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ELECTROMAGNETIC METHODS

On the Well-Posedness of Direct and Inverse Problems of Magnetostatics. Part I

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Abstract—In the first part of the article, the well-posedness of the problem of solving the fundamental equation of magnetostatics is analyzed. The errors in determining the resultant magnetic-field strength inside and outside a magnet from this equation are estimated depending on the inaccuracies in setting such initial equation parameters as external field strength, magnetic permeability, and the magnet shape.

Keywords: fundamental equation of magnetostatics, direct and inverse problem, well-posedness of problem, magnetic nondestructive testing

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INTRODUCTION

Constructing mathematical models of physical phenomena brings about the necessity for solving certain mathematical problems (differential or integral equations, extremum problems, etc.) that are formulated using some initial data in the form of input (numerical, functional, geometrical, etc.) parameters that determine the physical and geometrical characteristics of studied objects. However, these parameters are usually found from experimental data and can be taken to be known only approximately; one, therefore, needs to be confident that the problem solutions obtained with these parameters are close to the solutions that would have been obtained using the exact initial data. Thus, it is important that small perturbations in the initial data of the problem only cause small changes in the problem solution. These considerations lead to the notion of the well-posedness of the mathematical problem. Even more important (and more difficult) task is to derive particular estimates of errors in the problem solution depending on the inaccuracies in the input parameters.

When mathematical modeling is used to solve problems of magnetostatics for the needs of nondestructive magnetic testing of articles, examining the well-posedness of the relevant mathematical problem that arises from the chosen mathematical model of the phenomenon being studied plays an important role and is an integral part of the complete problem solution. In addition to the necessary requirement of the existence and uniqueness of the problem solution, the notion of correctness also includes the requirement of the so-called stability of this solution with respect to distortions in setting the initial data, which are, as a rule, obtained from measurements that have been taken to within a certain error. Proving the resilience of the solution to inaccuracies in the initial data is of especial importance when studying the well-posedness of the direct problem of magnetostatics (determining the resultant field strength outside and inside a magnet given the magnet shape, magnetic permeability, and external field strength), since the first two requirements are automatically met, as will be mentioned in what follows, for a wide class of similar problems. The sources of errors in the initial data can be of different nature. Devices that measure parameters of a current system that creates the external field produce results with a certain error. The magnet shape can be somewhat different from the geometrically ideal one; its inaccuracy can also be due to the presence of small internal or external flaws in the form of inclusions or cavities. Magnetic permeability (either constant or coordinate-dependent) cannot be determined precisely either. Therefore, when studying direct problems, of importance is not only the very fact of stability of the solution to the direct problem with respect to various inaccuracies in the initial data, but also, as noted above, the ability to produce *concrete estimates* for this stability that provide the possibility in principle for indicating what kind of *maximum* *error* may occur when obtaining the resultant magnetic-field strength within the employed mathematical model depending on the value of various inaccuracies in the initial data.

In what follows, we derive these estimates related to errors in setting of both the external filed and magnetic permeability (in particular, this should encompass the emergence of small inclusion flaws) as well as inaccuracy it setting the magnet shape due to its not being ideal or incorporating small cavities. In this work, some upper-bound estimates are provided for the error in calculating the magnetic field strength depending on the volume of inclusions or cavities that distort the shape.

The second part of this particle, which is being finalized for publication, is most important (and more voluminous) and devoted to the issues of the well-posedness (the uniqueness of the solution) of more complicated and topical *inverse* problems of magnetostatics that consist in reconstructing the "initial data" such as geometrical (magnet shape, the shape and dimensions of flaws and cavities) and/or physical characteristics, given the prescribed external and known (measured) resultant fields in a certain bounded domain outside the magnet that is accessible for measurements. The second part will make use of the results obtained in the present, first part of this work.

1. THE NOTION OF THE WELL-POSEDNESS OF PROBLEM

Physical models of actual phenomena are created based on generalized experience-based data, intuitive or logical inferences, that is, they are the product of human intellectual activity. Based on the physical model of a phenomenon, its mathematical models are formulated, i.e. relationships (differential, integral, or algebraic equations, inequalities, extremum problems, etc.) that link main parameters that characterize the studied phenomenon. Studying a mathematical model with appropriate tools makes it possible to enhance the understating of the phenomenon or to verify the adequacy of the model itself. Thus, the mathematical model is, to a degree, an approximation of the real phenomenon and, hence, does not have to inherit all of its properties (for example, the existence of the very phenomenon within this model, the unique predetermination of its development over time, the stability to small perturbations in the parameters that describe external conditions). Therefore, the first natural question to spring to mind when examining a mathematical model is that of the existence of a solution, as it does not make any sense to speak of the model adequacy given the negative answer to this question. The next question to arise is the uniqueness of the solution within the frame of the examined model. In the case of nonuniqueness, one needs either to introduce some additional formalized information on the phenomenon into the mathematical model or to develop an algorithm for selecting the wanted solution. The final important question to arise is that of the stability of the solution of the mathematical problem with respect to small changes in the initial data since these data are most often based on practical measurements that are taken to within a certain error. One thus needs to verify what is known in mathematics as the well-posedness (correctness of formulation) of a mathematical problem.

