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On Strong Solutions of Stochastic Differential Equations and Their Trajectory Analogs

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Abstract—We find new conditions for existence of strong solutions of ordinary differential equations with random right-hand side, stochastic differential equations with measurable random drift, and their trajectory analogs with symmetric integrals. We show that solutions of Itô equations satisfy a parabolic equation along trajectories of a Wiener process.

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1. INTRODUCTION

Let W be a standard Wiener process on a filtered probability space $\big(\Omega, F, (F_t)_{t\in[t_0,T]}, \mathbb{P}\big)$. We denote by $\mathcal{P}$ the sigma-algebra of predictable sets and by $B(D)$, where $D \subseteq \mathbb{R}^n$, the sigma-algebra of Borel subsets of D.

In stochastic analysis, two types of ordinary stochastic differential equations (SDEs) are usually considered. They are

(1) Itô equations

$$\xi(t) - \xi(t_0) = \int_{t_0}^{t} \sigma\big(s, W(s), \xi(s)\big)\, dW(s) + \int_{t_0}^{t} b\big(s, W(s), \xi(s)\big)\, ds, \tag{1.1}$$

where σ and b are measurable functions with respect to the σ-algebra $\mathcal{P} \times B(\mathbb{R}^2)$, $\sigma(s,v,x) = \sigma(\omega,s,v,x)$, $b(s,v,x) = b(\omega,s,v,x)$, and the first summand on the right-hand side is the stochastic Itô integral;

(2) Stratonovich equations

$$\xi(t)-\xi(t_0) = \int_{t_0}^{t} \sigma\big(s, W(s), \xi(s)\big) * dW(s) + \int_{t_0}^{t} B\big(s, W(s), \xi(s)\big)\, ds, \tag{1.2}$$

where the first summand on the right-hand side is the stochastic Fisk—Stratonovich integral. If σ is a “smooth” diffusion coefficient then equation (1.1) can be rewritten in the form with the Stratonovich integral.

The above equations are related to trajectory (deterministic) equations with symmetric integrals (see [9, 10]) of a similar form

$$\xi(t) - \xi(t_0) = \int_{t_0}^{t} \sigma\big(s, X(s), \xi(s)\big) * dX(s) + \int_{t_0}^{t} B\big(s, X(s), \xi(s)\big)\, ds, \tag{1.3}$$

where X is a continuous function (or a realization of a stochastic process with continuous trajectories). It is essential that equation (1.3) is deterministic. In stochastic analysis, it is a trajectory equation;

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hence, its coefficients need not be predictable. The following fact is important: If X is a typical trajectory of a Wiener process W then, within the framework of the Itô formula (to be more precise, a version of the Itô–Wentzell formula), the symmetric integral coincides with the Stratonovich integral with probability 1; therefore, we may use the same notation for such integrals.

We present necessary information on symmetric integrals. Let $\mathbb{R} = (-\infty, +\infty)$. We consider partitions T_n, $n \in \mathbb{N}$, of the segment $[0, t]$, where

$$T_n = \{t_k^{(n)}\}, \ \ 0 = t_0^{(n)} \le t_1^{(n)} \le \cdots \le t_k^{(n)} \le \cdots \le t_{m_n}^{(n)} = t, \ \ n \in \mathbb{N},$$

such that $T_n \subset T_{n+1}$, $n \in \mathbb{N}$, and

$$\lambda_n = \max_k \left| t_k^{(n)} - t_{k-1}^{(n)} \right| \to 0 \ \text{ as } \ n \to \infty.$$

We denote by $X^{(n)}$ the function on $[0, t]$ constructed from X according to T_n.

An integral

$$\int_0^t f\big(s, X(s)\big) * dX(s) = \lim_{n\to\infty} \int_0^t f\big(s, X^{(n)}(s)\big) \big(X^{(n)}\big)'(s) ds$$

is said to be *symmetric* if the limit on the right-hand side of the equality exists and is independent of the choice of the partition T_n, $n \in \mathbb{N}$.

