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On the Boundedness of Integral Operators in Morrey-Type Spaces with Variable Exponents

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Abstract—We consider the global Morrey-type spaces $GM_{p(\cdot),\theta(\cdot),w(\cdot)}(\Omega)$ with variable exponents $p(x)$, $\theta(x)$, and $w(x,r)$ defining these spaces. In the case of unbounded sets $\Omega\subset\mathbb{R}^n$, we prove the boundedness of the Hardy—Littlewood maximal operator and potential-type operator in these spaces. We prove Spanne-type results on the boundedness of the Riesz potential I^α in global Morrey-type spaces with variable exponents $GM_{p(\cdot),\theta(\cdot),w(\cdot)}(\Omega)$.

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INTRODUCTION

In this paper we consider the global Morrey-type spaces $GM_{p(\cdot),\theta(\cdot),w(\cdot)}(\Omega)$ with variable exponents $p(\cdot)$, $\theta(\cdot)$ and a general function $w(x,r)$ defining a Morrey-type norm. The Morrey spaces $M_{p,\lambda}$ are introduced in [1] in the frames of the study of partial differential equations. Many classical operators of harmonic analysis (e. g., maximal, fractional maximal, potential operators) were studied in the Morrey-type spaces with constant exponents p, θ [2—4]. The Morrey spaces also attracted attention of researchers in the area of variable exponent analysis; see [5—10]. The Morrey spaces $\mathcal{L}_{p(\cdot),\lambda(\cdot)}$ with variable exponent $p(\cdot)$, $\lambda(\cdot)$ were introduced and studied in [5]. The general versions $M_{p(\cdot),w(\cdot)}(\Omega)$, $\Omega\subset\mathbb{R}^n$, were introduced and studied in [11, 12]. The boundedness of maximal and potential type operators in the generalized Morrey-type spaces with a variable exponent were considered in [11] in the case of bounded sets $\Omega\subset\mathbb{R}^n$, in [12] in the case of unbounded sets $\Omega\subset\mathbb{R}^n$.

Let $f\in L_{\mathrm{loc}}(\mathbb{R}^n)$.The Hardy—Littlewood maximal operator is defined as

$$Mf(x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{\tilde B(x,r)}|f(y)|\,dy,$$

where $B(x,r)$ is a ball in $\mathbb{R}^n$ centered at a point $x\in\mathbb{R}^n$ and of a radius r, $\tilde B(x,r)=B(x,r)\cap\Omega,\Omega\subset\mathbb{R}^n$.

The fractional maximal operator of variable order $\alpha(x)$ is defined as

$$M^{\alpha(\cdot)}f(x)=\sup_{r>0}|B(x,r)|^{-1+\frac{\alpha(x)}{n}}\int_{\tilde B(x,r)}|f(y)|\,dy,\quad 0\leqslant\alpha(x)<n.$$

In the case $\alpha(x)=\alpha=\mathrm{const}$, this operator coincides with the classical fractional maximal operator M^α. If $\alpha(x)=0$ then $M^{\alpha(\cdot)}$ coincides with the operator M.

The Riesz potential $I^{\alpha(x)}$ of variable order $\alpha(x)$ is defined by the following equality:

$$I^{\alpha(x)}f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha(x)}}\,dy,\quad 0<\alpha(x)<n.$$

In the case $\alpha(x)=\alpha=\mathrm{const}$, this operator coincides with the classical Riesz potential I^α.

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1. LEBESGUE SPACES WITH VARIABLE EXPONENT. GENERALIZED MORREY-TYPE SPACES WITH VARIABLE EXPONENTS

Let $p(x)$ be a measurable function on an open set $\Omega \subset \mathbb{R}^n$ with values in $(1, \infty)$. Put

$$1 < p_- \leqslant p(x) \leqslant p_+ < \infty, \tag{1}$$

where $p_- = p_-(\Omega) = \operatorname{ess\,inf}_{x\in\Omega} p(x)$ and $p_+ = p_+(\Omega) = \operatorname{ess\,sup}_{x\in\Omega} p(x)$. We denote by $L_{p(\cdot)}(\Omega)$ the space of all measurable functions $f(x)$ on Ω such that

$$J_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\, dx < \infty,$$

where the norm is defined as

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 :\ J_{p(\cdot)}\left(\frac{f}{\eta}\right) \leqslant 1\right\}.$$