Without going deeply into the formalism, we just remind the mathematical notion of the well-posedness of a certain problem (see, for example, [1]). Any quantitative problem consists in finding a "solution" z from "initial data" u and is most often reduced to solving a certain operator equation $R(z) = u$. The mathematical formulation of the problem assumes that z and u are elements of some metric spaces Z and U (i.e., sets in which the distances between the elements are defined), while the operator R acts from Z into U. In this case, the problem (to be precise, the formulation of the problem) of finding z given u is called Hadamard correct provided the following three conditions are fulfilled:

- existence (there exists a solution $z \in Z$ for any $u \in U$);
- uniqueness (this is the only solution);
- stability [this solution continuously depends on the initial data, that is, small changes (perturbations) in the initial data u bring about a small change in the solution z (the smallness being understood in the sense of the distances in the above-mentioned metric spaces Z and U)].

The first two conditions guarantee the existence of an inverse operator R^{-1} that is defined on the entire U, while the third ensures the inverse operator continuity.

2. POSING DIRECT AND INVERSE PROBLEMS OF MAGNETOSTATICS

Many problems of mathematical physics are conventionally split into direct and inverse. The former are reduced to obtaining the consequences of a phenomenon based on its causes and the parameters of participating objects, while the latter consist in finding the characteristics of the cause of the phenomenon and/or the parameters of the objects given prescribed (measured) consequences. Inverse problems mainly arise due to the fact that not every object is accessible for direct immediate examination, and its properties can therefore be judged only from their indirect manifestations.

To solve direct and inverse problems of magnetostatics, we will use the relationship (see [2])

$$\mathbf{H}(\mathbf{r}) - \nabla \mathrm{div} \int_{\Omega} \frac{\mathbf{M}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' = \mathbf{H}^0(\mathbf{r}), \quad \mathbf{r} \in \Re^3 / (S \cup T), \tag{1}$$

and the known link between the vector-functions $\mathbf{H(r)}$ and $\mathbf{M(r)}$ for $\mathbf{r} \in \Omega$

$$\mathbf{M}(\mathbf{r}) = \chi \cdot \mathbf{H}(\mathbf{r}), \quad \chi := \mu - 1, \tag{2}$$

where Ω is a (singly connected or not) domain that has a boundary S and is occupied by a magnet; $\mathbf{H}^0(\mathbf{r})$ is the strength of an external field created by currents that occupy a certain closed domain T; $\mathbf{H(r)}$ is the strength of the resultant field; $\mathbf{M(r)}$ is the magnetization that appears in the magnet; μ is the magnet permeability; and χ is the magnetic susceptibility of the magnet. Depending on the situation, the permeability (and, consequently, the magnetic susceptibility) can be either constant (μ = const) or dependent on the coordinates ($\mu = \mu(\mathbf{r})$) and on the strength or magnetization ($\mu = \mu(\mathbf{r}, \mathbf{H}(\mathbf{r}))$ or $\mu = \mu(\mathbf{r}, \mathbf{M}(\mathbf{r}))$]. Relations (1), (2) are fully equivalent to the system of Maxwell equations for the case of magnetostatics, but they possess a number of undoubted advantages (discussed, for example, in [3–5]) that considerably facilitate both the examination and the very derivation of analytical formulae or numerical algorithms for solving direct and inverse problems of magnetostatics. In the present, first part of the work, we will only touch upon the well-posedness of the direct problem of magnetostatics in the form of Eqs. (1), (2), with a separate publication (Part II) being devoted to the well-posedness of the inverse problem of magnetostatics (which has been briefly formulated in the introduction).

The direct problem of magnetostatics consists in finding the resultant field strength $\mathbf{H(r)}$ both inside the magnet (in this case, it will be denoted as $\mathbf{H}^i(\mathbf{r})$) and outside it, in the domain $\Omega_1 := \Re^3/(\Omega \cup S \cup T)$ (denoted by $\mathbf{H}^e(\mathbf{r})$) for given "initial data", which are the magnet shape Ω, the external field strength $\mathbf{H}^0(\mathbf{r})$, and the permeability μ. Hereinafter, the magnet shape is understood to indicate not only its geometrical affiliation but also its dimensions and spatial position (localization). The well-posedness (in the above sense) of the direct problem has been proved under fairly general conditions (see below).

The following equation for determining the magnetization $\mathbf{M(r)}$ can be derived from relations (1) and (2):

$$\chi^{-1} \cdot \mathbf{M}(\mathbf{r}) + (A\mathbf{M})(\mathbf{r}) = \mathbf{H}^0(\mathbf{r}), \quad \mathbf{r} \in \Omega, \tag{3}$$

where the operator

$$(A\mathbf{M})(\mathbf{r}) := -\nabla \mathrm{div} \int_{\Omega} \frac{\mathbf{M}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}'. \tag{4}$$

Equation (3) and the operator A are considered in the real Hilbert space $\mathbf{L}_2(\Omega)$ of vector-functions with the modulus that is square-integrable over the domain Ω and the scalar product that is defined by the formula $(\mathbf{u}, \mathbf{v}) := \int_{\Omega} \mathbf{u} \cdot \mathbf{v}\, d\mathbf{r}$. This space proves to be most suitable for Eq. (3) as it contains a rather extensive class of vector-functions while the solution of Eq. (3) from this space ensures the finiteness of the magnet's free energy in external field (see [6, p. 51])

$$E = (\mu_0/2) \int_{\Omega} \mathbf{M} \cdot \mathbf{H}^0\, d\mathbf{r} < \infty \ (\mu_0 \text{ is the magnetic constant}),$$

which, of course, should always be the case.

It should be noted that after solving Eq. (3), the field $\mathbf{H}^i(\mathbf{r})$ inside the magnet is uniquely determined from relation (2), and afterwards the field $\mathbf{H}^e(\mathbf{r})$ outside of the magnet is found from relation (1) in which it should be assumed that $\mathbf{r} \in \Omega_1$.

