determined by Eq. (7) itself. Indeed, if $\mathbf{M}$ is a solution of Eq. (7), then, taking Eqs. (6) and (10) into account, we obtain the following equation for the projection $\mathbf{M}_U$ from (7):

$$(A\mathbf{M}_U)(\mathbf{r}) = \mathbf{H}^0(\mathbf{r}) - \mathbf{H}^e(\mathbf{r}), \;\; \mathbf{r} \in \Omega'. \tag{12}$$

Let us prove that the solution of this equation for the function $\mathbf{M}_U$ is unique. To do this, it suffices to show that the corresponding homogeneous equation

$$(A\mathbf{M}_U)(\mathbf{r}) = \mathbf{0}, \;\; \mathbf{r} \in \Omega' \tag{13}$$

has only the zero solution. In the above, after formula (10), it was proved that if $(A\mathbf{M})(\mathbf{r}) = \mathbf{0}$ for $\mathbf{r} \in \Omega'$ is satisfied for some vector $\mathbf{M}$, then the projection of this vector $\mathbf{M}_U = \mathbf{0}$. This implies that Eq. (13) has only the zero solution, and the projection $\mathbf{M}_U$ of any solution to Eq. (12) is, therefore, uniquely defined. As shown in [15], one of the possibilities for actually finding this projection reduces to successive solution of some classical integral equation at surface S (that has a unique solution) and the internal Neumann problem for the Laplace equation in domain Ω. However, the described path requires that the area Ω' in which the resultant field $\mathbf{H}^e(\mathbf{r})$ is measurable be "in contact" with the entire surface S of domain Ω (i.e., $\bar{\Omega}' \supset S$). If this condition is not fulfilled, then, using the uniqueness theorem for harmonic functions, it is theoretically possible to continue $\mathbf{H}^e(\mathbf{r})$ in a unique manner from Ω' on $\mathfrak{R}^3 \backslash \bar{\Omega}$, which will resume access to the surface S.

The foregoing allows the conclusion that all solutions $\mathbf{M}(\mathbf{r})$ of Eq. (7) have the form

$$\mathbf{M}(\mathbf{r}) = \mathbf{M}_U(\mathbf{r}) + \mathbf{M}_G(\mathbf{r}) + \mathbf{M}_J(\mathbf{r}), \;\; \mathbf{r} \in \Omega, \tag{14}$$

where $\mathbf{M}_U(\mathbf{r})$ is a function from $\mathbf{U}(\Omega)$ uniquely defined by the external field $\mathbf{H}^e(\mathbf{r})$, while $\mathbf{M}_G(\mathbf{r})$ and $\mathbf{M}_J(\mathbf{r})$ are *arbitrary* functions from $\mathbf{G}(\Omega)$ and $\mathbf{J}(\Omega)$. Equation (7) thus has a nonunique solution, which can be the reason for the nonuniqueness (in the general case) of solution to inverse problem A, which consists in reconstructing the magnetic permeability μ of a magnet given its shape Ω and a known resultant field $\mathbf{H}^e(\mathbf{r})$ outside it, in accordance with the above scheme for solving this problem.

SOME CASES OF UNAMBIGUOUS AND AMBIGUOUS DETERMINATION OF THE CHARACTERISTICS OF MAGNETIC SYSTEMS

In some cases, the nonuniqueness of determining the magnetization $\mathbf{M}(\mathbf{r})$ of a magnet from Eq. (7) based on the resultant field $\mathbf{H}^e(\mathbf{r})$ does not prevent the unambiguous determination of a number of important characteristics of investigated magnetic systems, namely, those determined only by the projection $\mathbf{M}_U(\mathbf{r})$ (uniquely defined, as shown above) of magnetization $\mathbf{M}(\mathbf{r})$ onto the subspace $\mathbf{U}(\Omega)$. One of such characteristics is, for example, the free energy of the magnet in an external field, calculated from the formula $E = (\mu_0/2)\int_\Omega \mathbf{M} \cdot \mathbf{H}^0 d\mathbf{r}$, which can be reduced to the form $E = (\mu_0/2)\int_\Omega \mathbf{M}_U \cdot \mathbf{H}^0 d\mathbf{r}$ [as follows from the orthogonality of the subspaces on the right-hand side of Eq. (5) and from $\mathbf{H}^0 \in \mathbf{U}(\Omega)$], while $\mathbf{M}_U(\mathbf{r})$ is uniquely defined by the resultant field.