This is a Banach space. The conjugate exponent p' is defined by the formula

$$p'(x) = \frac{p(x)}{p(x) - 1}.$$

Hölder's inequality for the variable exponents $p(\cdot)$ and $p'(\cdot)$ is of the form

$$\int_\Omega f(x)g(x)dx \leqslant C(p)\|f\|_{L_{p(\cdot)}(\Omega)}\|g\|_{L_{p'(\cdot)}(\Omega)},$$

where $C(p) = \frac{1}{p_-} + \frac{1}{p'_-}$. The Lebesgue spaces $L_{p(\cdot)}$ with variable exponents $p(\cdot)$ were introduced in [13] and studied in [14, 15].

Define $\mathcal{P}(\Omega)$ as the set of measurable functions $p : \Omega \to [1, \infty)$. Denote by $\mathcal{P}^{\log}(\Omega)$ the set of measurable functions $p(x)$ satisfying the local log-condition

$$|p(x) - p(y)| \leqslant \frac{A_p}{-\ln|x - y|}, \text{ with } |x - y| \leqslant \frac{1}{2}\ \forall x, y \in \Omega,$$

where A_p is independent of x and y. Next, put $\mathbb{P}^{\log}(\Omega)$ for the set of measurable functions $p(x)$ meeting both (1) and the log-condition. In the case of Ω is an unbounded set, we denote by $\mathbb{P}^{\log}_\infty(\Omega)$ the set of exponents which is a subset of the set of $\mathbb{P}^{\log}(\Omega)$ and satisfying the decay condition

$$|p(x) - p(\infty)| \leqslant A_\infty \ln(2 + |x|), \quad x \in \mathbb{R}^n.$$

Put $\mathbb{A}^{\log}(\Omega)$ for the set of bounded exponents $\alpha : \Omega \to \mathbb{R}$ satisfying the log-condition.

Let Ω be an open bounded set, $p \in \mathbb{P}^{\log}(\Omega)$, and $\lambda(x)$ be a measurable function on Ω with values in $[0, n]$. The variable Morrey space $\mathcal{L}_{p(\cdot),\lambda(\cdot)}(\Omega)$ with the norm

$$\|f\|_{\mathcal{L}_{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x\in\Omega, t>0} t^{-\frac{\lambda(x)}{p(x)}}\|f\|_{L_{p(\cdot)}(\tilde{B}(x,t))}.$$

was introduced in [3]. Let $w(x, r)$ be nonnegative measurable function on Ω, where $\Omega \subset \mathbb{R}^n$ is an open bounded set. The generalized Morrey-type space $M_{p(\cdot),w(\cdot)}(\Omega)$ with variable exponent and the norm

$$\|f\|_{M_{p(\cdot),w(\cdot)}(\Omega)} = \sup_{x\in\Omega, r>0} \frac{r^{-\frac{n}{p(x)}}}{w(x, r)}\|f\|_{L_{p(\cdot)}(\tilde{B}(x,r))}.$$

is defined in [11]. Let $w(x, r)$ be nonnegative measurable function on Ω, where $\Omega \subset \mathbb{R}^n$ is an open unbounded set. The generalized Morrey-type space $M_{p(\cdot),w(\cdot)}(\Omega)$ with variable exponent with the norm

$$\|f\|_{M_{p(\cdot),w(\cdot)}(\Omega)} = \sup_{x\in\Omega, r>0} \frac{\|f\|_{L_{p(\cdot)}(\tilde{B}(x,r))}}{w(x, r)}.$$

is defined in [12]. Put

$$\eta_p(x,r) = \begin{cases} \frac{n}{p(x)} & \text{if } r \leqslant 1; \\ \frac{n}{p(\infty)} & \text{if } r > 1. \end{cases}$$

