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A Priori Estimates and Fredholm Criteria for a Class of Regular Hypoelliptic Operators

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Abstract—We study the Fredholm property of regular hypoelliptic operators with special variable coefficients. In this paper, necessary and sufficient conditions are obtained for a priori estimates for differential operators acting in multianisotropic Sobolev spaces. Fredholm criteria are obtained for a wide class of regular hypoelliptic operators in multianisotropic weighted spaces in $\mathbb{R}^n$.

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1. INTRODUCTION, BASIC NOTIONS AND DEFINITIONS

The class of regular hypoelliptic operators is a special subclass of Hyormander's hypoelliptic operators (see [11]) and they are a natural generalization of elliptic and quasielliptic operators. These operators were introduced in the late 1960-70s and studied by many authors: Nikolsky [17], Mikhailov [16], Friberg [10], Volevich and Gindikin [23], and others. The analysis of regular hypoelliptic operators has certain difficulties as corresponding characteristic polynomials are neither homogeneous as in the elliptic case nor generalized homogeneous as in the quasielliptic case.

The Fredholm properties have been studied for some of the classes of hypoelliptic operators in different functional spaces but most of the result are for elliptic and quasielliptic operators.

For elliptic operators the Fredholm property is studied on various scales of weighted spaces in $\mathbb{R}^n$ in the papers by Bagirov [1], Lockhartand and McOwen [14, 15], Schrohe [19], and many others.

Bagirov [2], Karapetyan and Darbinyan [12] studied the Fredholm property of special classes of quiasielliptic operators in the weighted spaces in $\mathbb{R}^n$. Isomorphism properties are obtained in Demidenko's papers (see [6–8]) for quasielliptic operators with constant coefficients on the special scales of weighted spaces. Darbinyan and Tumanyan [4, 20] studied a priori estimates and Fredholm criteria for quasielliptic operators with special variable coefficients in anisotropic weighted Sobolev spaces. Index stability of quasielliptic operators on the special scales of weighted spaces is studied in [5, 21].

Rodino, Boggiatto, and Buzano (see [3]) studied the Fredholm properties and the spectrum of special classes of pseudodifferential operators acting in multianisotropic spaces with special polynomial weights. In Tumanyan's paper [22], Fredholm criteria are obtained for the special subclass of regular hypoelliptic operators.

In this paper, we obtain necessary and sufficient conditions for a priori estimates for differential operators acting in multianisotropic Sobolev spaces (Theorems 2.2 and 2.3). A regularizer is constructed for regular hypoelliptic operators with special variable coefficients and Fredholm criteria are established for the considered class of operators on the scale of weighted multianisotropic spaces in $\mathbb{R}^n$ (Theorems 2.5 and 2.6). The considered class of operators is more general than in the previous works (see, for example, [4, 20, 22]).

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Definition 1.1. A bounded linear operator A, acting from a Banach space X to a Banach space Y, is called an $n-$normal operator if the following conditions hold:

1. The image of operator A is closed $\left(Im(A) = \overline{Im(A)}\right)$;
2. The kernel of operator A is finite dimensional ($\dim Ker(A) < \infty$).

An operator A is called a Fredholm operator if conditions 1 and 2 hold and

3. The cokernel of the operator A is finite-dimensional
($\dim coker(A) = \dim Y/Im(A) < \infty$).

Definition 1.2. For a bounded linear operator A, acting from a Banach space X to a Banach space Y, bounded linear operators $R_1 : Y \to X$ and $R_2 : Y \to X$ are called respectively left and right regularizers if the following holds: $R_1A = I_X + T_1, AR_2 = I_Y + T_2$, where I_X, I_Y- are the identity operators, $T_1 : X \to X$ and $T_2 : Y \to Y$ are compact operators. A bounded linear operator $R : Y \to X$ is called a regularizer for operator A if it is a left and right regularizer.

Let $n \in \mathbb{N}$ and $\mathbb{R}^n$ be the Euclidean n-dimensional space, $\mathbb{Z}_+^n, \mathbb{N}^n$ be the sets of n-dimensional multi-indices and multi-indices with natural components respectively. Let $\mathcal{N} \subset \mathbb{Z}_+^n$ be a finite set of multi-indices, $\mathcal{R} = \mathcal{R}(\mathcal{N})$ be a minimum convex polyhedron containing all the points $\mathcal{N}$.

Definition 1.3. A polyhedron $\mathcal{R}$ is called completely regular if the following holds: a) $\mathcal{R}$ is a complete polyhedron: $\mathcal{R}$ has a vertex at the origin and further vertices on each coordinate axes in $\mathbb{R}^n$; b) all components of the outer normals of $(n-1)-$dimensional non-coordinate faces of $\mathcal{R}$ are positive.

Let $\mathcal{R}$ be a completely regular polyhedron. Denote by $\mathcal{R}_j^{n-1}$ $(j = 1, \ldots, I_{n-1})$ $(n-1)$-dimensional noncoordinate faces of $\mathcal{R}$ with the corresponding outer normals μ^j such that all multi-indices $\alpha \in \mathcal{R}_j^{n-1}$ satisfy

$$(\alpha : \mu^j) = \frac{\alpha_1}{\mu_1^j} + ... + \frac{\alpha_n}{\mu_n^j} = 1, \quad \partial\mathcal{R} = \bigcup_{j=1}^{I_{n-1}} \mathcal{R}_j^{n-1}.$$

For $k > 0$ denote by

$$k\mathcal{R} := \{k\alpha = (k\alpha_1, k\alpha_2 \ldots, k\alpha_n) : \alpha \in \mathcal{R}\}.$$

Consider the differential operator

$$P\left(x, \mathbb{D}\right) = \sum_{\alpha \in \mathcal{R}} a_\alpha(x) D^\alpha, \tag{1}$$

where $D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n},\ \ D_j = i^{-1}\frac{\partial}{\partial x_j},\ \ x = (x_1, ..., x_n) \in \mathbb{R}^n,\ \ a_\alpha(x) \in C(\mathbb{R}^n)$.

Denote by

$$P\left(x, \xi\right) = \sum_{\alpha \in \mathcal{R}} a_\alpha\left(x\right) \xi^\alpha. \tag{2}$$

For each $(n-1)$-dimensional noncoordinate face $\mathcal{R}_j^{n-1}$, $0j = 1, \ldots, I_{n-1}$, we denote

$$P_j\left(x, \mathbb{D}\right) = \sum_{\alpha \in \mathcal{R}_j^{n-1}} a_\alpha\left(x\right) D^\alpha,\ \ P_j\left(x, \xi\right) = \sum_{\alpha \in \mathcal{R}_j^{n-1}} a_\alpha\left(x\right) \xi^\alpha.$$

For $\xi \in \mathbb{R}^n$, we put

$$|\xi|_{\mathcal{R}} = \sum_{\alpha \in \mathcal{R}} |\xi^\alpha|,\ \ |\xi|_{\partial\mathcal{R}} = \sum_{\alpha \in \partial\mathcal{R}} |\xi^\alpha|.$$

Definition 1.4. A differential operator $P(x,\mathbb{D})$ is called regular at a point $x_0 \in \mathbb{R}^n$ if there exists a constant $\delta > 0$ such that
$$1 + |P(x_0,\xi)| \ge \delta |\xi|_{\mathfrak{R}},\ \forall \xi \in \mathbb{R}^n.$$