In addition, one can assert the uniqueness of determining the magnet permeability $\mathbf{H}^e(\mathbf{r})$ based on the resultant field μ in the case where this permeability is in advance known to be constant (μ = const). Let us prove this. The magnetization $\mathbf{M}(\mathbf{r})$ of the magnet must satisfy Eq. (3). Replacing the vector $\mathbf{M}(\mathbf{r})$ in this equation with its Weyl expansion (6), we arrive at

$$\chi^{-1}\mathbf{M}_U(\mathbf{r}) + \chi^{-1}\mathbf{M}_G(\mathbf{r}) + \chi^{-1}\mathbf{M}_J(\mathbf{r}) + (A\mathbf{M}_U)(\mathbf{r}) + (A\mathbf{M}_G(\mathbf{r})) + (A\mathbf{M}_J)(\mathbf{r}) = \mathbf{H}^0(\mathbf{r}). \tag{15}$$

Considering the above properties of operator A, we see that $(A\mathbf{M}_U)(\mathbf{r}) \in \mathbf{U}(\Omega)$, $(A\mathbf{M}_G)(\mathbf{r}) = \mathbf{M}_G(\mathbf{r})$, and $(A\mathbf{M}_J)(\mathbf{r}) = \mathbf{0}$, and by virtue of (15), we have

$$\left[\chi^{-1}\mathbf{M}_U(\mathbf{r}) + (A\mathbf{M}_U)(\mathbf{r})\right] + \left(\chi^{-1} + 1\right)\mathbf{M}_G(\mathbf{r}) + \chi^{-1}\mathbf{M}_J(\mathbf{r}) = \mathbf{H}^0(\mathbf{r}). \tag{16}$$

If μ = const then χ^{-1} = const, too. Since multiplication by a constant does not withdraw vector functions from subspaces they originally belonged to, the terms on the left-hand side of Eq. (16) belong to

$\mathbf{U}(\Omega)$, $\mathbf{G}(\Omega)$, and $\mathbf{J}(\Omega)$, respectively. Therefore, Eq. (16) can be regarded as a Weyl expansion for $\mathbf{H}^0(\mathbf{r})$. Since the external field $\mathbf{H}^0(\mathbf{r}) \in \mathbf{U}(\Omega)$, the uniqueness of the Weyl decomposition implies that the last two terms in Eq. (16) become zero, and therefore $\mathbf{M}_G(\mathbf{r}) = \mathbf{M}_J(\mathbf{r}) = \mathbf{0}$. Then, we find from Eq. (16) that the magnetization $\mathbf{M}(\mathbf{r})$ coincides with its projection $\mathbf{M}_U(\mathbf{r})$, which is uniquely defined, as shown above. This leads to an important conclusion about the uniqueness of the solution to inverse problem A in the case of a constant permeability μ of the magnet.