Definition 1.1. Let $p \in P^{\log}(\Omega)$, $w(x,r)$ be a positive function on $\Omega \times [0,\infty]$, where $\Omega \in \mathbb{R}^n$. The global Morrey-type space $GM_{p(\cdot),\theta(\cdot),w(\cdot)}(\Omega)$ with variable exponents is defined as the set of functions $f \in L^{\rm loc}_{p(\cdot)}(\Omega)$ with the finite norm

$$\|f\|_{GM_{p(\cdot),\theta(\cdot),w(\cdot)}(\Omega)} = \sup_{x\in\Omega} \Big\| w(x,r) r^{-\eta_p(x,r)} \|f\|_{L_{p(\cdot)}(\tilde{B}(x,r))} \Big\|_{L_{\theta(\cdot)}(0,\infty)}.$$

We assume that the positive measurable function $w(x,r)$ satisfies the condition

$$\sup_{x\in\Omega} \|w(x,r)\|_{L_{\theta(\cdot)}(0,\infty)} < \infty.$$

Then this space contains at least any bounded functions and thereby is nonempty. In the case $w(x,r) = r^{-\frac{\lambda(x)}{p(x)}+\eta_p(x,r)}$, the corresponding space is denoted by $GM^{\lambda(\cdot)}_{p(\cdot),\theta(\cdot)}$:

$$GM^{\lambda(\cdot)}_{p(\cdot),\theta(\cdot)}(\Omega) = GM_{p(\cdot),w(\cdot),\theta}\big|_{w(x,r)=r^{-\frac{\lambda(x)}{p(x)}+\eta_p(x,r)}},$$

and

$$\|f\|_{GM^{\lambda(\cdot)}_{p(\cdot),\theta(\cdot)}(\Omega)} = \sup_{x\in\Omega} \Big\| w(x,r) r^{-\frac{\lambda(x)}{p(x)}} \|f\|_{L_{p(\cdot)}(\tilde{B}(x,r))} \Big\|_{L_{\theta(\cdot)}(0,\infty)}.$$

In the case $\theta = \infty$, the space $GM_{p(\cdot),\infty,w(\cdot)}(\Omega)$ coincides with the generalized Morrey space with variable exponent $M_{p(\cdot),w(\cdot)}(\Omega)$ with the finite quasi-norm

$$\|f\|_{M_{p(\cdot),w(\cdot)}(\Omega)} = \sup_{x\in\Omega} \Big\{ w(x,r) r^{-\eta_p(x,r)} \|f\|_{L_{p(\cdot)}(\tilde{B}(x,r))} \Big\}.$$

If $p(\cdot) = p = \text{const}$ and $\theta(x) = \theta = \text{const}$ then the space $GM_{p(\cdot),\theta(\cdot),w(\cdot)}(\Omega)$ coincides with the ordinary global Morrey space $GM_{p,\theta,w}(\Omega)$, considered by V.I. Burenkov and others [2–4]. Some Spanne- and Adams-type theorems were proved in [5] for bounded sets Ω. Also, various results on the boundedness of operators are obtained in [11, 12, 16].

We need the following results from [12] for our arguments.

Lemma 1.2. *Assume that* $p \in \mathbb{P}^{\log}_{\infty}(\Omega)$ *and* $f \in L^{p(\cdot)}_{\rm loc}(\mathbb{R}^n)$.*Then*

$$\|f\|_{L_{p(\cdot)}(B(x,t))} \leqslant C t^{\eta_p(x,t)} \int_t^\infty r^{-\eta_p(x,r)-1} \|f\|_{L_{p(\cdot)}(B(x,r))}\, dr.$$

Theorem 1.3. *Suppose that* $p \in \mathbb{P}^{\log}_{\infty}(\Omega)$. *Then for each* $f \in L_{p(\cdot)}(\Omega)$ *we have*

$$\|Mf\|_{L_{p(\cdot)}(\tilde{B}(x,t))} \leqslant C t^{\eta_p(x,t)} \sup_{r>2t} \Big\{ r^{-\eta_p(x,r)} \|f\|_{L_{p(\cdot)}(\tilde{B}(x,r))} \Big\},$$

where C *is independent of* $f, x \in \Omega$ *and* $t > 0$.

Let us prove the next necessary inequality.

Theorem 1.4. *Assume that* $p \in \mathbb{P}^{\log}_{\infty}(\Omega)$. *Then*

$$\|Mf\|_{L_{p(\cdot)}(\tilde{B}(x,t))} \leqslant C t^{\eta_p(x,t)} \int_t^\infty s^{-\eta_p(x,s)-1} \|f\|_{L_{p(\cdot)}(\tilde{B}(x,s))}\, ds, \tag{2}$$

where C *is independent of* f, x, t.