3. A BRIEF REVIEW OF THE PREVIOUSLY OBTAINED RESULTS

In [3], Eq. (3), for example, was studied in the case of isotropic magnets, when the magnetic permeability (and, hence, susceptibility) depends only on the coordinates $\mu = \mu(\mathbf{r})$, with the function μ(r) being continuous in Ω along with its partial derivatives (i.e., it is a magnet without foreign inclusions) and bounded $1 < \mu_1 < \mu(\mathbf{r}) \le \mu_2 < \infty$. It was proved that for a linear operator $R := \chi^{-1}(\mathbf{r}) + A$ in the left-hand side of Eq. (3) there exists a bounded inverse operator R^{-1}, with its norm $\left\|R^{-1}\right\| < \mu_2 - 1$. It then follows that in this case, all the three conditions of the correctness of the direct problem for Eq. (3) are satisfied, with the stability of the solution being understood with respect to the external field $\mathbf{H}^0(\mathbf{r})$.

In [7], Eq. (3) was studied for the case in which the permeability μ is a bounded piecewise-linear function of coordinates, a case that corresponds to the possible presence of foreign inclusions in the magnet. In addition, the magnet domain Ω can be both bounded and unbounded (for example, a half-space, an infinite cylinder, etc.). In this paper, Eq. (3) is reduced to the operator equation $T\mathbf{M} = \mathbf{M}$ with the operator $T\mathbf{M} := (\mu - 1)/(\mu + 1)\,(2\mathbf{H}^0 + \mathbf{M} - 2A\mathbf{M})$, where the operator A is as defined in Eq. (4). It has been proved that the operator T is contracting in $\mathbf{L}_2(\Omega)$, and, therefore, Eq. (3) has the unique solution in this space; this satisfies the first two conditions of the well-posedness of the relevant direct problem. A similar conclusion was drawn in [5, p. 160] for the case of anisotropic magnets in which the magnetic permeability is a tensor.

A great deal of attention is being paid to the issue of the well-posedness of the direct problem for Eqs. (3), (4) in [8–10] and in the references therein. In [8], Eq. (3) is considered in the more general form (including anisotropic materials)

$$\mathbf{g}(\mathbf{M}(\mathbf{r}),\mathbf{r}) + (A\mathbf{M})(\mathbf{r}) = \mathbf{H}^0(\mathbf{r}), \quad \mathbf{r} \in \Omega. \tag{5}$$

However, the domain Ω filled with the magnet is taken to be bounded and singly connected (i.e., without "holes"). Equation (3) can be obtained from Eq. (5) at $\mathbf{g}(\mathbf{M}(\mathbf{r}),\mathbf{r}) = \chi^{-1}(\mathbf{M}(\mathbf{r}),\mathbf{r})\,\mathbf{M}(\mathbf{r})$. Under rather weak and natural restrictions on the function $\mathbf{g}(\mathbf{M}(\mathbf{r}),\mathbf{r})$, the so-called strong monotonicity of the operator in the left-hand side of Eq. (5) was proved in [8]. This ensures the presence of a continuous inverse operator, thereby guaranteeing the existence and uniqueness of the solution of Eq. (5) and its continuous dependence on the external field $\mathbf{H}^0(\mathbf{r})$, thereby satisfying the third condition of the correctness, viz., the stability of the solution with respect to external field. In his subsequent works, Friedman [9, 10] derived estimates for the perturbation of the solution of Eq. (5) depending on the perturbation in the external field $\mathbf{H}^0(\mathbf{r})$ for the case of an isotropic magnet. He also justified the convergence of approximate general methods of the Ritz and Galerkin type for numerical solution of this equation.

The works [11–14] also deal with the questions of the well-posedness of problems of magnetostatics. However, they discuss the well-posedness (with respect to the solution uniqueness) of inverse problems of magnetostatics, whereas the present, first part of this work is devoted to direct problems. The questions of the well-posedness of inverse problems of magnetostatics are considered in the second part of this particle, which is being finalized for publication.

4. ESTIMATING THE ERROR OF CALCULATING THE RESULTANT FIELD STRENGTH

When elucidating the well-posedness of the direct problem of magnetostatics, it is important not only to establish the fact of the continuous dependence of the problem solution $\mathbf{H}(\mathbf{r})$ on the change (error) of the external field $\mathbf{H}^0(\mathbf{r})$ but also the continuous dependence of the solution on changes in other initial data such as the magnetic permeability $\mu(r)$ and the shape of the magnet domain Ω. Moreover, for practical applications it is not only establishing the stability of the problem solution with respect to one or other initial data that is of importance (the third condition of well-posedness) but also deriving concrete estimates that demonstrate the maximum possible error of the solution depending on the errors in setting these data. In this respect, of importance are the results of the theoretical work [15] (duplicated in the monograph [5, pp. 167–174]) devoted to studying changes in the resultant field strength depending on changes in either the external field strength or the magnet permeability or the magnet shape. Without going into any comprehensive explanations, let us provide herein estimates (which can be obtained from the results of this work by easy manipulations) for the error in the resultant field $\mathbf{H}(\mathbf{r})$ [obtained from the solution of Eq. (3) with subsequent application of relations (2) and (1)] depending on the errors of the external field $\mathbf{H}^0(\mathbf{r})$ and magnetic permeability μ that are set in these equations as well as on the degree of change in the magnet shape Ω due to, for example, the appearance of an internal or surface flaw. The above error estimates are of especial importance with respect to determining the resultant field $\mathbf{H}^i(\mathbf{r})$ inside the studied magnet from Eqs. (1) or (3). The point is that instrument-assisted determination of this field inside the magnet is impossible in practical terms, and, consequently, it can only be determined from the above-indicated equations. This is why it is important to be able to estimate the resultant error in solving these equations versus errors in the initial data used therein. The resultant field $\mathbf{H}^e(\mathbf{r})$ outside the magnet can, in principle, be measured with a device. However, one needs to realize that the error in the thus-obtained value of the resultant field is comprised of two components; these are the measurement error and the error due to the fact that, in reality, we measure the true resultant field that corresponds to the true initial data rather than a field that corresponds to the declared initial data that contain certain errors. The instrumental error is usually known, whereas the second contribution to the error can only be derived from the estimates given below.