Note that the conclusion about the uniqueness of determining permeability in the case of its constancy was drawn because multiplication of vector functions $\mathbf{M}_U(\mathbf{r})$, $\mathbf{M}_G(\mathbf{r})$, and $\mathbf{M}_J(\mathbf{r})$ by χ^{-1} (which is also constant, together with μ) in the first three terms of Eq. (15) did not withdraw these functions from the corresponding subspaces $\mathbf{U}(\Omega)$, $\mathbf{G}(\Omega)$, and $\mathbf{J}(\Omega)$. Let us show that this property holds *only* in the case of constant permeability, and, therefore, susceptibility, implying that any dependence of permeability on coordinates may prove to be a mathematical reason for the possible ambiguity of its reconstruction. Let us find out, for example, in which case multiplying projection $\mathbf{M}_U(\mathbf{r})$ by $\chi^{-1}(\mathbf{r})$ does not withdraw it from $\mathbf{U}(\Omega)$, that is, $\chi^{-1}(\mathbf{r}) \cdot \mathbf{M}_U(\mathbf{r}) \in \mathbf{U}(\Omega)$. If the latter is satisfied, then $\operatorname{div}\left[\chi^{-1}(\mathbf{r}) \cdot \mathbf{M}_U(\mathbf{r})\right] = \mathbf{0}$ and $\operatorname{rot}\left[\chi^{-1}(\mathbf{r}) \cdot \mathbf{M}_U(\mathbf{r})\right] = 0$. Using formulas for the divergence and rotor of this combination, we obtain $\chi^{-1}(\mathbf{r}) \cdot \operatorname{div}\mathbf{M}_U(\mathbf{r}) + \nabla\chi^{-1}(\mathbf{r}) \cdot \mathbf{M}_U(\mathbf{r}) = 0$ and $\chi^{-1}(\mathbf{r}) \cdot \operatorname{rot}\mathbf{M}_U(\mathbf{r}) + \nabla\chi^{-1}(\mathbf{r}) \times \mathbf{M}_U(\mathbf{r}) = \mathbf{0}$. Considering that $\operatorname{div}\mathbf{M}_U(\mathbf{r}) = 0$ and $\operatorname{rot}\mathbf{M}_U(\mathbf{r}) = \mathbf{0}$, we have $\nabla\chi^{-1}(\mathbf{r}) = \mathbf{0}$, and therefore $\chi^{-1}(\mathbf{r}) = \text{const}$, that is, $\mu(\mathbf{r}) = \text{const}$.

It was shown above that if the magnetization $\mathbf{M}(\mathbf{r})$ is an irrotational and solenoidal vector-function $\mathbf{M}(\mathbf{r}) \in \mathbf{U}(\Omega)$ in Ω and, hence, in the expansion in Eq. (14) $\mathbf{M}(\mathbf{r}) = \mathbf{M}_U(\mathbf{r})$, it can be uniquely reconstructed. As proved above, such a situation arises, for example, in the case of a constant permeability in a magnet. The following suspicion arises. Maybe, being a solution to Eq. (3), the magnetization $\mathbf{M}(\mathbf{r})$ in any case has only the projection $\mathbf{M}_U(\mathbf{r})$ in the expansion in Eq. (14) that is nonzero. In other words, is it not the common property that a solution $\mathbf{M}(\mathbf{r})$ to Eq. (3) belongs to $\mathbf{U}(\Omega)$ in the case where the right-hand side of the equation $\mathbf{H}^0(\mathbf{r}) \in \mathbf{U}(\Omega)$? With the positive answer to this question, inverse problem A would always have a unique solution. Unfortunately, it is not the case. It was shown in [19] that if, for example, the magnet is a ball centered at the coordinate origin that has a model permeability of the form $\mu(\mathbf{r}) = c_0 e^{\alpha r}$ ($r := |\mathbf{r}|$, c_0, and α are parameters), then, even in the case of a constant external field $\mathbf{H}^0(\mathbf{r}) = \{0, 0, H_0\}$, the solution $\mathbf{M}(\mathbf{r})$ of Eq. (3) has (of course, with $\alpha \neq 0$) nonzero projections $\mathbf{M}_G(\mathbf{r})$ and $\mathbf{M}_J(\mathbf{r})$ on subspaces $\mathbf{G}(\Omega)$ and $\mathbf{J}(\Omega)$ and, as shown above, these projections, clearly, cannot be uniquely reconstructed. This *may* be the main reason for the possible ambiguity of solution to inverse problem A. It has already been rigorously proved in [19] that in a constant external field, unique reconstruction of the ball's magnetic permeability $\mu(\mathbf{r})$ of the indicated model type, based on the measured resultant field is, indeed, impossible. This is due to the fact that for a constant external field $\mathbf{H}^0(\mathbf{r}) = \{0, 0, H_0\}$, the so-called (measured) reaction field $\mathbf{H}^R(\mathbf{r}) := \mathbf{H}^e(\mathbf{r}) - \mathbf{H}^0(\mathbf{r})$ is calculated at an arbitrary point $\mathbf{r}$ outside the spherical magnet of radius R by the formula [19]