An operator $P(x,\mathbb{D})$ is called regular in $\mathbb{R}^n$ if $P(x,\mathbb{D})$ is regular at each point $x \in \mathbb{R}^n$. An operator $P(x,\mathbb{D})$ is called uniformly regular in $\mathbb{R}^n$ if there exists a constant $\delta > 0$ such that:
$$1 + |P(x,\xi)| \ge \delta |\xi|_{\mathfrak{R}},\ \forall \xi \in \mathbb{R}^n,\ \forall x \in \mathbb{R}^n.$$

Example 12. Examples of regular differential operators:

1. Let $m \in \mathbb{N}$ and $\mathfrak{R}$ be a Newton polyhedron for the set of points
$$(0,0,\dots,0),(m,0,\dots,0),\dots,(0,0,\dots,m).$$
In this case conditions from definition 1.4 coincide with ellipticity conditions.
2. Let $\nu \in \mathbb{N}^n$ and $\mathfrak{R}$ be a Newton polyhedron for the set of points
$$(0,0,\dots,0),(\nu_1,0,\dots,0),\dots,(0,0,\dots,\nu_n).$$
In this case, the conditions from definition 1.4 coincide with quasiellipticity conditions.
3. Let $n = 2$ and $\mathfrak{R}$ be a Newton polyhedron for the points $(0,0)$, $(8,0)$, $(0,8)$ and $(6,4)$. Then
$$P(x,\mathbb{D}) = a_1 D_1^8 + a_2 D_1^6 D_2^4 + a_3 D_2^8 + q(x)$$
is a regular differential operator in $\mathbb{R}^2$ with some $a_1, a_2, a_3 > 0$ and $q \in C(\mathbb{R}^2)$.
4. Let $n = 3$ and $\mathfrak{R}$ be a Newton polyhedron for the points $(0,0,0)$, $(8,0,0)$, $(0,8,0)$, $(6,4,0)$, $(6,0,6)$, $(0,6,6)$ and $(0,0,12)$. Then
$$P(x,\mathbb{D}) = D_1^8 + D_1^6 D_2^4 + D_2^8 + D_1^6 D_3^6 + D_2^6 D_3^6 + D_3^{12} + q(x)$$
is a regular differential operator in $\mathbb{R}^3$ with $q \in C(\mathbb{R}^3)$.

Let the sequence $\{a_i\}_{i=0}^{\infty} \subset \mathbb{R}_+$ be such that the series $\sum_{i=0}^{\infty} a_i$ diverges and the inequality $a_{i+1} < \gamma a_i$ holds, where $\gamma > 0$ and $i = 0,1,\dots$. Let's define a sequence $\{b_i\}_{i=0}^{\infty}$ the following way: $b_0 = 0$, $b_i = \sum\limits_{j=0}^{i} a_j, i = 1,2,\dots$. Consider a system of intervals
$$V_0 = \left\{ r : |r - b_0| < \frac{2\gamma+1}{\gamma+1} a_0 \right\},$$
$$V_i = \left\{ r : |r - b_i| < \frac{\gamma}{\gamma+1} a_i \right\},\ i = 1,2,\dots.$$

The system $\{V_i\}_{i=0}^{\infty}$ is an open covering for $\mathbb{R}_+$. Let us also consider a system of open sets U_j $(j = 1,\dots,l)$, which covers a unit sphere $|x| = 1$. Similarly to the paper by Bagirov [1] we construct a system $\{W_p\}_{p=1}^{\infty}$ and the corresponding partition of unity. We define $\{W_p\}_{p=1}^{\infty}$ as follows:
$$W_p = V_{\left[\frac{p-1}{l}\right]} \times U_{p-\left[\frac{p-1}{l}\right]l},\ \ p = 1,2,\dots.$$
It is obvious that the system of sets $\{W_p\}_{p=1}^{\infty}$ is an open covering for $\mathbb{R}^n$ and $\min\limits_{x\in\overline{W}_p} |x| \to \infty$ when $p \to \infty$.

Let $\theta^1, \theta^2 \in C^{\infty}(\mathbb{R})$ be nonnegative functions defined as follows: $\theta^1(t) = 1$ if $|t| \le \frac{\gamma}{2(\gamma+1)}$, $\theta^1(t) = 0$ if $|t| \ge \frac{\gamma}{\gamma+1}$, $\theta^2(t) = 1$ if $|t| \le \frac{\gamma}{\gamma+\frac{3}{4}}$, $\theta^2(t) = 0$ if $|t| \ge \frac{\gamma}{\gamma+\frac{1}{2}}$. Obviously, $\theta^2(t)\theta^1(t) = \theta^1(t)$ for all $t \in \mathbb{R}$. Consider the functions
$$\theta_0^1(t) = \theta^1\left(\frac{t}{\frac{2\gamma+1}{\gamma} a_0}\right), \theta_i^1(t) = \theta^1\left(\frac{t-b_i}{a_i}\right),\ i = 1,2,\dots;$$

$$\kappa_i^1(t)=\theta_i^1(t)\left(\sum_{j=0}^{\infty}\theta_j^1(t)\right)^{-1},\ i=0,1,2,\ldots;$$

$$\kappa_0^2(t)=\theta^2\left(\frac{t}{\frac{2\gamma+1}{\gamma}a_0}\right),\kappa_i^2(t)=\theta^2\left(\frac{t-b_i}{a_i}\right),\ i=1,2\ldots.$$

These functions have the following properties:

1. At each point $t\in\mathbb{R}_+$, the values of only one or two functions $\kappa_i^1(t)$ and $\kappa_i^2(t)$ are nonzero;
2. $supp\kappa_i^1\subset supp\kappa_i^2\subset\{t:|t-b_i|\le\frac{\gamma}{\gamma+\frac{1}{2}}a_i\}$;
3. $\kappa_i^2(t)\kappa_i^1(t)=\kappa_i^1(t)$ for all $t\in\mathbb{R}_+$;
4. For any $r\in\mathbb{N}$, there exists a constant $C_r>0$ such that
$$\left|D^r\kappa_i^1(t)\right|\le C_ra_i^{-r},\ \left|D^r\kappa_i^2(t)\right|\le C_ra_i^{-r},\ i=0,1,2,\ldots;$$
5. $\sum\limits_{i=0}^{\infty}\kappa_i^1(t)\equiv1.$

Consider the unity partition $\{v_j^1\}_{j=1}^l$, subordinated to $\{U_j\}_{j=1}^l$: $\sum\limits_{j=1}^{l}v_j^1(\omega)\equiv1$, where $\omega=\dfrac{x}{|x|}$. Let us also consider $\{v_j^2\}_{j=1}^l$ system of functions, which satisfies $suppv_j^2\subset U_j$ and $v_j^2(\omega)v_j^1(\omega)=v_j^1(\omega)$ for $j=1,\ldots,l$.