Proof. Using Theorem 1.3 and Lemma 1.2, we deduce

$$\begin{aligned}\|Mf\|_{L_{p(\cdot)}(\tilde B(x,t))} &\leqslant Ct^{\eta_p(x,t)}\sup_{r>2t}\Big\{r^{-\eta_p(x,r)}\|f\|_{L_{p(\cdot)}(\tilde B(x,r))}\Big\}\\ &\leqslant Ct^{\eta_p(x,t)}\sup_{r>t}\Big\{r^{-\eta_p(x,r)}\|f\|_{L_{p(\cdot)}(\tilde B(x,r))}\Big\}\\ &\leqslant Ct^{\eta_p(x,t)}\sup_{r>t}\Big\{\int_r^\infty s^{-\eta_p(x,s)-1}\|f\|_{L_{p(\cdot)}(B(x,s))}\,ds\Big\}\\ &= Ct^{\eta_p(x,t)}\int_t^\infty s^{-\eta_p(x,s)-1}||f||_{L_{p(\cdot)}(\tilde B(x,s))}\,ds.\end{aligned}$$

The theorem is proved. □

The next result generalizes an inequality proved in [12].

Theorem 1.5. *Let $p\in\mathbb{P}^{\log}_\infty(\Omega)$ and let the function $\alpha(x)$ and $q(x)$ satisfy the condition $\frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha(x)}{n}$. Then, for each $x\in\mathbb{R}^n$ and $t>0$,*

$$\left\|\frac{1}{(1+|y|)^{\gamma(y)}}I^{\alpha(\cdot)}f\right\|_{L_{q(\cdot)}(\tilde B(x,t))}\leqslant Ct^{\eta_q(x,t)}\int_t^\infty r^{-\eta_q(x,r)-1}\|f\|_{L_{p(\cdot)}(\tilde B(x,r))}\,dr. \tag{3}$$

Proof. We represent the function f as $f(x)=f_1(x)+f_2(x)$, where $f_1(x)=f(x)\chi_{\tilde B(x,2t)}$ and $f_2(x)=f(x)\chi_{\Omega\setminus\tilde B(x,2t)}$. Then

$$\frac{1}{(1+|y|)^{\gamma(y)}}I^{\alpha(\cdot)}f(y)=\frac{1}{(1+|y|)^{\gamma(y)}}I^{\alpha(\cdot)}f_1(y)+\frac{1}{(1+|y|)^{\gamma(y)}}I^{\alpha(\cdot)}f_2(y).$$

From the results obtained in [16] it follows that

$$\begin{aligned}\left\|\frac{1}{(1+|y|)^{\gamma(y)}}I^{\alpha(\cdot)}f_1\right\|_{L_{q(\cdot)}(\tilde B(x,t))} &\leqslant \left\|\frac{1}{(1+|y|)^{\gamma(y)}}I^{\alpha(\cdot)}f_1\right\|_{L_{q(\cdot)}(\mathbb{R}^n)}\\ &\leqslant C\|f_1\|_{L_{p(\cdot)}(\mathbb{R}^n)}=C\|f\|_{L_{p(\cdot)}(B(x,2t))}.\end{aligned}$$

Lemma 1.2 implies

$$\left\|\frac{1}{(1+|y|)^{\gamma(y)}}I^{\alpha(\cdot)}f_1\right\|_{L_{q(\cdot)}(\tilde B(x,t))}\leqslant Ct^{\eta_q(x,t)}\int_{2t}^\infty r^{-\eta_q(x,r)-1}\|f\|_{L_{p(\cdot)}(\tilde B(x,r))}\,dr. \tag{4}$$