$$\begin{cases} \mathbf{H}^R(\mathbf{r}) = \nabla D(\mathbf{r}) \\ D(\mathbf{r})\big|_{\text{sphere}} := -H_0 \dfrac{a-1}{a+2} \dfrac{R^3}{r^2} \cos\theta \end{cases}, \tag{17}$$

where the parameter a is expressed as follows in terms of the parameters c_0 and α of magnetic permeability and the ball radius R:

$$a := -c_0 e^b \left[\frac{b\left(e^{-b} - 1 + b\right)}{e^{-b} - 1 + b - b^2/2} + 2 \right], \quad b := \alpha R, \tag{18}$$

while $D(\mathbf{r})|_{\text{sphere}}$ stands for an expression that is obtained after switching from Cartesian $\mathbf{r} = (x, y, z)$ to spherical (r, θ, φ) coordinates in the expression $D(\mathbf{r})$, using the known formulas $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, and $z = r\cos\theta$. If the domain Ω' where the reaction field $\mathbf{H}^R(\mathbf{r})$ is measured contains at least one point, then given a known radius R of the ball magnet, Eq. (17) only allows one to determine the value of the parameter a, i.e., of a combination of the unknown parameters c_0 and α of permeability $\mu(\mathbf{r})$ that has the form in Eq. (18), and it is impossible to find the values of individual parameters c_0 and α from the value of this combination. However, if the magnetic-permeability parameters c_0 and α are known, and it is required to determine the size R of the ball magnet (this case corresponds now to inverse problem B), with the position of its center known, then this problem, in principle, is solvable since it reduces to solving Eq. (18) for R, given known values of a, c_0, and α.

It follows from the above that, as expected, in the *general* case, inverse problem C (simultaneous reconstruction of both the shape of a magnet and its permeability based on the measured resultant field) has no unique solution. However, in this sense, the situation is far from clear in every specific case.

A typical example of the equivocality of the discussed situation is given in [15]. Consider a ball of radius R with a constant magnetic permeability μ placed in a constant external field $\mathbf{H}^0(\mathbf{r}) = \{0, 0, H_0\}$. In this case, passing to the limit $\alpha \to 0$ in formulas (17) and (18), it is easy to obtain that the measured reaction field $\mathbf{H}^R(\mathbf{r}) := \mathbf{H}^e(\mathbf{r}) - \mathbf{H}^0(\mathbf{r})$ has the following form in the spherical coordinate system (r, θ, φ):

$$\mathbf{H}^R(\mathbf{r})\Big|_{\text{sphere}} = -\frac{\mu - 1}{\mu + 2} R^3 H_0 \nabla\left(\frac{\cos\theta}{r^2}\right), \quad r > R, \tag{19}$$

where the gradient is calculated in the spherical coordinate system. The same expression was also derived in [15]. Thus, based on the measured reaction field, one can uniquely reconstruct only the parameter combination $\frac{\mu - 1}{\mu + 2} R^3$ using formula (19) but not μ and R separately. Consequently, two different concentric balls with parameter pairs μ_1, R_1 and μ_2, R_2 that produce the same value for the combination $\frac{\mu - 1}{\mu + 2} R^3$ yield the same reaction field at the same distance from their center. Therefore, with a limited access to the ball surface for field measurements, it is impossible to uniquely reconstruct both the permeability and radius of the ball simultaneously. However, there are some encouraging circumstances here. If the resultant field $\mathbf{H}^e(\mathbf{r})$ [and, hence, the reaction field $\mathbf{H}^R(\mathbf{r})$] can be measured at least at one point located immediately at the ball surface (i.e., if the measurement domain Ω' touches the surface of the ball at least at one point), separate determination of μ and R becomes possible. Indeed, from (19) it is easy to obtain an expression for the r-component H_r^R of the reaction field in the spherical coordinate system as

$$H_r^R = 2\frac{\mu - 1}{\mu + 2} H_0 \left(\frac{R}{r}\right)^3 \cos\theta, \quad r > R, \tag{20}$$