Let functions $\{\varphi_p\}_{p=1}^{\infty}$ and $\{\psi_p\}_{p=1}^{\infty}$ be such that

$$\varphi_p(x)=\kappa^1_{\left[\frac{p-1}{l}\right]}(|x|)v^1_{p-\left[\frac{p-1}{l}\right]l}\left(\frac{x}{|x|}\right),$$

$$\psi_p(x)=\kappa^2_{\left[\frac{p-1}{l}\right]}(|x|)v^2_{p-\left[\frac{p-1}{l}\right]l}\left(\frac{x}{|x|}\right),\ p=1,2,\ldots.$$

These systems of functions have the following properties:

1. $\mathrm{supp}\varphi_p\subset\mathrm{supp}\psi_p\subset W_p$;
2. $\psi_p(x)\varphi_p(x)=\varphi_p(x)$ for all $x\in\mathbb{R}^n$;
3. For any $\alpha\in\mathbb{Z}_+$ exists a constant $C_\alpha>0$ such that:
$$|D^\alpha\psi_p(x)|\le C_\alpha\left(a_{\left[\frac{p-1}{l}\right]}\right)^{-|\alpha|},\ \ |D^\alpha\varphi_p(x)|\le C_\alpha\left(a_{\left[\frac{p-1}{l}\right]}\right)^{-|\alpha|},\ \ \forall x\in\mathbb{R}^n,p=1,2,\ldots;$$
4. $\sum\limits_{p=1}^{\infty}\varphi_p(x)\equiv1.$

Denote

$$Q:=\left\{g\in C\left(\mathbb{R}^n\right):\exists c>0\text{ such that }g(x)\ge c>0,\ \forall x\in\mathbb{R}^n\right\}.$$

For $m\in\mathbb{Z}_+$ and a completely regular polyhedron $\mathcal{R}$, we denote by $Q^{m,\mathcal{R}}$ a set of weight functions $g\in Q$, which satisfy the following conditions:

1. $\frac{1}{g(x)}\rightrightarrows0$ as $|x|\to\infty$;
2. For $\beta\in m\mathcal{R},\ \beta\neq0$ $D^\beta g(x)\in C(\mathbb{R}^n)$ and there exists $C_\beta>0$ such that $\frac{|D^\beta g(x)|}{g(x)^{1+(\beta:\mu^j)}}\le C_\beta$ for all $x\in\mathbb{R}^n,j=1,\ldots,I_{n-1}$;

3. For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and $p_0 = p_0(\varepsilon) > 0$ such that, for all $p > p_0$, under the condition $\max\limits_{j=1,\dots,l} \operatorname{diam} U_j < \delta$ the following holds:

$$\max_{x,y\in\overline{W}_p} \frac{|g(x)-g(y)|}{g(y)} < \varepsilon, \qquad \max_{x,y\in\overline{W}_p} \frac{1}{g(x)^{\frac{1}{\mu_{\max}}} a_{\left[\frac{p-1}{l}\right]}} < \varepsilon,$$

where $\mu_{\max} = \max\limits_{1\le i\le I_{n-1}} \max\limits_{1\le s\le n} \{\mu_s^i\}$.

The considered class $Q^{m,\mathcal{R}}$ includes polynomial functions and special exponential functions such as $(1+|x|_{\mathcal{R}})^l, \exp(1+|x|_{\mathcal{R}})^r$, with $l, r > 0$.

For $k \in \mathbb{R}$ and a completely regular polyhedron $\mathcal{R}$, denote

$$H^{k,\mathcal{R}}(\mathbb{R}^n) := \left\{ u \in S' : \|u\|_{k,\mathcal{R}} := \left(\int |\widehat{u}(\xi)|^2 (1+|\xi|_{\partial\mathcal{R}})^{2k}\, d\xi \right)^{\frac{1}{2}} < \infty \right\},$$

where S' is the set of tempered distributions, $\widehat{u}$ is a Fourier transformation of u. For $k \in \mathbb{Z}_+$, $q \in Q$, a completely regular polyhedron $\mathcal{R}$, and $\Omega \subset \mathbb{R}^n$, denote

$$H_q^{k,\mathcal{R}}(\mathbb{R}^n) := \left\{ u : \|u\|_{H_q^{k,\mathcal{R}}(\mathbb{R}^n)} := \|u\|_{k,\mathcal{R},q} := \sum_{\alpha\in k\mathcal{R}} \left\| D^\alpha u \cdot q^{k-\max\limits_i(\alpha:\mu^i)} \right\|_{L_2(\mathbb{R}^n)} < \infty \right\},$$

$$H_q^{k,\mathcal{R}}(\Omega) := \left\{ u : \|u\|_{H_q^{k,\mathcal{R}}(\Omega)} := \sum_{\alpha\in k\mathcal{R}} \left\| D^\alpha u \cdot q^{k-\max\limits_i(\alpha:\mu^i)} \right\|_{L_2(\Omega)} < \infty \right\}.$$

2. MAIN RESULTS

Let $k \in \mathbb{Z}_+$ and $q \in Q$. Consider the differential operator $P(x,\mathbb{D})$ (see (1)) with the coefficients that satisfy the following conditions:

$$P(x,\mathbb{D}) = \sum_{\alpha\in\mathcal{R}} a_\alpha(x) D^\alpha = \sum_{\alpha\in\mathcal{R}} \left(a_\alpha^0(x) q(x)^{1-\max\limits_i(\alpha:\mu^i)} + a_\alpha^1(x) \right) D^\alpha, \tag{3}$$

where $a_\alpha(x) = a_\alpha^0(x) q(x)^{1-\max\limits_i(\alpha:\mu^i)} + a_\alpha^1(x)$, $D^\beta(a_\alpha^0(x)) = O\left(q(x)^{\min\limits_i(\beta:\mu^i)} \right)$, and $D^\beta(a_\alpha^1(x)) = o\left(q(x)^{1-\max\limits_i(\alpha-\beta:\mu^i)} \right)$ as $|x| \to \infty$ for all $\alpha \in \mathcal{R}$, $\beta \in k\mathcal{R}$.

It is easy to check that $P(x,\mathbb{D})$ generates a bounded linear operator acting from $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$.

For $N > 0$ and $x_0 \in \mathbb{R}^n$, denote

$$K_N(x_0) := \{x \in \mathbb{R}^n : |x - x_0| \le N\}, K_N := K_N(0).$$

Further we will use the following result, which is a consequence of Theorem 7.1 in [13].

Theorem 2.1. *Let $k \in \mathbb{Z}_+$, $q \in Q$ and $P(x,\mathbb{D})$ be the differential operator* (3). *Then the differential operator $P(x,\mathbb{D}) : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ is an n-normal operator if and only if there exist constants $\kappa > 0$ and $N > 0$ such that*

$$\|u\|_{k+1,\mathcal{R},q} \le \kappa \left(\|Pu\|_{k,\mathcal{R},q} + \|u\|_{L_2(K_N)} \right), \quad \forall u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n).$$

Theorem 2.2. *Let* $k \in \mathbb{Z}_+, q \in Q^{k,\mathcal{R}}$ *and* $P(x, \mathbb{D})$ *be the differential operator* (3) *with coefficients satisfying the condition* $\lim\limits_{p\to\infty} \max\limits_{x,y\in\overline{W_p}} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ *for all* $\alpha \in \mathcal{R}$. *Let there exist a constant* $\kappa > 0$ *such that*

$$\|u\|_{k+1,\mathcal{R},q} \le \kappa \left(\|Pu\|_{k,\mathcal{R},q} + \|u\|_{L_2(\mathbb{R}^n)}\right), \ \ \forall u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n). \tag{4}$$

Then $P(x, \mathbb{D})$ *is regular in* $\mathbb{R}^n$ *and there exist constants* $\delta > 0$ *and* $M > 0$ *such that*

$$\left|\sum_{\alpha\in\mathcal{R}} a^0_\alpha(x)\lambda^{1-\max\limits_i(\alpha:\mu^i)}\xi^\alpha\right| \ge \delta(\lambda + |\xi|_{\partial\mathcal{R}}), \ \forall\xi \in \mathbb{R}^n, \ \lambda > 0, \ |x| > M. \tag{5}$$

Proof. From Theorem 2.1 and [22] it follows that $P(x, \mathbb{D})$ is regular in $\mathbb{R}^n$, so it remains to prove the lower bound (5). Let $\{x_m\}_{m=1}^\infty \subset \mathbb{R}^n$ be such a sequence that $|x_m| \to \infty$ as $m \to \infty$. Without loss of generality we assume that $x_m \in W_m$.