If $|x-z|\leqslant t$ and $|z-y|\geqslant 2t$, we have $\frac{1}{2}|z-y|\leqslant|x-y|\leqslant\frac{3}{2}|z-y|$. Since $\frac{1}{(1+|y|)^{\gamma(y)}}\leqslant 1$, we infer

$$\begin{aligned}\left\|\frac{1}{(1+|y|)^{\gamma(y)}}I^{\alpha(\cdot)}f_2\right\|_{L_{q(\cdot)}(\tilde B(x,t))} &\leqslant \left\|\int_{\mathbb{R}^n\setminus B(x,2t)}|z-y|^{\alpha-n}f(y)\,dy\right\|_{L_{q(\cdot)}(\tilde B(x,t))}\\ &\leqslant C\int_{\mathbb{R}^n\setminus B(x,2t)}|x-y|^{\alpha-n}|f(y)|\,dy\cdot\|\chi_{B(x,t)}\|_{L_{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Choosing $\beta>\frac{n}{q_-}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B(x,2t)}|x-y|^{\alpha-n}|f(y)|\,dy &= \beta\int_{\mathbb{R}^n\setminus B(x,2t)}|x-y|^{\alpha-n+\beta}|f(y)|\Big(\int_{|x-y|}^\infty s^{-\beta-1}\,ds\Big)\,dy\\ &=\beta\int_{2t}^\infty s^{-\beta-1}\Big(\int_{y\in\mathbb{R}^n:2t\leqslant|x-y|\leqslant s}|x-y|^{\alpha-n+\beta}|f(y)|\,dy\Big)\,ds\end{aligned}$$

$$\leqslant C\int_{2t}^{\infty} s^{-\beta-1}\|f\|_{L_{p(\cdot)}(B(x,s))}\cdot \||x-y|^{\alpha-n+\beta}\|_{L_{p'(\cdot)}(B(x,s))}\,ds$$
$$\leqslant C\int_{2t}^{\infty} s^{\alpha-\eta_p(x,s)-1}\|f\|_{L_{p(\cdot)}(B(x,s))}\,ds.$$

Therefore,

$$\left\|\frac{1}{(1+|y|)^{\gamma(y)}}I^{\alpha(\cdot)}f_2\right\|_{L_{q(\cdot)}(\tilde{B}(x,t))}\leqslant Ct^{\eta_p(x,t)}\int_{2t}^{\infty} s^{-\eta_q(x,s)-1}\|f\|_{L_{p(\cdot)}(B(x,s))}\,ds.$$

The latter together with (4) yields (3). The theorem is proved. □

Let u and v be positive measurable functions. The dual Hardy operator is defined as

$$\tilde{H}_{v,u}f(x)=v(x)\int_{x}^{\infty} f(t)u(t)\,dt,\ x\in\mathbb{R}^n.$$

Suppose that a is a positive fixed number. Put $\theta_{1,a}(x)=\operatorname{ess\,inf}_{y\in[x,a)}\theta_1(y)$,

$$\tilde{\theta}_1(x)=\begin{cases}\theta_{1,a}(x) & \text{if } x\in[0,a],\\ \overline{\theta}_1=\text{const} & \text{if } x\in[a,\infty).\end{cases}$$

Moreover, we denote $\theta_1=\operatorname{ess\,inf}_{x\in\mathbb{R}_+}\theta_1(x)$ and $\Theta_2=\operatorname{ess\,sup}_{x\in\mathbb{R}_+}\theta_2(x)$ for $\theta_1(x)$ and $\theta_2(x)$.

The next theorem is proved in [17].

Theorem 1.6. *Let $\theta_1(x)$ and $\theta_2(x)$ be measurable functions on $\mathbb{R}_+$. Suppose that there exists a positive number a such that $\theta_1(x)=\overline{\theta}_1=\text{const}$, $\theta_2(x)=\overline{\theta}_2=\text{const}$ holds for all $x>a$, and $1<\theta_1\leqslant\tilde{\theta}_1(x)\leqslant\theta_2(x)\leqslant\Theta_2<\infty$ almost everywhere. If*

$$G=\sup_{t>0}\int_0^t [v(x)]^{\theta_2(x)}\Big(\int_t^{\infty} u^{\tilde{\theta}_1'(x)}(\tau),d\tau\Big)^{\frac{\theta_2(x)}{(\theta_1)'(x)}}dx<\infty$$

then the operator $\tilde{H}_{v,u}$ from $L_{\theta_1(\cdot)}(\mathbb{R}^+)$ to $L_{\theta_2(\cdot)}(\mathbb{R}^+)$ is bounded.