At the ball surface [when passing to the limit $r \to R$ in Eq. (20)], the component H_r^R coincides with the normal component of the field at this surface, and its value can be calculated by the formula

$$H_r^R\Big|_{r=R} = 2\frac{\mu - 1}{\mu + 2} H_0 \cos\theta,$$

which no longer incorporates the ball radius. The magnetic permeability μ can therefore be determined based on the measured reaction field at some point at the ball surface. After taking the second measurement of the reaction field at some point outside the magnet (for $r > R$), one can obtain the ball radius R from Eq. (20). Thus, it is still sometimes possible to resolve the issue of the nonuniqueness of solution to inverse problem C for simply shaped magnets by means of specially selected repeated measurements or (as will be shown below) a well-chosen external field $\mathbf{H}^0(\mathbf{r})$.

As for the above example of a ball magnet of radius R with a model permeability of the form $\mu(\mathbf{r}) = c_0 e^{\alpha r}$, here again there exists a fundamental opportunity for uniquely reconstructing this permeability based on the measured reaction field outside the magnet (that is, for separately determining the values of parameters c_0 and α) if we place the magnet in an *inhomogeneous* external field $\mathbf{H}^0(\mathbf{r})$ (as shown above, this is impossible in the case of a homogeneous external field). Let us demonstrate this. As we know, the external field $\mathbf{H}^0(\mathbf{r})$ outside domains bearing currents that create it is always solenoidal and irrotational and can therefore be represented as the gradient of a harmonic function $\mathbf{H}^0(\mathbf{r}) = \nabla\eta^0(\mathbf{r})$, $\Delta\eta^0(\mathbf{r}) = 0$. Since the function $\eta^0(\mathbf{r})$ is harmonic outside the current-bearing domains, it can be represented in the spherical coordinate system inside the investigated ball Ω as an expansion in terms of solid spherical harmonics as

$$\eta^0(\mathbf{r})\big|_{\text{sphere}} = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\eta^0_{lm} r^l Y_{lm}(\theta,\varphi),$$

with known coefficients $\{\eta^0_{lm}\}$ ($\{Y_{lm}(\theta,\varphi)\}$ are classical spherical harmonics). The external field thus has the form

$$\mathbf{H}^0(\mathbf{r}) = \nabla\eta^0(\mathbf{r}),\ \eta^0(\mathbf{r})\big|_{\text{sphere}} = \sum_{l=1}^{\infty} r^l Q_l(\theta,\varphi),\ Q_l(\theta,\varphi) := \sum_{m=-l}^{l}\eta^0_{lm}Y_{lm}(\theta,\varphi). \tag{21}$$

It can be shown [21] that the reaction field $\mathbf{H}^R(\mathbf{r})$ outside the ball magnet can be calculated in this case using the formula

$$\begin{cases}\mathbf{H}^R(\mathbf{r}) = \nabla D(\mathbf{r}) \\ D(\mathbf{r})\big|_{\text{sphere}} := -\sum\limits_{l=1}^{\infty}\dfrac{a_l - l}{a_l + l + 1}\dfrac{R^{2l+1}}{r^{l+1}}Q_l(\theta,\varphi)\end{cases}, \tag{22}$$

where the numbers $\{a_l\}$ are expressed as follows in terms of the parameters c_0 and α of magnetic permeability and the ball-magnet radius R:

$$a_l := -c_0 e^b\left[bl + \frac{(bl)^{2l+1}/(2l)!}{e^{-bl} - \sum\limits_{k=0}^{2l}(-1)^k(bl)^k/k!} + l + 1\right],\ b := \alpha R. \tag{23}$$