The above error estimates will be, from the very beginning, obtained in the mean-square norm (i.e., in the norm of the space $\mathbf{L}_2$ in the relevant domain). However, to interpret and understand the meaning of the derived estimates, it is more convenient and familiar for practicians to formulate those in the language of the average value of the error in the relevant domain rather than in the form of the above norm. For this reason, the final estimates will be given precisely in this language. Let us recollect that in mathematics (and physics), the average value of a certain function $f(\mathbf{r})$ defined in some three-dimensional domain Ω of volume V is the quantity $\frac{1}{V}\int_{\Omega} f(\mathbf{r})\,d\mathbf{r}$. The mean-square norm of an arbitrary vector-function $\mathbf{a}(\mathbf{r})$ in the space $\mathbf{L}_2(\Omega)$ can be represented in the form

$$\|\mathbf{a}\| := \sqrt{\int_{\Omega} |\mathbf{a}(\mathbf{r})|^2\,d\mathbf{r}} = \sqrt{V}\cdot\sqrt{\frac{\int_{\Omega} |\mathbf{a}(\mathbf{r})|^2\,d\mathbf{r}}{V}}. \tag{6}$$

The second multiplier in the right-hand side of this formula is the square root of the average value of the square of the value (that is, the modulus) of the vector-function $\mathbf{a}(\mathbf{r})$ over the domain Ω; therefore, it can be naturally interpreted as the average of the value (modulus) of the vector-function $\mathbf{a}(\mathbf{r})$ itself over the domain Ω (the probability theory similarly introduces the root-mean-square of a random quantity as the square root of its variance, which itself is defined as the average value of the squared deviation of the random quantity from its average value). Thus, the determined average value of the modulus of the vector $\mathbf{a}(\mathbf{r})$ over the domain Ω will be denoted as $|\mathbf{a}|_{\mathrm{av},\Omega}$ and we have

$$|\mathbf{a}|_{\mathrm{av},\Omega} := \sqrt{\frac{\int_{\Omega} |\mathbf{a}(\mathbf{r})|^2\,d\mathbf{r}}{V}}.$$

Then we can use equality (6) to derive the following expression for the average value of the modulus of the vector $\mathbf{a}(\mathbf{r})$ over the domain Ω in terms of the mean-square norm of the vector $\mathbf{a}(\mathbf{r})$ itself in the space $\mathbf{L}_2(\Omega)$ and the volume V of the domain Ω:

$$|\mathbf{a}|_{\mathrm{av},\Omega} := \frac{\|\mathbf{a}\|}{\sqrt{V}}. \tag{7}$$

It should be noted that if the vector $\mathbf{a}(\mathbf{r})$ stays constant within the domain Ω (and, hence, the modulus of this vector is also constant), then calculation of the average value of the modulus of this vector by the formula in Eq. (7) yields, as one should expect, this constant value of the modulus itself; in this case, this indeed coincides with its average value.

Let $\mu_{\min}$ and $\mu_{\max}$ be the greatest lower and least upper bounds of the coordinate-dependent magnetic permeability $\mu(\mathbf{r})$ of a magnet that occupies a domain Ω with a volume V, that is, $1 < \mu_{\min} \le \mu(\mathbf{r}) \le \mu_{\max}$. Let $\Delta\mathbf{H}^0(\mathbf{r})$ stand for the value of an error to within which the external field strength $\mathbf{H}^0(\mathbf{r})$ is set in Eq. (1), i.e., $\Delta\mathbf{H}^0(\mathbf{r})$ is essentially the pointwise difference between the approximate value of the external field strength, which appears in the right-hand side of Eqs. (1) or (3) when they are solved, and its true value. Then, the following upper and lower estimates hold true for the error $\Delta\mathbf{H}^i(\mathbf{r})$ induced in the value of the resultant field strength inside the magnet when solving the direct problem:

$$\frac{\mu_{\min}-1}{\mu_{\min}(\mu_{\max}-1)}\|\Delta\mathbf{H}^0\| \le \|\Delta\mathbf{H}^i\| \le \frac{\mu_{\max}-1}{\mu_{\min}-1}\|\Delta\mathbf{H}^0\|. \tag{8}$$

Where the norms of the vector-functions are taken in the space $\mathbf{L}_2(\Omega)$. Taking Eq. (7) into account, let us write down the obtained estimates (8) in terms of the average values of the modulus of errors in the form

$$\frac{\mu_{\min}-1}{\mu_{\min}(\mu_{\max}-1)}|\Delta\mathbf{H}^0|_{\mathrm{av},\,\Omega} \le |\Delta\mathbf{H}^i|_{\mathrm{av},\,\Omega} \le \frac{\mu_{\max}-1}{\mu_{\min}-1}|\Delta\mathbf{H}^0|_{\mathrm{av},\,\Omega}. \tag{9}$$

If the magnetic permeability μ stays constant inside the magnet, the estimates in Eq. (9) are significantly simplified and become

$$\frac{1}{\mu}\left|\Delta\mathbf{H}^0\right|_{\text{av},\,\Omega} \le \left|\Delta\mathbf{H}^i\right|_{\text{av},\,\Omega} \le \left|\Delta\mathbf{H}^0\right|_{\text{av},\,\Omega}. \tag{10}$$

This estimate shows that in this case, the average error in determining the resultant field inside the magnet from the fundamental equation of magnetostatics does not exceed the average error in setting the external field. The estimates in Eqs. (9) and (10) are helpful not only as error estimates but can be used when allowing for the effect of any alterations to the device that generates the external field on the resultant field inside the magnet (for example, when changing the number of turns in the coil that generates the external field).