2. MAIN RESULTS

Theorem 2.1. *Assume that $p(\cdot)\in\mathbb{P}^{\log}_{\infty}(\Omega)$, and $\theta_1(x)$ and $\theta_2(x)$ are measurable functions on $\mathbb{R}_+$. Suppose that there exists a positive number a such that we have $\theta_1(x)=\overline{\theta}_1=\text{const}$, $\theta_2(x)=\overline{\theta}_2=\text{const}$ for all $t>a$, and $1<\theta_1\leqslant\tilde{\theta}_1(x)\leqslant\theta_2(x)\leqslant\Theta_2<\infty$ a. e. Let the positive measurable functions w_1 ш w_2 satisfy the condition*

$$A=\sup_{x\in\Omega,t>0}\int_0^t (w_2(x,r))^{\theta_2(r)}\Big(\int_t^{\infty}\Big(\frac{1}{w_1(x,s)s}\Big)^{[\tilde{\theta}_1(r)]'}ds\Big)^{\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}}dr<\infty. \tag{5}$$

Then the maximal operator M from $GM_{p(\cdot),\theta_1(\cdot),w_1(\cdot)}(\Omega)$ to $GM_{p(\cdot),\theta_2(\cdot),w_2(\cdot)}(\Omega)$ is bounded.

Proof. According to the definition and to Theorem 1.6, Hölder's inequality with variable exponents θ, θ' infers

$$\|Mf\|_{GM_{p(\cdot),\theta_2(\cdot),w_2(\cdot)}(\Omega)}=\sup_{x\in\Omega}\left\|\frac{w_2(x,r)}{r^{\eta_p(x,r)}}\|Mf\|_{L_{p(\cdot)}(B(x,r))}\right\|_{L_{\theta_2(\cdot)}(0,\infty)}$$
$$\leqslant C\sup_{x\in\Omega}\left\|w_2(x,r)\int_r^{\infty} t^{-\eta_p(x,t)-1}\|f\|_{L_{p(\cdot)}(B(x,t))}\,dt\right\|_{L_{\theta_2(\cdot)}(0,\infty)}.$$

Denote

$$\tilde{H}_{v,u} f(r) = v(r) \int_r^\infty g(t) u(t)\, dt,$$

where $v(r) = w_2(x,r)$, $g(t) = \frac{w_1(x,t)}{t^{\eta_p(x,t)}} \|f\|_{L_{p(\cdot)}(B(x,t))}$, and $u(t) = \frac{1}{w_1(x,t)t}$ for every fixed $x \in \Omega$. Then condition (2) has the form (5), from which the boundedness of the operator $\tilde{H}_{v,w} f(r)$ from $L_{\theta_1(\cdot)}(0,\infty)$ to $L_{\theta_2(\cdot)}(0,\infty)$ follows. Consequently, we have

$$\|Mf\|_{GM_{q(\cdot),\theta_2(\cdot),w_2(\cdot)}(\Omega)} \leqslant A \cdot \sup_{x\in\Omega} \Big\| w_1(x,t) t^{-\eta_p(x,t)} \|f\|_{L_{p(\cdot)}(B(x,t))} \Big\|_{L_{\theta_1(\cdot)}(0,\infty)} = A \cdot \|f\|_{GM_{p(\cdot),\theta_1(\cdot),w_1(\cdot)}}.$$

The latter infers the boundedness of the operator M from $GM_{p(\cdot),\theta_1(\cdot),w_1(\cdot)}$ to $GM_{q(\cdot),\theta_2(\cdot),w_2(\cdot)}$. □

Corollary 2.2. *Let $p(\cdot) \in \mathbb{P}_\infty^{\log}(\Omega)$, $w_1(x,r) = w_2(x,r) = r^{\beta(x)}$. If*

$$\inf_{x\in\Omega, r>0} (\beta(x)+1)[\tilde{\theta}_1(r)]' > 1 \tag{6}$$

and

$$\sup_{x\in\Omega,t>0} \int_0^t r^{\theta_2(r)\beta(x)} \frac{t^{[-(\beta(x)+1)[\tilde{\theta}_1(r)]'+1]\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}}}{[(\beta(x)+1)[\tilde{\theta}_1(r)]-1]^{\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}}}\, dr < \infty \tag{7}$$

then the maximal operator M from $GM_{p(\cdot),\theta_1(\cdot),r^{\beta(\cdot)}}(\Omega)$ to $GM_{p(\cdot),\theta_2(\cdot),r^{\beta(\cdot)}}(\Omega)$ is bounded.