Let $m_0 \in \mathbb{N}$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$, with $\mathrm{supp}\varphi \subset \mathbb{R}^n \setminus \left(\bigcup\limits_{i=1}^{m_0} W_i\right)$, $\|\varphi\|_{L_2(\mathbb{R}^n)} = 1$. Denote $\tilde\varphi_m = \varphi_m\varphi$. Let $j \in \{1, \dots, I_{n-1}\}$ and $\xi \in \mathbb{R}^n$. Consider the function

$$\tilde u_j(x) = \exp\left(i\left(q(x_m)^{\frac{1}{\mu^j}}\xi, x\right)\right)\tilde\varphi_m(x),$$

where μ^j is an outer normal of noncoordinate face $\mathcal{R}_j^{n-1}$ such that, for all $\alpha \in \mathcal{R}_j^{n-1}$, the equality $(\alpha : \mu^j) = 1$ is satisfied.

Denote $\mathcal{R}_j = \{\alpha \in \mathcal{R} : (\alpha : \mu^j) = \max\limits_{1\le i\le I_{n-1}} (\alpha : \mu^i)\}$. Since $q \in Q^{m,\mathcal{R}}$, then for any $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ and $m_0(\varepsilon) > 0$ such that, for all $m > m_0$ and $\max\limits_{j=1,\dots,l} \mathrm{diam}U_j < \delta$,

$$|q(x) - q(y)| \le \varepsilon q(y), \ \forall x, y \in W_m.$$

Then for any $r > 0$ the following holds

$$|q(x)^r - q(x_m)^r| \le \tau_r(\varepsilon)q(x_m)^r, \ \forall x \in W_m, \tag{6}$$

where $\tau_r(\varepsilon) \to 0$ as $\varepsilon \to 0$.

From inequality (6) and the inclusion $\mathrm{supp}\tilde u_{j,m} \subset W_m$ it follows that there exists $\tau(\varepsilon)$ such that $\tau(\varepsilon) \to 0$ as $\varepsilon \to 0$ and the following inequalities hold:

$$\|\tilde u_{j,m}\|_{k+1,\mathcal{R},q} \ge (1 - \tau(\varepsilon))\,\|\tilde u_{j,m}\|_{k+1,\mathcal{R},q(x_m)}, \tag{7}$$

$$\|P\tilde u_{j,m}\|_{k,\mathcal{R},q} \le (1 + \tau(\varepsilon))\,\|P\tilde u_{j,m}\|_{k,\mathcal{R},q(x_m)}. \tag{8}$$

Then for $m_0 \in \mathbb{N}$ large enough and a sufficiently small $\max\limits_{j=1,\dots,l} \mathrm{diam}U_j$ for $m > m_0$ the following holds:

$$\|\tilde u_{j,m}\|_{k+1,\mathcal{R},q} \ge \frac{1}{2}\|\tilde u_{j,m}\|_{k+1,\mathcal{R},q(x_m)}, \tag{9}$$

$$\|P\tilde u_{j,m}\|_{k,\mathcal{R},q} \le \frac{1}{2}\|P\tilde u_{j,m}\|_{k,\mathcal{R},q(x_m)}. \tag{10}$$

Taking into account the conditions on the functions $\{\varphi_m\}_{m=1}^\infty$ and the weight function $q \in Q^{k,\mathcal{R}}$, we conclude that, for all $\gamma \in k\mathcal{R}$ and $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ and $m_0(\varepsilon) > 0$ such that for all $m > m_0$ and $\max\limits_{j=1,\dots,l} \mathrm{diam}U_j < \delta$ the following inequality holds:

$$\frac{|D^\gamma\varphi_m(x)|}{q(x)^{(\gamma:\mu_i)}} = \frac{|D^\gamma\varphi_m(x)|a^{|\gamma|}_{\left[\frac{m-1}{l}\right]}}{q(x)^{(\gamma:\mu_i)-\frac{|\gamma|}{\mu_{\max}}}q(x)^{\frac{|\gamma|}{\mu_{\max}}}a^{|\gamma|}_{\left[\frac{m-1}{l}\right]}} \le \omega_\gamma(\varepsilon), \tag{11}$$

where $\omega_\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then it is easy to derive the lower estimate

$$\|\tilde{u}_{j,m}\|_{k+1,\mathcal{R},q(x_m)} \geq \sum_{\beta\in(k+1)\mathcal{R}_j} |\xi^\beta| q(x_m)^{k+1} \|\tilde{\varphi}_m\|_{L_2(\mathbb{R}^n)} - \omega_1(\varepsilon) \sum_{\gamma\in(k+1)\left(\mathcal{R}\setminus\mathcal{R}_j^{n-1}\right)} |\xi^\gamma| q(x_m)^{k+1} \|\varphi\|_{H^{k+1,\mathcal{R}}(W_m)}, \quad (12)$$

where $\omega_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Similarly to the proof of Theorem 2.4 in [22] it can be shown that for $\beta \in k(\mathcal{R} \setminus \mathcal{R}_j)$ and all $m > m_0$, with m_0 large enough, the following holds:

$$\left\|D^\beta(P(x,\mathbb{D})\tilde{u}_{j,m})\right\|_{L_2(\mathbb{R}^n)} q(x_m)^{k-\max\limits_i(\beta:\mu^i)} \leq \omega_2(\varepsilon) \sum_{\gamma\in(k+1)\left(\mathcal{R}\setminus\mathcal{R}_j^{n-1}\right)} |\xi^\gamma| q(x_m)^{k+1} \|\varphi\|_{H^{k+1,\mathcal{R}}(W_m)}, \quad (13)$$

where $\omega_2(\varepsilon) \to 0$ as $\varepsilon \to 0$. For $\beta \in k\mathcal{R}_j$ we have

$$\begin{aligned}\left\|D^\beta\left(P(x,\mathbb{D})\tilde{u}_{j,m}\right)\right\|_{L_2(\mathbb{R}^n)} q(x_m)^{k-(\beta:\mu^j)} &\leq \left\|D^\beta\left(\sum_{\alpha\in\mathcal{R}} a_\alpha^0(x) q(x)^{1-(\alpha:\mu^j)} D^\alpha \tilde{u}_{j,m}\right)\right\|_{L_2(\mathbb{R}^n)} q(x_m)^{k-(\beta:\mu^j)} \\ &+ \left\|D^\beta\left(\sum_{\alpha\in\mathcal{R}} a_\alpha^1(x) D^\alpha \tilde{u}_{j,m}\right)\right\|_{L_2(\mathbb{R}^n)} q(x_m)^{k-(\beta:\mu^j)}. \quad (14)\end{aligned}$$

Taking into account that, for all fixed $\alpha \in \mathcal{R}$ and $\beta \in k\mathcal{R}$, one has $D^\beta\left(a_\alpha^1(x)\right) = o\left(q(x)^{1-\max\limits_i(\alpha-\beta:\mu^i)}\right)$ as $|x| \to \infty$, it is easy to check that for $\beta \in k\mathcal{R}_j$ and $m > m_0$, with a sufficiently large m_0, the following holds:

$$\left\|D^\beta\left(\sum_{\alpha\in\mathcal{R}} a_\alpha^1(x) D^\alpha \tilde{u}_{j,m}\right)\right\|_{L_2(\mathbb{R}^n)} q(x_m)^{k-(\beta:\mu^j)} \leq \omega_3(\varepsilon) \sum_{\gamma\in(k+1)\mathcal{R}} |\xi^\gamma|\, q(x_m)^{k+1} \|\varphi\|_{H^{k+1,\mathcal{R}}(W_m)}, \quad (15)$$

where $\omega_3(\varepsilon) \to 0$ as $\varepsilon \to 0$.