Let us turn to estimating the error $\Delta\mathbf{H}^e(\mathbf{r})$ in the resultant field outside the magnet. Let some domain $\tilde{\Omega}$ contain the magnet domain Ω) (i.e., $\Omega \subset \tilde{\Omega}$), while we are only interested in the behavior of the resultant field $\mathbf{H}^e(\mathbf{r})$ within the domain $\Omega_2 := \tilde{\Omega}/\Omega$ that has a volume V_2, which is usually small. Then the following error estimate $\Delta\mathbf{H}^e(\mathbf{r})$ versus the error $\Delta\mathbf{H}^0(\mathbf{r})$ in setting the external field is valid:

$$\left\|\Delta\mathbf{H}^e\right\|_2 \le (\mu_{\max}-1)\left\|\Delta\mathbf{H}^0\right\| + \left\|\Delta\mathbf{H}^0\right\|_2,$$

where $\|\ldots\|_2$ denotes the norm in the space $\mathbf{L}_2(\Omega_2)$. Taking Eq. (7) into account, this inequality entails the following inequalities for the average error values:

$$\left|\Delta\mathbf{H}^e\right|_{\text{av},\,\Omega} \le (\mu_{\max}-1)\sqrt{\frac{V}{V_2}}\left|\Delta\mathbf{H}^0\right|_{\text{av},\,\Omega} + \left|\Delta\mathbf{H}^0\right|_{\text{av},\,\Omega_2}. \tag{11}$$

The above estimates automatically imply continuous dependence of the solution to the direct problem of magnetostatics on the external field.

The formulae in Eqs. (9) and (11) become especially transparent in the case of a constant external field. Let a magnet be placed in a constant field $\mathbf{H}^0 = \text{const}$ that can be measured (or calculated) only with a certain accuracy $\Delta\mathbf{H}^0$. For this reason, the fields inside and outside the magnet were calculated from Eqs. (1), (2) not with the precise value of the field $\mathbf{H}^0$ but also with a constant field $\mathbf{H}^0 + \Delta\mathbf{H}^0$that was obtained as the result of the above measurements (or calculations). Then, according to (9) and (11), we have the following visual estimates of the errors arising in the resultant fields inside and outside the magnet in terms of the value of the error $|\Delta\mathbf{H}^0|$ in the external field $\mathbf{H}^0$ that also remains constant:

$$\left|\Delta\mathbf{H}^i\right|_{\text{av},\,\Omega} \le \frac{\mu_{\max}-1}{\mu_{\min}-1}\left|\Delta\mathbf{H}^0\right|, \quad \left|\Delta\mathbf{H}^e\right|_{\text{av},\,\Omega_2} \le \left[(\mu_{\max}-1)\sqrt{\frac{V}{V_2}}+1\right]\left|\Delta\mathbf{H}^0\right|.$$

If the magnetic permeability μ remains constant too, the above estimates transform into the following simple relationships:

$$\left|\Delta\mathbf{H}^i\right|_{\text{av},\,\Omega} \le \left|\Delta\mathbf{H}^0\right|, \quad \left|\Delta\mathbf{H}^e\right|_{\text{av},\,\Omega_2} \le \left[(\mu-1)\sqrt{\frac{V}{V_2}}+1\right]\left|\Delta\mathbf{H}^0\right|.$$

Now let $\Delta\mu(\mathbf{r})$ denote the value of the error with which the true magnetic permeability $\mu(\mathbf{r})$ is set in Eqs. (1), (2), with the external field $\mathbf{H}^0(\mathbf{r})$ considered to be set precisely. Suppose that the magnetic permeability $\mu = \mu(\mathbf{r}) + \Delta\mu(\mathbf{r})$ set (with the error) in Eq. (2) stays within certain limits $1 < \mu_{\min} \le \mu(\mathbf{r}) + \Delta\mu(\mathbf{r}) \le \mu_{\max}$ as before. The following estimates then hold for the error $\Delta\mathbf{H}^i(\mathbf{r})$ induced in the value of the resultant field strength inside the magnet when solving the direct problem from the fundamental equation of magnetostatics:

$$\begin{aligned}\left\|\Delta\mathbf{H}^i\right\| &\le \left(1+\frac{\mu_{\max}-1}{\mu_{\min}-1}\right)\left\|\frac{\Delta\mu}{\mu+\Delta\mu-1}\mathbf{H}^i\right\| \le \left(1+\frac{\mu_{\max}-1}{\mu_{\min}-1}\right)\max_{\Omega}\left(\frac{|\Delta\mu(\mathbf{r})|}{\mu(\mathbf{r})+\Delta\mu(\mathbf{r})-1}\right)\cdot\left\|\mathbf{H}^i\right\| \\ &\le \frac{1}{\mu_{\min}-1}\left(1+\frac{\mu_{\max}-1}{\mu_{\min}-1}\right)\max_{\Omega}|\Delta\mu(\mathbf{r})|\cdot\left\|\mathbf{H}^i\right\|.\end{aligned} \tag{12}$$