In the above case of a constant external field $\mathbf{H}^0(\mathbf{r}) = \{0, 0, H_0\}$, in Eq. (21) we have $\eta^0_{10} = \sqrt{4\pi/3}H_0$, with the rest of $\eta^0_{lm} = 0$; therefore, $Q_1(\theta,\varphi) = H_0\cos\theta$ and the remaining $Q_l(\theta,\varphi) = 0$ ($l = 2, 3, \ldots$). Therefore, formulas (22) and (23) transform, as it should be, into formulas (17) and (18) for a constant external field and, as shown above, separate determination of the parameters c_0 and α of magnetic permeability becomes in this case impossible. This is due to the fact that for a constant magnetic field $\mathbf{H}^0(\mathbf{r}) = \{0, 0, H_0\}$ in the expression (21) for $\eta^0(\mathbf{r})$, only the function $Q_1(\theta,\varphi)$ is nonzero in the sum, and, therefore, only the value of parameter a_1 [expressed as in (18)] can be determined from the measured reaction field $\mathbf{H}^R(\mathbf{r})$ using Eq. (22); as indicated, this parameter does not allow one to determine the inidividual values of parameters c_0 and α, i.e., to reconstruct the magnetic permeability. If the external field $\mathbf{H}^0(\mathbf{r})$ is chosen not constant and such that, at least, two of the system of functions $\{Q_l(\theta,\varphi)\}_{l=1}^{\infty}$ in Eq. (21) are nonzero, separate determination of c_0 and α becomes fundamentally possible. For example, if, at least, $Q_1(\theta,\varphi)$ and $Q_2(\theta,\varphi)$ are different from zero, then it will be possible to determine the values of, at least, two parameters a_1 and a_2 using Eq. (22). Then from Eq. (23) we find that the ratio of these parameters

$$\frac{a_1}{a_2} = \left[b + \frac{b^3/2}{e^{-b} - 1 + b - b^2/2} + 2\right]\Bigg/\left[2b + \frac{4b^5/3}{e^{-2b} - 1 + 2b - 2b^2 + 4b^3/3 - 2b^4/3} + 3\right]$$

contains only one unknown $b := \alpha R$, which can be determined from this equation, and, hence, will make it possible to determine the value of parameter $\alpha = b/R$. After this, the parameter c_0 can be immediately determined from Eq. (23) for $l = 1$, and the magnetic permeability in the considered model $\mu(\mathbf{r}) = c_0 e^{\alpha r}$ can therefore be reconstructed completely.

Note that for the considered ball magnet, separate determination of permeability and ball radius R (this now refers to inverse problem C) is impossible whatever the configuration of the external field $\mathbf{H}^0(\mathbf{r})$. This follows from the fact that formulas (22) and (23) contain parameters α and R only in the form of their product, $b = \alpha R$. Therefore, only the value of this product can be determined but not the values of α and R individually. This confirms one more time that in the general case, inverse problem C is not uniquely solvable.

TRICKS THAT ALLOW ONE TO OBTAIN UNIQUE SOLUTIONS OF INVERSE PROBLEM IN CERTAIN CASES

Here is a brief description of some interesting examples containing well-founded algorithms for the unambiguous determination of the position, the dimensions, and the value of constant magnetic permeability for ball-shaped magnets, the detailed description being available in the recent monograph [16]. The first example concerns the possibility of uniquely determining the coordinates of the center of a ball with known radius R and permeability $\mu = \text{const}$, placed in a constant external field in such a way that it cannot be observed visually. Let us only suppose that we know that this ball is located somewhere inside a certain domain, with the measurement of the reaction field being available only outside this domain. As indicated above, the reaction field in this case is of the form in Eq. (19) and is inhomogeneous, since the gradient appearing in this formula has the following form in the Cartesian coordinate system:

$$\nabla\left(\frac{\cos\theta}{r^2}\right) = \nabla\frac{z}{r^3} = -\frac{3xz}{r^5}\mathbf{e}_x - \frac{3yz}{r^5}\mathbf{e}_y + \left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right)\mathbf{e}_y.$$

In [16, pp. 279, 280], it is shown how this inhomogeneity can be utilized to allow unique reconstruction of the coordinates of the ball magnet center by measuring the reaction field at two points.