From conditions (3), the estimate $\lim\limits_{p\to\infty} \max\limits_{x,y\in\overline{W_p}} |a_\alpha^0(x) - a_\alpha^0(y)| = 0$ for all $\alpha \in \mathcal{R}$ and $q \in Q^{k,\mathcal{R}}$, and from inequality (6) we obtain that, for $\alpha \in \mathcal{R}$ and $\beta \in k\mathcal{R}$ with a sufficiently large m_0 and small $\max\limits_{j=1,\dots,l} \operatorname{diam} U_j$ for $m > m_0$, the following holds:

$$\left|D^\beta\left(a_\alpha^0(x) q(x)^{1-(\alpha:\mu^j)} - a_\alpha^0(x_m) q(x_m)^{1-(\alpha:\mu^j)}\right)\right| \leq \tau_{\alpha,\beta}(\varepsilon) q(x_m)^{1-(\alpha:\mu^j)+(\beta:\mu^j)}, \quad (16)$$

where $\tau_{\alpha,\beta}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Using (16), similarly to the proof of Theorem 2.4 in [22], it can be shown that, for a large enough m_0 and for any $m > m_0$, the following estimate holds:

$$\left\|D^\beta\left(\sum_{\alpha\in\mathcal{R}} a_\alpha^0(x) q(x)^{1-(\alpha:\mu^j)} D^\alpha \tilde{u}_{j,m}\right)\right\|_{L_2(\mathbb{R}^n)} q(x_m)^{k-(\beta:\mu^j)}$$

$$\leq \left|\sum_{\alpha\in\mathcal{R}_j} a^0_\alpha(x_m)\xi^\alpha\right| \left|\xi^\beta\right| q(x_m)^{k+1}\|\tilde{\varphi}_m\|_{L_2(\mathbb{R}^n)} + \omega_4(\varepsilon)\sum_{\gamma\in(k+1)\left(\mathcal{R}\setminus\mathcal{R}_j^{n-1}\right)} |\xi^\gamma|\, q(x_m)^{k+1}\|\varphi\|_{H^{k+1,\mathcal{R}}(W_m)}, \quad (17)$$

where $\omega_4(\varepsilon)\to 0$ as $\varepsilon\to 0$.

From estimates (13)–(17) for all $m > m_0$ we get

$$\|P\tilde{u}_{j,m}\|_{k,\mathcal{R},q(x_m)} \leq \left|\sum_{\alpha\in\mathcal{R}_j} a^0_\alpha(x_m)\xi^\alpha\right| \sum_{\beta\in k\mathcal{R}_j}\left|\xi^\beta\right| q(x_m)^{k+1}\|\tilde{\varphi}_m\|_{L_2(\mathbb{R}^n)} + \omega_5(\varepsilon)\sum_{\gamma\in(k+1)\mathcal{R}} |\xi^\gamma|\, q(x_m)^{k+1}\|\varphi\|_{H^{k+1,\mathcal{R}}(W_m)}, \quad (18)$$

where $\omega_5(\varepsilon)\to 0$ as $\varepsilon\to 0$. Then from (4), (12), and (18) we get

$$\sum_{\beta\in(k+1)\mathcal{R}_j}\left|\xi^\beta\right| q(x_m)^{k+1}\|\tilde{\varphi}_m\|_{L_2(\mathbb{R}^n)} - \omega_1(\varepsilon)\sum_{\gamma\in(k+1)\left(\mathcal{R}\setminus\mathcal{R}_j^{n-1}\right)} |\xi^\gamma|\, q(x_m)^{k+1}\|\varphi\|_{H^{k+1,\mathcal{R}}(W_m)}$$
$$\leq \kappa\left(\left|\sum_{\alpha\in\mathcal{R}_j} a^0_\alpha(x_m)\xi^\alpha\right| \sum_{\beta\in k\mathcal{R}_j}\left|\xi^\beta\right| q(x_m)^{k+1}\|\tilde{\varphi}_m\|_{L_2(\mathbb{R}^n)} + \omega_5(\varepsilon)\sum_{\gamma\in(k+1)\mathcal{R}} |\xi^\gamma|\, q(x_m)^{k+1}\|\varphi\|_{H^{k+1,\mathcal{R}}(W_m)} + \|\tilde{\varphi}_m\|_{L_2(\mathbb{R}^n)}\right).$$

Since $\left\{a^0_\alpha(x) : \alpha\in\mathcal{R}_j\right\}$ are bounded functions and $x_m\to\infty$ as $m\to\infty$, there exist convergent subsequences of sequences $\left\{a^0_\alpha(x_m) : \alpha\in\mathcal{R}_j\right\}$. Without loss of generality we assume that the sequences $\left\{a^0_\alpha(x_m) : \alpha\in\mathcal{R}_j\right\}$ are convergent, so for each $\alpha\in\mathcal{R}_j$ there exists a constant $\tilde{a}^0_\alpha$ such that $a^0_\alpha(x_m)\rightrightarrows\tilde{a}^0_\alpha$ as $m\to\infty$. Then for a large enough m_0 and for $m > m_0$ we get

$$\sum_{\beta\in(k+1)\mathcal{R}_j}\left|\xi^\beta\right| q(x_m)^{k+1}\|\tilde{\varphi}_m\|_{L_2(\mathbb{R}^n)} - \omega_6(\varepsilon)\sum_{\gamma\in(k+1)\mathcal{R}} |\xi^\gamma|\, q(x_m)^{k+1}\|\varphi\|_{H^{k+1,\mathcal{R}}(W_m)}$$
$$\leq \kappa\left|\sum_{\alpha\in\mathcal{R}_j} \tilde{a}^0_\alpha\xi^\alpha\right| \sum_{\beta\in k\mathcal{R}_j}\left|\xi^\beta\right| q(x_m)^{k+1}\|\tilde{\varphi}_m\|_{L_2(\mathbb{R}^n)},$$

where $\omega_6(\varepsilon)\to 0$ as $\varepsilon\to 0$. Let us divide the last inequality by $q(x_m)^{k+1}$ and sum up it over $m > m_0$. Taking into account that each set W_m intersects with a fixed number of other sets, for some constant $C_1 > 0$, we get

$$C_1\sum_{\beta\in(k+1)\mathcal{R}_j}\left|\xi^\beta\right| - \omega_7(\varepsilon)\sum_{\gamma\in(k+1)\mathcal{R}} |\xi^\gamma|$$
$$\leq \left|\sum_{\alpha\in\mathcal{R}_j} \tilde{a}^0_\alpha\xi^\alpha\right| \sum_{\beta\in k\mathcal{R}_j}\left|\xi^\beta\right|,$$

where $\omega_7(\varepsilon)\to 0$ as $\varepsilon\to 0$. Choosing an appropriate ε and tending $m_0\to\infty$, for some constant $C_2 > 0$, we get the inequality

$$C_2 \sum_{\alpha\in(k+1)\mathcal{R}_j} |\xi^\alpha| \le \left|\sum_{\alpha\in\mathcal{R}_j} \tilde{a}^0_\alpha \xi^\alpha\right| \sum_{\beta\in k\mathcal{R}_j} \left|\xi^\beta\right|.$$

Using the last inequality and estimates (2.4) from the proof of Theorem 2.1 in [22], we get that for $j\in\{1,\dots,I_{n-1}\}$ there exists a constant $\delta_j>0$ such that

$$\left|\sum_{\alpha\in\mathcal{R}_j} \tilde{a}^0_\alpha \xi^\alpha\right| \ge \delta_j(1+|\xi|_{\mathcal{R}_j^{n-1}}),$$

where $|\xi|_{\mathcal{R}_j^{n-1}} = \sum\limits_{\beta\in\mathcal{R}_j^{n-1}} \left|\xi^\beta\right|$.