In order to rewrite relations (12) in terms of average errors, let us introduce, in a natural manner, a *relative* average error of determining a certain vector-function in the domain Ω as the ratio of the average value of the modulus of the error $|\Delta \mathbf{a}|_{\mathrm{av},\Omega}$ in determining this quantity and the mean value of the modulus of the vector-function $|\mathbf{a}|_{\mathrm{av},\Omega}$, viz.

$$|\Delta \mathbf{a}|_{\mathrm{av.rel},\,\Omega} := \frac{|\Delta \mathbf{a}|_{\mathrm{av},\,\Omega}}{|\mathbf{a}|_{\mathrm{av},\,\Omega}}. \tag{13}$$

Taking this into account, inequality (12) can be rewritten as

$$\left|\Delta \mathbf{H}^i\right|_{\mathrm{av.rel},\,\Omega} \le \frac{1}{\mu_{\min}-1}\left(1+\frac{\mu_{\max}-1}{\mu_{\min}-1}\right)\max_{\Omega}|\Delta\mu(\mathbf{r})|. \tag{14}$$

Relation (14) demonstrates an estimate from above for the average relative error in determining the resultant field inside the magnet from the fundamental equation of magnetostatics (1) or (3) in terms of the maximum pointwise estimate of the error in setting the magnetic permeability in this equation.

The following estimate is valid for the induced $\Delta \mathbf{H}^e(\mathbf{r})$ error in the value of the resultant field within the measurement domain Ω_2 outside the magnet:

$$\begin{aligned}\left\|\Delta \mathbf{H}^e\right\|_2 &\le (\mu_{\max}-1)\left\|\frac{\Delta\mu}{\mu+\Delta\mu-1}\mathbf{H}^i\right\| \le (\mu_{\max}-1)\max_{\Omega}\left(\frac{|\Delta\mu(\mathbf{r})|}{\mu(\mathbf{r})+\Delta\mu(\mathbf{r})-1}\right)\cdot\left\|\mathbf{H}^i\right\| \\ &\le \frac{\mu_{\max}-1}{\mu_{\min}-1}\max_{\Omega}|\Delta\mu(\mathbf{r})|\cdot\left\|\mathbf{H}^i\right\|.\end{aligned} \tag{15}$$

When expressed in terms of average errors, this estimate can be rewritten in the form

$$\left|\Delta \mathbf{H}^e\right|_{\mathrm{av},\,\Omega_2} \le \frac{\mu_{\max}-1}{\mu_{\min}-1}\sqrt{\frac{V}{V_2}}\left|\mathbf{H}^i\right|_{\mathrm{av},\,\Omega}\cdot\max_{\Omega}|\Delta\mu(\mathbf{r})|. \tag{16}$$

It then immediately follows that the solution of the fundamental equation of magnetostatics continuously depends on the magnetic permeability with respect to both the internal resultant field and the external field in the measurement domain Ω_2.

Based on these inequalities, it is easy to derive an estimate for the change in the solution of the fundamental equation of magnetostatics $\mathbf{H}^i(\mathbf{r})$ [and then $\mathbf{H}^e(\mathbf{r})$ as well] induced by the appearance of a flaw in the magnet. Let a flaw with a magnetic permeability $\mu_d(\mathbf{r})$ emerge in a domain ω_d with a volume V_d that belongs to the magnet domain Ω that has the magnetic permeability $\mu(\mathbf{r})$. The initial magnetic permeability $\mu(\mathbf{r})$ inside Ω has thus changed into a magnetic permeability of the form

$$\mu = \begin{cases}\mu(\mathbf{r}), & \mathbf{r}\in\Omega/\omega_d \\ \mu_d(\mathbf{r}), & \mathbf{r}\in\omega_d\end{cases}. \tag{17}$$

Such a transformation can be equivalently described by the emergence of an error $\Delta\mu(\mathbf{r})$ in setting the magnetic permeability of the form

$$\Delta\mu(\mathbf{r}) = \begin{cases}0, & \mathbf{r}\in\Omega/\omega_d \\ \mu_d(\mathbf{r})-\mu(\mathbf{r}), & \mathbf{r}\in\omega_d\end{cases}. \tag{18}$$

Now, allowing for Eq. (18), we can derive the following estimates for relevant changes $\Delta\mathbf{H}^i$ and $\Delta\mathbf{H}^e$ in the initial resultant field $\mathbf{H}^i$ inside Ω and $\mathbf{H}^e$ in Ω_2 induced by the emergence of the above-described flaw:

$$\begin{aligned}\left\|\Delta \mathbf{H}^i\right\| &\le \left(1+\frac{\mu_{\max}-1}{\mu_{\min}-1}\right)\left\|\frac{\mu_d-\mu}{\mu_d-1}\mathbf{H}^i\right\|_d \le \left(1+\frac{\mu_{\max}-1}{\mu_{\min}-1}\right)\max_{\omega_d}\left(\frac{|\mu-\mu_d|}{\mu_d-1}\right)\cdot\left\|\mathbf{H}^i\right\|_d \\ &\le \frac{1}{\mu_{d\,\min}-1}\left(1+\frac{\mu_{\max}-1}{\mu_{\min}-1}\right)\max_{\omega_d}|\mu-\mu_d|\cdot\left\|\mathbf{H}^i\right\|_d;\end{aligned} \tag{19}$$