The following *condition* (S) is necessary and sufficient for the existence of a symmetric integral. Consider a pair of functions X and f, where $X = X(s)$ with $s \in \mathbb{R}^+$ and $f = f(s, u)$ with $s \in \mathbb{R}^+$ and $u \in \mathbb{R}$. We say that this pair *satisfies condition* (S) *on* $[0, t]$ if

(a) the function X is continuous on $[0, t]$;

(b) for almost all u, the function $f = f(s, u)$ is a function of bounded variation for $s \in [0, t]$ and is right-continuous with respect to $s \in [0, t]$;

(c) for almost all u, we have

$$\int_0^t \mathbf{1}\big(X(s) = u\big) |f|(ds, u) = 0,$$

where $|f|(s, u)$ denotes, for each u, the total variation of the function $f = f(\tau, u)$ with respect to the variable τ on $[0, s]$;

(d) the total variation $|f|(t, u)$ of the function $f = f(s, u)$ with respect to the variable s on $[0, s]$ is locally summable with respect to u.

If $F = F(s, u)$ is a function whose partial derivatives F'_s and F'_u are continuous with respect to u and the pair $\big(X, F'_u\big)$ of functions satisfies condition (S) on $[0, t]$ then the following equality holds:

$$F\big(t, X(t)\big) - F\big(0, X(0)\big) = \int_0^t F'_u\big(s, X(s)\big) * dX(s) + \int_0^t F'_s\big(s, X(s)\big)\, ds.$$

If $X(t) = W(t, \omega)$, the latter formula coincides with the formula for the stochastic Itô differential with the Stratonovich integral.

We fix a probability space $\big(\Omega, F, (F_t)_{t\in[t_0,T]}, \mathbb{P}\big)$ and a Wiener process W. Recall (see [2, 4]) that a solution of SDE (1.1) is said to be *strong* if it is continuous with probability 1, agrees with the filtration (F_t), and substitution of the solution into equation (1.1) leads to a valid equality with probability 1. The definitions of strong solutions of equations (1.2) and (1.3) are similar.

We describe the results of the present article. First, we consider conditions for existence of strong solutions of SDEs (1.1) and (1.2) with measurable random drift. The methods presented below are based on the structure of solutions of SDEs and allow us to prove existence results for strong solutions of equation (1.3), where X is a stochastic process with continuous trajectories that agrees with (F_t). The problem on finding existence conditions for strong solutions of ODEs with random right-hand side and SDEs attracted the attention of many researchers (see, for example, [7, 12, 13]). We also mention surveys on this topic [2, 11] with detailed references.

Second, we succeeded to understand better the well-known connection between a solution of a Itô SDE and the mathematical expectation of such a solution (which is a solution of a parabolic equation). We show that solutions of SDE (1.1) satisfy with probability 1 a parabolic equation along trajectories of a Wiener process. This assertion is a consequence of the Itô formula, i.e., the (trajectory based) connection with parabolic equations is already built in the definition of the Itô integral.

2. MAIN RESULTS

1. We present necessary information on ordinary differential equations (ODEs). We consider the Cauchy problem

$$y' = f(t, y), \ \ t \in [t_0, T], \ \ y(t_0) = y_0. \tag{2.1}$$

By Peano's theorem (see [3]), this problem is locally solvable if f is a continuous function. For discontinuous functions, a generalization of the notion of a "solution" is needed. Such a generalization was suggested by Carathéodory (see [6]).

A *Carathéodory solution* of problem (2.1) is an absolutely continuous function y such that

$$y(t) - y_0 = \int_{t_0}^{t} f\big(s, y(s)\big)\, ds, \ \ t \in [t_0, T].$$

Assume that the following Carathéodory conditions hold:

- for almost all t, the function $f = f(t, y)$ is defined and continuous with respect to y;
- for every y, the function $f = f(t, y)$ is measurable with respect to t;
- for every bounded interval of the t-axis, the inequality $\big|f(t, y)\big| \le n(t)$ holds, where n is a Lebesgue integrable function.