For $\lambda>0$, substituting $\lambda^{-\frac{1}{\mu^j}}\xi = \left(\lambda^{-\frac{1}{\mu_1^j}}\xi_1,\dots,\lambda^{-\frac{1}{\mu_n^j}}\xi_n\right)$ for $\xi=(\xi_1,\dots,\xi_n)$, we obtain

$$\left|\sum_{\alpha\in\mathcal{R}_j} \tilde{a}^0_\alpha \xi^\alpha \lambda^{1-(\alpha:\mu^j)}\right| \ge \delta_j(\lambda+|\xi|_{\mathcal{R}_j^{n-1}}).$$

The same can be done for all $j\in\{1,\dots,I_{n-1}\}$. Then, using Theorem 6.1 in [16], we get the following inequality:

$$\left|\sum_{\alpha\in\mathcal{R}} \tilde{a}^0_\alpha \xi^\alpha \lambda^{1-\max\limits_i(\alpha:\mu^i)}\right| \ge \delta(\lambda+|\xi|_{\partial\mathcal{R}}),\ \forall\lambda>0,\ \forall\xi\in\mathbb{R}^n.$$

Since the last inequality holds for all partial limits of sequences $\left\{a^0_\alpha(x_m):\alpha\in\mathcal{R}\right\}$, where $x_m\to\infty$, we conclude that there exist constants $\delta>0$ and $M>0$ such that

$$\left|\sum_{\alpha\in\mathcal{R}} a^0_\alpha(x) \xi^\alpha \lambda^{1-\max\limits_i(\alpha:\mu^i)}\right| \ge \delta(\lambda+|\xi|_{\partial\mathcal{R}}),\ \forall\lambda>0,\ \forall\xi\in\mathbb{R}^n,\ \forall|x|>M. \qquad \square$$

Remark 5. For $q\in Q$ based on Theorem 2.1 in [22] uniform regularity in $\mathbb{R}^n$ is a necessary condition for the fulfillment of the a priori estimate (4). From Theorem 2.2 it follows that in the spaces $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ with a weight function from the class $Q^{k,\mathcal{R}}$, the condition (5) is also necessary along with uniform regularity in $\mathbb{R}^n$ for fulfillment of the a priori estimate (4). Theorem 2.3 shows that these conditions on the symbol of operator are also sufficient conditions for a priori estimate (4) in the spaces under consideration.

Theorem 2.3. *Let $k\in\mathbb{Z}_+,\ q\in Q^{k,\mathcal{R}}$ and let $P(x,\mathbb{D})$ be the differential operator (3) with coefficients satisfying the relations $\lim\limits_{p\to\infty}\max\limits_{x,y\in\overline{W_p}}|a^0_\alpha(x)-a^0_\alpha(y)|=0$ for all $\alpha\in\mathcal{R}$. Let $P(x,\mathbb{D})$ be regular in $\mathbb{R}^n$ and let there exist constants $\delta>0$ and $M>0$ such that*

$$\left|\sum_{\alpha\in\mathcal{R}} a^0_\alpha(x) \lambda^{1-\max\limits_i(\alpha:\mu^i)} \xi^\alpha\right| \ge \delta(\lambda+|\xi|_{\mathcal{R}}),\ \forall\xi\in\mathbb{R}^n,\ \lambda>0,\ |x|>M. \tag{19}$$

Then there exist constants $\kappa>0$ and $N>0$ such that

$$\|u\|_{k+1,\mathcal{R},q} \le \kappa\left(\|Pu\|_{k,\mathcal{R},q}+\|u\|_{L_2(K_N)}\right),\ \forall u\in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n). \tag{20}$$

Proof. Let $m_0\in\mathbb{N}$. Using properties of the functions $\{\varphi_m\}_{m=0}^\infty$, it is easy to check that, for some constant $C>0$, the following estimate holds:

$$\|u\|^2_{k+1,\mathcal{R},q} \le C\left(\sum_{m=0}^{m_0}\|\varphi_m u\|^2_{k+1,\mathcal{R},q}+\sum_{m=m_0+1}^{\infty}\|\varphi_m u\|^2_{k+1,\mathcal{R},q}\right),\quad \forall u\in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n). \tag{21}$$

From a priori estimates for bounded domains in [18], for some constants $C_1 > 0$ and $N_1 > 0$, we have

$$\sum_{m=1}^{m_0} \|\varphi_m u\|^2_{k+1,\mathcal{R},q} \le C_1 \left(\|Pu\|^2_{k,\mathcal{R},q} + \|u\|^2_{L_2(K_{N_1})} \right), \forall u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n), \tag{22}$$

where N_1 is such that $\bigcup_{i=0}^{m_0} W_i \subset K_{N_1}$.

Denote

$$P_0(x,\mathbb{D}) := \sum_{\alpha\in\mathcal{R}} a^0_\alpha(x) D^\alpha,$$

$$L(x,\mathbb{D}) := \sum_{\alpha\in\mathcal{R}} a^1_\alpha(x) D^\alpha,$$

$$P^m(x,\mathbb{D}) := \sum_{\alpha\in\mathcal{R}} \Bigg[\psi_m(x) \left(a^0_\alpha(x) q(x)^{1-\max\limits_i(\alpha:\mu^i)} - a^0_\alpha(x_m) q(x_m)^{1-\max\limits_i(\alpha:\mu^i)} \right) + a^0_\alpha(x_m) q(x_m)^{1-\max\limits_i(\alpha:\mu^i)} \Bigg] D^\alpha, \quad m = 1,2,\ldots.$$

Using properties of the functions $\{\psi_m\}_{m=0}^\infty$, the conditions $\lim\limits_{p\to\infty} \max\limits_{x,y\in\overline{W_p}} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for $\alpha \in \mathcal{R}$, $q \in Q^{k,\mathcal{R}}$, and inequality (6), it is easy to check that, for $\alpha \in \mathcal{R}$ and $\beta \in k\mathcal{R}$, by choosing m_0 large enough and $\max\limits_{j=1,\ldots,l} \operatorname{diam} U_j$ small enough (for $m > m_0$), we obtain the following estimate:

$$\left| D^\beta \left(\psi_m(x) \left(a^0_\alpha(x) q(x)^{1-\max_i(\alpha:\mu^i)} - a^0_\alpha(x_m) q(x_m)^{1-\max_i(\alpha:\mu^i)} \right) \right) \right| \le \tau_{\alpha,\beta}(\varepsilon) q(x_m)^{1-\max_i(\alpha-\beta:\mu^j)},$$

where $\tau_{\alpha,\beta}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

From the last estimate and Theorem 2.2 in [12] it follows that, for m_0 large enough and for $m > m_0$, the operators $P^m(x,\mathbb{D}) : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ have bounded inverse operators. Since (19) holds, we conclude that they have uniformly bounded norms and, for some constant $C_2 > 0$, the following upper bound is valid:

$$\|\varphi_m u\|^2_{k+1,\mathcal{R},q} \le C_2 \|P^m(\varphi_m u)\|^2_{k,\mathcal{R},q}, \ \forall u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n),$$

where C_2 does not depend on m.