Proof. Condition (5) has the form

$$\sup_{x\in\Omega,t>0} \int_0^t r^{\theta_2(r)\beta(x)} \Big(\int_t^\infty s^{-[\beta(x)+1][\tilde{\theta}_1(r)]'}\, ds \Big)^{\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}} dr < \infty.$$

By the convergence of the inner integral, we obtain conditions (6) and (7). □

The next theorems give Spanne-type results on the boundedness of the Riesz potential I^α in global Morrey-type spaces with variable exponent $GM_{p(\cdot),\theta(\cdot),w(\cdot)}(\Omega)$. In the following theorem $\alpha = \text{const}$.

Theorem 2.3. *Assume that $p(\cdot) \in \mathbb{P}_\infty^{\log}(\Omega)$, the constant number α is positive, and $(\alpha p(\cdot))_+ = \sup_{x\in\Omega} \alpha p(x) < n$. Let $\theta_1(x)$ and $\theta_2(x)$ be measurable functions on $\mathbb{R}_+$. Suppose that there exists a positive number a such that, for all $x > a$, we have $\theta_1(x) = \overline{\theta}_1 = \text{const}$, $\theta_2(x) = \overline{\theta}_2 = \text{const}$, and $1 < \theta_1 \leqslant \tilde{\theta}_1(x) \leqslant \theta_2(x) \leqslant \Theta_2 < \infty$ almost everywhere. Let the functions $p_1(x)$ and $p_2(x)$ satisfy the equality $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\alpha(x)}{n}$ and let the functions w_1 and w_2 meet the condition*

$$T = \sup_{x\in\Omega,t>0} \int_0^t (w_2(x,r))^{\theta_2(r)} \Big(\int_t^\infty \Big(\frac{s^{\alpha-1}}{w_1(x,s)} \Big)^{[\tilde{\theta}_1(r)]'} ds \Big)^{\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}} dr < \infty. \tag{8}$$

Then the operators I_α and M_α from $GM_{p_1(\cdot),\theta_1(\cdot),w_1(\cdot))}(\Omega)$ to $GM_{p_2(\cdot),\theta_2(\cdot),w_2(\cdot)}(\Omega)$ are bounded.

Proof. Using the definition and results from [12] (see also Theorem 1.5), we have

$$\|I^\alpha\|_{GM_{q(\cdot),\theta_2(\cdot),w_2(\cdot)}(\Omega)} = \sup_{x\in\Omega} \Big\| w_2(x,r) r^{-\eta_q(x,r)} \|I_\alpha f\|_{L_{q(\cdot)}(B(x,r))} \Big\|_{L_{\theta_2(\cdot)}(0,\infty)}$$

$$\leqslant C \sup_{x\in\Omega} \Big\| w_2(x,r) \int_r^\infty t^{-\eta_q(x,t)-1} ||f||_{L_{p(\cdot)}(B(x,t))}\, dt \Big\|_{L_{\theta_2(\cdot)}(0,\infty)}.$$

Denote

$$\tilde{H}_{v,u}f(r) = v(r)\int_r^{\infty} g(t)u(t)\,dt,$$

where $v(r) = w_2(x,r)$, $g(t) = \frac{w_1(x,t)}{t^{\eta_p(x,t)}}\|f\|_{L_{p(\cdot)}(B(x,t))}$, and $u(t) = \frac{t^{\eta_p(x,t)-\eta_q(x,t)-1}}{w_1(x,t)}$ for every fixed $x \in \Omega$. Then condition (2) has the form (8), from which the boundedness of the operator $\tilde{H}_{v,w}f(r)$ acting from $L_{\theta_1(\cdot)}(0,\infty)$ to $L_{\theta_2(\cdot)}(0,\infty)$ follows. Consequently, we have

$$\|I^{\alpha}f\|_{GM_{q(\cdot),\theta_2(\cdot),w_2(\cdot)}(\Omega)} \leqslant T \cdot \sup_{x\in\Omega}\Big\|w_1(x,t)t^{-\eta_p(x,t)}\|f\|_{L_{p(\cdot)}(B(x,t))}\Big\|_{L_{\theta_1(\cdot)}(0,\infty)} = T\cdot\|f\|_{GM_{p(\cdot),\theta_1(\cdot),w_1(\cdot)}}.$$