As is known, they imply existence of a Carathéodory solution of problem (2.1). The following fact is essential for further considerations: This solution y is constructed as the uniform limit of a sequence $\{y_k\}$ of functions, where y_k is defined on $\big[t_0 + ih, t_0 + i(h+1)\big]$ with $h = \frac{T-t_0}{k}$ by the relation

$$y_k(t) - y_0 = \int_{t_0}^{t} f\big(s, y_k(s-h)\big)\, ds \tag{2.2}$$

and $y_k(t) = y_0$ for $t \le t_0$.

By a *differential inclusion* we mean a relation of the form $y'(t) \in F(t, y)$, where F is a multivalued mapping that takes each point $(t, y) \in [t_0, T] \times \mathbb{R}^n$ into a set $F(t, y)$ (here by $y = y(t)$ we mean an unknown function). If, for each point (t, y), the set $F(t, y)$ is a singleton then the corresponding differential inclusion is an ODE. A function y is called a *solution* if it is absolutely continuous and satisfies the relation $y'(t) \in F(t, y)$ almost everywhere. Such a function y is also called a Filippov solution of problem (2.1). It is clear that differential properties of a solution y depend on the properties of the multivalued mapping.

In the sequel, by $F(t, y)$ we denote the least closed convex set containing all values $f(t, y^*)$ as y^* tends to y ranging over a neighborhood of y (except for a set of zero measure). By Theorems 8 and 9 in [6, Ch. 2, Sec. 7], it is possible to construct a solution as the limit of Carathéodory solutions of (2.1) even if f is not continuous with respect to y. Instead, we require that the following conditions be valid:

- the function $f = f(t, y)$ is measurable and almost everywhere satisfies the inequality

$$\big|f(t, y)\big| \le n(t) \ \text{ with a summable function } n. \tag{2.3}$$

The method of the proof is important for the further study. A Filippov solution is constructed as the limit of a sequence of solutions of ODEs of the form $y' = f_{\varepsilon_m}(t,y)$, $\varepsilon_m \searrow 0$, that are obtained by averaging f with respect to the spatial variable, i.e., we have

$$f_\varepsilon(t,y) = \frac{1}{2\varepsilon}\int_{\{|y-x|<\varepsilon\}} f(t,x)\,dx.$$

The above-mentioned Carathéodory's theorem can be applied to these functions.

2. We consider a Wiener process W on a probability space $\big(\Omega, F, (F_t)_{t\in[t_0,T]}, \mathbb{P}\big)$. The main aim of this subsection is to study conditions for existence of a solution of an ODE with random right-hand side of the form

$$y' = f\big(t, W(t), y + W(t)\big),\ \ t \in [t_0, T],\ \ y(t_0) = y_0, \tag{2.4}$$

where f is a random $\mathcal{P}\times B(\mathbb{R}^2)$-measurable function, $f(t,v,y) = f(\omega,t,v,y)$. We are also interested in conditions for existence of strong solutions of equation (2.4).

Let $\widetilde{f}(t,y) = f\big(t, W(t), y + W(t)\big)$. Assume that $\widetilde{f}$ satisfies conditions (2.3) with probability 1 for almost all t. By the above, there exists a Filippov solution of equation (2.4) for almost all ω. It remains to clarify when this solution agrees with F_t. Analyzing the proofs of Carathéodory's and Filippov's theorems and taking into account the fact that the limit of a sequence of measurable functions is again a measurable functions, we find that solutions in Filippov's theorem are constructed in two stages. First, a sequence of functions y_k is constructed according to (2.2) with the averaged right-hand side $f_\varepsilon(t,y)$. The uniform limit of these functions is a Carathéodory solution y_ε. Second, the limit is taken of a subsequence y_{ε_m}, where ε_m tends to zero, which leads to a Filippov solution of the initial equation. If, at the first stage of construction of solutions with the use of formula (2.2), the function on the right-hand side is measurable with respect to some σ-algebra (for example, F_t-measurable) then so is the constructed Filippov solution.