Taking into account that $P^m(\varphi_m u) = P_0(\varphi_m u)$ for all $u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ and $m = 1,2,\ldots$, we get

$$\|\varphi_m u\|^2_{k+1,\mathcal{R},q} \le C_2 \|P^m(\varphi_m u)\|^2_{k,\mathcal{R},q} \le C_2 \|P_0(\varphi_m u)\|^2_{k,\mathcal{R},q}, \ \forall u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n).$$

Using properties of the functions $\{\varphi_m\}_{m=1}^\infty$ and estimate (11), it can be shown that for a sufficiently large m_0 and a small $\max\limits_{j=1,\ldots,l} \operatorname{diam} U_j$ for $m > m_0$, and some constants $C_3, C_4 > 0$, the following estimate holds:

$$\|\varphi_m P_0 u - P_0(\varphi_m u)\|^2_{k,\mathcal{R},q}$$

$$\leq C_3 \left\| \sum_{\alpha\in\mathcal{R}} \sum_{\beta+\gamma=\alpha,|\gamma|>0} a^0_\alpha(x) D^\beta u D^\gamma \varphi_m q(x)^{1-\max\limits_i(\alpha:\mu^i)} \right\|^2_{k,\mathcal{R},q}$$

$$\leq C_4 \left\| \sum_{\alpha\in\mathcal{R}} \sum_{\beta+\gamma=\alpha,|\gamma|>0} a^0_\alpha(x) D^\beta u D^\gamma \varphi_m \frac{1}{q(x)^{\min\limits_i(\gamma:\mu^i)}} q(x)^{1-\max\limits_i(\beta:\mu^i)} \right\|^2_{k,\mathcal{R},q} \leq \omega(\varepsilon)\|u\|^2_{H^{k+1,\mathcal{R}}_q(W_m)},$$

where $\omega(\varepsilon)\to 0$ as $\varepsilon \to 0$. From the last two estimates, for some constant $C_5>0$, we get

$$\|\varphi_m u\|^2_{k+1,\mathcal{R},q} \leq C_5\left(\|\varphi_m P_0 u\|^2_{k,\mathcal{R},q} + \omega(\varepsilon)\|u\|^2_{H^{k+1,\mathcal{R}}_q(W_m)}\right), \ \forall u \in H^{k+1,\mathcal{R}}_q(\mathbb{R}^n).$$

Summing up over all $m>m_0$ both sides of this inequality and taking into account the properties of $\{W_m\}^\infty_{m=1}$, for some constant $C_6>0$, we obtain

$$\sum_{m=m_0+1}^{\infty} \|\varphi_m u\|^2_{k+1,\mathcal{R},q} \leq C_6\left(\|P_0 u\|^2_{k,\mathcal{R},q} + \omega(\varepsilon)\|u\|^2_{k+1,\mathcal{R},q}\right), \ \forall u \in H^{k+1,\mathcal{R}}_q(\mathbb{R}^n). \qquad (23)$$

From (21), (22), and (23) we deduce the following estimate:

$$\|u\|^2_{k+1,\mathcal{R},q} \leq CC_1\left(\|Pu\|^2_{k,\mathcal{R},q} + \|u\|^2_{L_2(K_{N_1})}\right)$$
$$+ CC_6\left(\|P_0 u\|^2_{k,\mathcal{R},q} + \omega(\varepsilon)\|u\|^2_{k+1,\mathcal{R},q}\right), \forall u \in H^{k+1,\mathcal{R}}_q(\mathbb{R}^n).$$

We can choose m_0 large enough and $\max\limits_{j=1,\dots,l} \operatorname{diam} U_j$ small enough such that

$$CC_6\omega(\varepsilon) < \frac{1}{2}.$$

Then, for some constant $C_7>0$, the following holds

$$\|u\|^2_{k+1,\mathcal{R},q} \leq C_7\left(\|Pu\|^2_{k,\mathcal{R},q} + \|u\|^2_{L_2(K_{N_1})} + \|P_0 u\|^2_{k,\mathcal{R},q}\right), \ \forall u \in H^{k+1,\mathcal{R}}_q(\mathbb{R}^n). \qquad (24)$$

We have $P_0(x,\mathbb{D}) = P(x,\mathbb{D}) - L(x,\mathbb{D})$. Then

$$\|P_0 u\|_{k,\mathcal{R},q} \leq \|Pu\|_{k,\mathcal{R},q} + \|Lu\|_{k,\mathcal{R},q}, \forall u \in H^{k+1,\mathcal{R}}_q(\mathbb{R}^n).$$

Since $D^\beta(a^1_\alpha(x)) = o\left(q(x)^{1-\max\limits_i(\alpha-\beta:\mu^i)}\right)$ as $|x|\to\infty$, for any $\alpha\in\mathcal{R}$ and $\beta \in k\mathcal{R}$, it is easy to check that for $N_2>0$ we have

$$\|Lu\|^2_{k,\mathcal{R},q} \leq \tau(N_2)\|u\|^2_{k+1,\mathcal{R},q} + C_8\|u\|^2_{H^{k+1,\mathcal{R}}(K_{N_2})}, \ \forall u \in H^{k+1,\mathcal{R}}_q(\mathbb{R}^n),$$

where $\tau(N_2)\to 0$ as $N_2\to\infty$ and $C_8 = C_8(N_2)>0$. Then, using an a priori estimate similar to (22), with some constant $C_9 = C_9(N_2)>0$, we have

$$\|Lu\|^2_{k,\mathcal{R},q} \leq \tau(N_2)\|u\|^2_{k+1,\mathcal{R},q} + C_9\left(\|Pu\|^2_{k,\mathcal{R},q} + \|u\|^2_{L_2(K_{N_2})}\right).$$

Substituting the last estimates in (24), we get

$$\|u\|^2_{k+1,\mathcal{R},q} \leq C_7\left(\|Pu\|^2_{k,\mathcal{R},q} + \|u\|^2_{L_2(K_{N_1})}\right) + 2C_7\|Pu\|^2_{k,\mathcal{R},q}$$
$$+2C_7\tau(N_2)\|u\|^2_{k+1,\mathcal{R},q} + 2C_7C_9\left(\|Pu\|^2_{k,\mathcal{R},q} + \|u\|^2_{L_2(K_{N_2})}\right), \forall u \in H^{k+1,\mathcal{R}}_q(\mathbb{R}^n).$$

We can take N_2 such that $C_7\tau(N_2)<1/4$. Then, for some $C_{10}>0$ and $N=\max(N_1,N_2)>0$, we obtain the following estimate:

$$\|u\|_{k+1,\mathcal{R},q} \leq C_{10}\left(\|Pu\|_{k,\mathcal{R},q} + \|u\|_{L_2(K_N)}\right), \forall u \in H^{k+1,\mathcal{R}}_q(\mathbb{R}^n). \qquad \square$$

In what follows, we need the following result (see Theorem 3.14 [9]):

Theorem 2.4. *Let A be a bounded linear operator acting from a Banach space X to a Banach space Y. Then the following holds:*

1. *If operator A has a left regularizer, then kernel of operator A in X is finite dimensional;*
2. *If operator A has a right regularizer, then the image of operator A is closed in Y and the cokernel is finite dimensional;*
3. *The operator A has left and right regularizers if and only if A is a Fredholm operator.*

The following assertion is valid.