$$\begin{aligned}\left\|\Delta \mathbf{H}^e\right\|_2 &\le (\mu_{\max}-1)\left\|\frac{\mu_d-\mu}{\mu_d-1}\mathbf{H}^i\right\|_d \le (\mu_{\max}-1)\max_{\omega_d}\left(\frac{|\mu-\mu_d|}{\mu_d-1}\right)\cdot\left\|\mathbf{H}^i\right\|_d \\ &\le \frac{\mu_{\max}-1}{\mu_{d\,\min}-1}\max_{\omega_d}|\mu-\mu_d|\cdot\left\|\mathbf{H}^i\right\|_d,\end{aligned} \tag{20}$$

where $\mu_{\max}$ and $\mu_{\min}$ are the common upper and lower bounds for the "flawless" magnetic permeability $\mu(\mathbf{r})$ and the "flawed" magnetic permeability (17) in the magnet domain Ω, while $\mu_{d\,\min}$ is the lower bound of the magnetic permeability of the flaw within its domain ω_d. When expressed in terms of average error, these inequalities become

$$\left|\Delta\mathbf{H}^i\right|_{\text{av},\,\Omega} \le \frac{1}{\mu_{d\,\min}-1}\left(1+\frac{\mu_{\max}-1}{\mu_{\min}-1}\right)\max_{\omega_d}\left|\mu-\mu_d\right|\sqrt{\frac{V_d}{V}}\left|\mathbf{H}^i\right|_{\text{av},\,\omega_d}; \quad (21)$$

$$\left|\Delta\mathbf{H}^e\right|_{\text{av},\,\Omega_2} \le \frac{\mu_{\max}-1}{\mu_{\min}-1}\max_{\omega_d}\left|\mu-\mu_d\right|\sqrt{\frac{V_d}{V_2}}\left|\mathbf{H}^i\right|_{\text{av},\,\omega_d}. \quad (22)$$

Equations (21) and (22) demonstrate the type of dependence of the change in the resultant fields $\mathbf{H}^i(\mathbf{r})$ and $\mathbf{H}^e(\mathbf{r})$ inside and outside the magnet induced by the appearance of a flaw versus both the degree of change in the magnetic permeability inside this flaw (due to the presence of the multiplier $\max|\mu-\mu_d|$) and the volume V_d of the flawed domain.

If the magnetic permeabilities μ and μ_d remain constant, the estimates for relevant changes $\Delta\mathbf{H}^i$ and $\Delta\mathbf{H}^e$ in the initial resultant fields $\mathbf{H}^i$ in Ω and $\mathbf{H}^e$ in Ω_2 become much simpler. In this case, one can easily obtain from Eqs. (19) and (20) that

$$\left\|\Delta\mathbf{H}^i\right\| \le \frac{1}{\mu_d-1}\left(1+\frac{\mu_{\max}-1}{\mu_{\min}-1}\right)\left|\mu-\mu_d\right|\left\|\mathbf{H}^i\right\|_d; \quad \left\|\Delta\mathbf{H}^e\right\|_2 \le \frac{\mu_{\max}-1}{\mu_d-1}\left|\mu-\mu_d\right|\left\|\mathbf{H}^i\right\|_d.$$

where $\mu_{\max}\{\mu, \mu_d\}$ and $\mu_{\min}\{\mu, \mu_d\}$. When expressed in terms of average errors, these estimates have the form

$$\left|\Delta\mathbf{H}^i\right|_{\text{av},\Omega} \le \frac{1}{\mu_d-1}\left(1+\frac{\mu_{\max}-1}{\mu_{\min}-1}\right)\left|\mu-\mu_d\right|\sqrt{\frac{V_d}{V}}\left|\mathbf{H}^i\right|_{\text{av},\omega_d};$$

$$\left|\Delta\mathbf{H}^e\right|_{\text{av},\Omega_2} \le \frac{\mu_{\max}-1}{\mu_d-1}\left|\mu-\mu_d\right|\sqrt{\frac{V_d}{V_2}}\left|\mathbf{H}^i\right|_{\text{av},\omega_d}.$$

Let us now consider perturbations in the resultant fields $\mathbf{H}^i$ and $\mathbf{H}^e$ caused by the appearance of a cavity in the magnet (this can be a surface or internal flaw). Let us assume that initially we had a magnet that occupies a domain Ω and has a magnetic permeability $\mu(\mathbf{r})$. Let us consider perturbations in the above fields caused by a cavity located inside a domain ω_d with a volume V_d within the magnet domain Ω with the volume V. Let the magnetic permeability inside the cavity be $\mu_d = 1$. Let $\Omega_c = \Omega/\omega_d$ be the magnet domain outside the cavity. Then the perturbation $\Delta\mathbf{H}^i$ in the resultant field $\mathbf{H}^i$ inside this domain satisfies the inequality

$$\left\|\Delta\mathbf{H}^i\right\|_c \le \frac{\mu_{\max}-1}{\mu_{\min}-1}\left\|\Delta\mathbf{H}^i\right\|_d,$$

where $\|\ldots\|_c$ is the norm in the space $\mathbf{L}_2(\Omega_2)$, while $\mu_{\min}$ and $\mu_{\max}$ are the greatest lower and least upper bounds of the initial permeability $\mu(\mathbf{r})$ of the magnet in the domain Ω. When expressed in domain-average errors, this inequality can be rewritten in the form

$$\left|\Delta\mathbf{H}^i\right|_{\text{av},\Omega_c} \le \frac{\mu_{\max}-1}{\mu_{\min}-1}\sqrt{\frac{V_d}{V-V_d}}\left|\mathbf{H}^i\right|_{\text{av},\omega_d}.$$