As proved above, if the position of the center of the ball is known but the ball is inaccessible for visual observation (i.e., no measurements can be taken at the ball surface), then in a constant external field $\mathbf{H}^0(\mathbf{r})$, by measuring the reaction field it is impossible to simultaneously reconstruct the unknown radius R of the ball and its unknown magnetic permeability μ even if it is constant. However, even here the right choice of the inhomogeneous external field can save the situation. In [16, pp. 281–289], algorithms are specified for the unique reconstruction of unknown parameters R and μ in the cases where the inhomogeneous external field $\mathbf{H}^0(\mathbf{r})$ is created either by a specially oriented magnetic dipole or by a thin-walled solenoid of finite length, with its center hosting the ball with unknown radius R and unknown constant magnetic permeability μ.

In cases that are more challenging for closed-form analysis, the possibility of uniquely solving inverse problems similar to A, B, and C can sometimes be investigated by in-depth study of the behavior of the solution to the direct problem of determining the reaction field as a function of numerical parameters characterizing the model form of magnetic permeability and/or the localization and dimensions of a model defect. In this case, the character of the dependence of various components of the reaction field on the mentioned numerical parameters (monotonicity, typical arrangement of extrema, inflection points, etc.) is studied. For example, in [22], based on the algorithm for solving the direct problem of determining the reaction field due to a ball-shaped article with an arbitrary internal defect in an arbitrary external field, curves have been derived and investigated for the dependence of various components of the reaction field on the parameters characterizing the center position and radius of a homogeneous spherical defect within a nonmagnetic ball-shaped article. This analysis showed how, by rotating this article in a constant external field, one can determine the direction from the defect center to the article center based on the position of a maximum in the measured tangential component of the reaction field on its surface. Further, by pointing the constant external field in this direction, it is possible to uniquely determine the distance from the defect center to the center of the article based on the position of the maximum of the x-component of the reaction field, and hence the position of the defect center inside the article. Then, from the measured normal component of the reaction field at the top pole of the article, one can also find the radius of the ball-shaped defect.

In [23–26], based on a similar analysis of the graphs of the dependence of the reaction-field components on numerical parameters characterizing the position and size of internal defect, an algorithm is proposed for the unambiguous determination (based on the measured reaction field) of the depth and radius of a spherical cavity inside a magnetic half-space for cases of a constant external field, as well as the external field created by a current loop or a current coil.

MAIN RESULTS

In conclusion, let us briefly formulate the main results of this paper.

The uniqueness of solution has been investigated for the proposed statements of the inverse problems of magnetostatics, which are to restore the "input data", i.e., the geometrical (the shape of a magnet and the shape and dimensions of defects and cavities) and/or physical (magnetic permeability) characteristics of the magnet, given an external field and a known (measured) resultant field in some finite region outside the magnet available for measurements. It has been shown that the main reason for the possible (in the general case) nonuniqueness of the solution of this problem is the presence of a whole subspace constituting the kernel of operator $(A\mathbf{M})(\mathbf{r})$ in Eq. (7); this implies the nonuniqueness (in the general case) of determining the magnetization $\mathbf{M}(\mathbf{r})$of the magnet from this equation based on the resultant field measured outside it. A class of this nonuniqueness is described that the determined magnetization can be (additively) correctly attributed to. It has been proved that in the case of the constancy of the magnetic permeability of the investigated magnet, the above-mentioned class of nonuniqueness reduces to zero, and this permeability can therefore be reconstructed unambiguously.

Various examples are provided of both unique and (fundamentally) nonunique solution to the inverse problem of magnetostatics stated as discussed.

Some techniques of eliminating the nonuniqueness of solution to the inverse problem have been demonstrated for a number of cases, including ensuring the possibility of measurement of resultant field in the close proximity to the magnet; choosing "correct" points for taking measurements of this field; using specially configured external fields; and taking advantage of regularities revealed when solving the direct problems of magnetostatics for the configuration at hand.

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