Theorem 2.5. *Let $k \in \mathbb{Z}_+$, $q \in Q^{k,\mathcal{R}}$ and $P(x,\mathbb{D})$ be the differential operator* (3) *with the coefficients that satisfy $\lim\limits_{p\to\infty} \max\limits_{x,y\in\overline{W_p}} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for all $\alpha \in \mathcal{R}$.*

Then the operator $P(x,\mathbb{D}) : H^{k+1,\mathcal{R}}_q(\mathbb{R}^n) \to H^{k,\mathcal{R}}_q(\mathbb{R}^n)$ is a Fredholm operator if and only if $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that

$$\left|\sum_{\alpha\in\mathcal{R}} a^0_\alpha(x)\lambda^{1-\max\limits_i(\alpha:\mu^i)}\xi^\alpha\right| \geq \delta(\lambda + |\xi|_{\partial\mathcal{R}}), \forall\xi \in \mathbb{R}^n, \lambda > 0, |x| > M. \tag{25}$$

Proof. Let us first prove the sufficiency. Let $m_0 \in \mathbb{N}$ and $x_m \in W_m, m = 1, 2 \ldots.$ For $m \leq m_0$, denote

$$P^m(x,\mathbb{D}) := \sum_{\alpha\in\mathcal{R}} \left(\psi_m(x)\left(a_\alpha(x) - a_\alpha(x_m)\right) + a_\alpha(x_m)\right) D^\alpha,$$

$$P^{m,0}(x,\mathbb{D}) := \sum_{\alpha\in\partial\mathcal{R}} \left(\psi_m(x)\left(a_\alpha(x) - a_\alpha(x_m)\right) + a_\alpha(x_m)\right) D^\alpha,$$

$$R^{m,0} := F^{-1}\frac{|\xi|_{\partial\mathcal{R}}}{(1 + |\xi|_{\partial\mathcal{R}})P^{m,0}\left(x_m, \xi\right)}F.$$

Since $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$, for sufficiently small diameters of $\{W_m\}_{m=1}^{m_0}$, from Lemma 3.1 in [22] it follows that for $m \leq m_0$ the following representation holds:

$$P^m(x,\mathbb{D})R^{m,0} = I + T^m_1 + T^m_2, \tag{26}$$

where $T^m_1 : H^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H^{k+1+\sigma,\mathcal{R}}(\mathbb{R}^n)$, with $\sigma = \sigma(\mathcal{R}) > 0$, and the operator $T^m_2 : H^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H^{k+1,\mathcal{R}}(\mathbb{R}^n)$ satisfies the inequality $\|T^m_2\| < 1$. Denote

$$R^m := R^{m,0}(I + T^m_2)^{-1}.$$

Then we have

$$P^m R^m = I + T^m, \tag{27}$$

where $T^m : H^{k,\mathcal{R}}(\mathbb{R}^n) \to H^{k+\sigma,\mathcal{R}}(\mathbb{R}^n)$, with some $\sigma = \sigma(\mathcal{R}) > 0$.

For $m > m_0$, denote

$$P^m(x,\mathbb{D}) := \sum_{\alpha\in\mathcal{R}} \left[\psi_m(x)\left(a^0_\alpha(x)q(x)^{1-\max\limits_i(\alpha:\mu^i)} - a^0_\alpha(x_m)q(x_m)^{1-\max\limits_i(\alpha:\mu^i)}\right)\right.$$
$$\left. + a^0_\alpha(x_m)q(x_m)^{1-\max\limits_i(\alpha:\mu^i)}\right] D^\alpha.$$

Taking into account that $q \in Q^{k,\mathcal{R}}$ and $\lim\limits_{p\to\infty} \max\limits_{x,y\in\overline{W_p}} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$, from Theorem 2.2 in [12] we can choose a number m_0 such that, for all $m > m_0$, the operators $P^m : H^{k+1,\mathcal{R}}_q(\mathbb{R}^n) \to H^{k,\mathcal{R}}_q(\mathbb{R}^n)$ have the inverse operators $R^m : H_q^{k,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$. Since (25) holds, the inverse operators have uniformly bounded norms. Consider

$$Rf := \sum_{l=0}^{\infty} \psi_l R^l(\varphi_l f), f \in H_q^{k,\mathcal{R}}(\mathbb{R}^n).$$

□

Taking into account that the norms of the operators R^l acting from $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ have uniformly bounded norms, the properties of the function q and the functions $\{\varphi_m\}_{m=1}^{\infty}$ and $\{\psi_m\}_{m=1}^{\infty}$, it is easy to check that R is a bounded linear operator acting from $H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ to $H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$. Similarly to the proof of Theorem 2.6 in [22] it can be checked that $R : H_q^{k,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k+1,\mathcal{R}}(\mathbb{R}^n)$ is a right regularizer. Then, applying Theorem 2.4, we get that cokernel of the operator $P(x,\mathbb{D}) : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ is finite-dimensional. From Theorem 2.3 it follows that the operator $P(x,\mathbb{D}) : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ is n-normal. So, $P(x,\mathbb{D}) : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ is a Fredholm operator.

Let us prove the necessity of the theorem. Since a Fredholm operator is n-normal, Theorem 2.1 implies the fulfillment of the a priori estimate (4). Applying Theorem 2.2, we get that $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$ and (25) holds.

Theorem 2.6. *Let $k \in \mathbb{Z}_+, q \in Q^{k,\mathcal{R}}$ and $P(x,\mathbb{D})$ be differential operator* (3) *with the coefficients that satisfy* $\lim\limits_{p\to\infty} \max\limits_{x,y\in\overline{W_p}} |a_\alpha^0(x) - a_\alpha^0(y)| = 0$ *for all* $\alpha \in \mathcal{R}$. *Then the following statements are equivalent:*

1. *The operator* $P(x,\mathbb{D}) : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ *is a Fredholm operator.*
2. *There exist constants* $\kappa > 0$ *and* $N > 0$ *such that*

$$\|u\|_{k+1,\mathcal{R},q} \le \kappa \left(\|Pu\|_{k,\mathcal{R},q} + \|u\|_{L_2(K_N)}\right), \forall u \in H_q^{k+1,\mathcal{R}}(\mathbb{R}^n). \tag{28}$$

3. $P(x,\mathbb{D})$ *is regular in* $\mathbb{R}^n$ *and there exist constants* $\delta > 0$ *and* $M > 0$ *such that*

$$\left|\sum_{\alpha\in\mathcal{R}} a_\alpha^0(x) \lambda^{1-\max\limits_i(\alpha:\mu^i)} \xi^\alpha\right| \ge \delta(\lambda + |\xi|_{\partial\mathcal{R}}),\ \forall \xi \in \mathbb{R}^n,\ \lambda > 0,\ |x| > M. \tag{29}$$

Proof. Let $P(x,\mathbb{D}) : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ is a Fredholm operator, then it is n-normal. By Theorem 2.1, the a priori estimate (28) is valid. So it is proved that condition 2 follows from condition 1. Since the a priori estimate is valid, from Theorem 2.2 it follows that a) $P(x,\mathbb{D})$ is regular in $\mathbb{R}^n$, and b) condition (29) holds for the symbol of operator, which means that condition 2 implies condition 3. Having the condition for the symbol of $P(x,\mathbb{D})$ and applying Theorem 2.5, we conclude that $P(x,\mathbb{D}) : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ is a Fredholm operator, which means that condition 1 follows from condition 3. Thus, the equivalence of all of the conditions is proved. □

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