This means that the operator I^{α} from $GM_{p_1(\cdot),\theta_1(\cdot),w_1(\cdot)}$ to $GM_{p_2(\cdot),\theta_2(\cdot),w_2(\cdot)}$ is bounded. The theorem is proved. □

Corollary 2.4. *Let $p(\cdot) \in P_{\infty}^{\log}(\Omega)$ and $w_1(x,r) = w_2(x,r) = r^{\beta(x)}$. If*

$$\sup_{x\in\Omega, r>0}(\alpha - \beta(x) - 1)[\tilde{\theta}_1(r)]' < -1 \tag{9}$$

and

$$\sup_{x\in\Omega,t>0}\int_0^t r^{\theta_2(r)\beta(x)}\frac{t^{[[\alpha-\beta(x)-1][\tilde{\theta}_1(r)]'+1]\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}}}{[\beta(x)+1-\alpha][\tilde{\theta}_1(r)]'-1}\,dr < \infty \tag{10}$$

then the operators I_{α} and M_{α} from $GM_{p(\cdot),\theta_1(\cdot),r^{\beta}}(\Omega)$ to $GM_{p(\cdot),\theta_2(\cdot),r^{\beta}}(\Omega)$ are bounded.

Proof. Condition (8) takes the form

$$\sup_{x\in\Omega,t>0}\int_0^t r^{\theta_2(r)\beta(x)}\Big(\int_t^{\infty} s^{(\alpha-1-\beta(x))[\tilde{\theta}_1(r)]'}\,ds\Big)^{\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}}\,dr < \infty.$$

By the convergence of the inner integral, we deduce conditions (9) and (10). □

In the following theorem, $\alpha(x)$ is a variable exponent.

Theorem 2.5. *Assume that $p(\cdot) \in \mathbb{P}_{\infty}^{\log}(\Omega)$, the function $\alpha(x)$ satisfies the condition $\alpha(x) > 0$ and $(\alpha(\cdot)p(\cdot))_+ = \sup_{x\in\Omega}\alpha(x)p(x) < n$. Let $\theta_1(x)$ and $\theta_2(x)$ be measurable functions on $\mathbb{R}_+$. Suppose that there exists a positive number a such that, for all $x > a$, we have $\theta_1(x) = \overline{\theta}_1 = \mathrm{const}$, $\theta_2(x) = \overline{\theta}_2 = \mathrm{const}$, and $1 < \theta_1 \leqslant \tilde{\theta}_1(x) \leqslant \theta_2(x) \leqslant \Theta_2 < \infty$ almost everywhere. Let the functions $p_1(x)$ and $p_2(x)$ satisfy the equality $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\alpha(x)}{n}$ and let the functions w_1 and w_2 meet the condition*

$$T = \sup_{x\in\Omega,t>0}\int_0^t (w_2(x,r))^{\theta_2(r)}\Big(\int_t^{\infty}\Big(\frac{s^{\alpha(x)-1}}{w_1(x,s)}\Big)^{[\tilde{\theta}_1(r)]'}ds\Big)^{\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}}\,dr < \infty.$$

Then the operators $\frac{1}{(1+|x|)^{\gamma(x)}}I^{\alpha(\cdot)}$ and $\frac{1}{(1+|x|)^{\gamma(x)}}M^{\alpha(\cdot)}$ acting from the space $GM_{p_1(\cdot),\theta_1(\cdot),w_1(\cdot))}(\Omega)$ to the space $GM_{p_2(\cdot),\theta_2(\cdot),w_2(\cdot)}(\Omega)$ are bounded.

Proof. The proof of this theorem is the same as that of Theorem 2.3: It is sufficient to substitute $\frac{1}{(1+|x|)^{\gamma(x)}}I^{\alpha(\cdot)}f(x)$ for $I^{\alpha}f(x)$. □

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