While when expressed in terms of average relative errors, it becomes

$$\left|\Delta\mathbf{H}^i\right|_{\text{av,rel},\Omega_c} \le \frac{\mu_{\max}-1}{\mu_{\min}-1}\sqrt{\frac{V_d}{V-V_d}}\frac{\left|\mathbf{H}^i\right|_{\text{av},\omega_d}}{\left|\mathbf{H}^i\right|_{\text{av},\Omega_c}}.$$

These inequalities demonstrate dependence of the errors on the volume of the emerged cavity. The following estimate can be produced for the field change outside the magnet (in the measurement domain Ω_2):

$$\left\|\Delta\mathbf{H}^e\right\|_2 \le (\mu_{\max}-1)\left(1+\frac{\mu_{\max}-1}{\mu_{\min}-1}\right)\left\|\mathbf{H}^i\right\|_d,$$

or

$$\left|\Delta\mathbf{H}^e\right|_{\mathrm{av},\Omega_2} \le (\mu_{\max}-1)\left(1+\frac{\mu_{\max}-1}{\mu_{\min}-1}\right)\sqrt{\frac{V_d}{V_2}}\left|\mathbf{H}^i\right|_{\mathrm{av},\omega_d}.$$

This estimate also indicates the type of dependence of the above error on the volume of the cavity ω_d.

CONCLUSIONS

In conclusion, let us briefly summarize the main results that have been obtained in this work.

1. The stability of the solution to the direct problem of magnetostatics [finding the strength $\mathbf{H}(\mathbf{r})$ of the resultant magnetic field inside and outside a magnet from Eqs. (1), (2)] has been proved not only with respect to the measurement error of the set external field $\mathbf{H}^0(\mathbf{r})$ (this has already been done in other works cited herein) but also with respect to errors in setting other initial data, including errors in setting the magnet shape due to possible (geometrical) imperfection of this shape or the appearance of small cavities that distort the magnet shape and errors in setting the value of magnetic permeability that are induce, for example, by the appearance of small inclusion-type flaws.

2. Formulae are provided for concrete estimates of the above stability. These formulae make it possible to find out in principle what the *maximum error* can be within the frame of the used mathematical model of the resultant magnetic-field strength depending on the values of various errors in the initial data that have been enumerated in the item above. In this work, the upper estimates are also given for the errors in calculating the resultant magnetic-field strength depending on the volume of possible inclusions or cavities that distort the magnet shape. All these estimates are provided in terms of both absolute and relative errors, a representation that is more visual, common, and convenient in practical usage.

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REFERENCES

1. Ivanov, V.K., Vasin, V.V., and Tanana, V.P., *Teoriya lineinykh nekorrektnykh zadach i ee prilozheniya* (Theory of Linear Ill-Posed Problems and Its Applciations), Moscow: Nauka, 1978.
2. Khizhnyak, N.A., *Integral'nye uravneniya makroskopicheskoi elektrodinamiki* (Integral Equations of Macroscopic Electrodynamics), Kiev: Naukova dumka, 1986.
3. Dyakin, V.V., Direct and inverse problem of magnetostatics, *Defektoskopiya*, 1996, no. 3, pp. 3–6.
4. Dyakin, V.V. and Raevskii, V.Ya., Direct and inverse problem of classical electrodynamics, *Defektoskopiya*, 1996, no. 10, pp. 31–39.
5. Dyakin, V.V., *Matematicheskie osnovy klassicheskoi magnitostatiki* (Mathematical Foundations of Classical Magnetostatics), Yekaterinburg: Ural Branch, Russ. Acad. Sci., 2016.
6. Vonsovskii, S.V., *Magnetizm (Magnetism)*, Moscow: Nauka, 1971.
7. Dyakin, V.V. and Umergalina, O.V., Calculation of the field of a flaw in three-dimensional half-space, *Russ. J. Nondestr. Test.*, 2003, vol. 39, no. 4, pp. 297–309.
8. Friedman, M.J., Mathematical study of the nonlinear singular integral magnetic field equation. 1, *SIAM J. Appl. Math.*, 1980, vol. 39, no. 1, pp. 14–20.
9. Friedman, M.J., Mathematical study of the nonlinear singular integral magnetic field equation. 2, *SIAM J. Numer. Anal*, 1981, vol. 18, no. 4, pp. 644–653.
10. Friedman, M.J., Mathematical study of the nonlinear singular integral magnetic field equation. 3, *SIAM J. Math. Anal.*, 1981, vol. 12, no. 4, pp. 536–540.
11. Shur, M.L., Novoslugina, A.P., and Smorodinskii, Ya.G., On the inverse problem of magnetostatics, *Russ. J. Nondestr. Test.*, 2013, vol. 49, no. 8, pp. 465–473.
12. Pechenkov, A.N. and Shcherbinin, V.E., On the solution of the inverse problem of magnetostatic tomography, *Russ. J. Nondestr. Test.*, 2009, vol. 45, no. 3, pp. 176–190.
13. Pechenkov, A.N. and Shcherbinin, V.E., On the nonuniqueness of the solution of the inverse problem of magnetostatic nondestructive evaluation, *Kontrol'. Diagn.*, 2006, no. 9, pp. 59–60.
14. Pechenkov, A.N., Effect of the shape of a body on the uniqueness of the solution to the inverse problem of magnetostatic flaw detection, *Russ. J. Nondestr. Test.*, 2006, vol. 42, no. 10, pp. 653–655.
15. Dyakin, V.V., Raevskii, V.Ya., and Kudryashova, O.V., Estimates of solutions to certain problems of magnetostatics, *Russ. J. Nondestr. Test.*, 2015, vol. 51, no. 10, pp. 601–609.

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