Theorem 1. *Let*

$$\widetilde{f}(t,y) = f\big(t, W(t), y + W(t)\big)$$

and let $\widetilde{f}$ be a $\mathcal{P}\times B(\mathbb{R})$-measurable function satisfying conditions (2.3) *with probability* 1 *for almost all t. Then there exists a strong solution of equation* (2.4).

Remark. In [13], a similar assertion was proven for ODEs with $f\big(t, y + W(t)\big)$ on the right-hand side, where f is a bounded function that is not random.

3. Filippov's theorem allows us to find simple sufficient conditions for existence of solutions of equations with symmetric integrals of the form (1.3).

As is known $\big($see [1, 10]$\big)$, a solution of equation (1.3) $\big($and, consequently, of the corresponding Stratonovich equation of the form (1.2) with $X(t) = W(t)\big)$ can be found in the form $\xi(t) = \varphi^*\big(t, X(t), C(t)\big)$, where the partial derivatives $(\varphi^*)'_t$, $(\varphi^*)'_v$, and $(\varphi^*)'_u$ of the function $\varphi^* = \varphi^*(t,v,u)$ are continuous and satisfy the following two conditions for almost all t:

$$(\varphi^*)'_u\big(t, X(t), C(t)\big) = \sigma\Big(t, X(t), \varphi^*\big(t, X(t), C(t)\big)\Big), \tag{2.5}$$

$$\Big(\varphi^*\big(t, v, C(t)\big)\Big)'_t\Big|_{v=X(t)} = B\Big(t, X(t), \varphi^*\big(t, X(t), C(t)\big)\Big). \tag{2.6}$$

We present simple arguments justifying this claim. We rewrite equation (1.3):

$$\Big[(\varphi^*)'_u\big(t, X(t), C(t)\big) - \sigma\Big(t, X(t), \varphi^*\big(t, X(t), C(t)\big)\Big)\Big] * dX(s)$$
$$+ \Big[\Big(\varphi^*\big(t, v, C(t)\big)\Big)'_t\Big|_{v=X(t)} - B\Big(t, X(t), \varphi^*\big(t, X(t), C(t)\big)\Big)\Big]\,dt = 0.$$

Taking into account results of [1], we conclude that conditions (2.5) and (2.6) hold.

We find functions φ^* and C. We consecutively solve two equations. First, for each t, we find a general solution of the equation

$$(\varphi^*)'_v = \sigma(t, v, \varphi^*) \tag{2.7}$$

in the form $\varphi^*\big(t, v, C(t)\big)$, where $C = C(t)$ is an arbitrary "constant" that depends on t. According to [3, pp. 158–160], it is sufficient to require that the partial derivative σ'_φ of $\sigma = \sigma(t, v, \varphi)$ be continuous for all t and v. Then condition (2.5) holds. Second, we find a solution of the Cauchy problem for an ODE of the form

$$C'(t) = f\big(t, C(t)\big), \;\; \varphi^*\big(t_0, X(t_0), C(t_0)\big) = \xi_0, \;\; t \in [t_0, T], \tag{2.8}$$

where

$$f(t, c) = \frac{B\Big(t, X(t), \varphi^*\big(t, X(t), c\big)\Big) - (\varphi^*)'_t\big(t, X(t), c\big)}{\sigma\Big(t, X(t), \varphi^*\big(t, X(t), c\big)\Big)}.$$

Equation (2.8) is obtained from (2.6) with the use of (2.5). A similar (but different) approach to construction of solutions of SDEs can be found in [5]; yet another method for solving SDEs is presented in [8].

Theorem 2. *Assume that the following conditions hold*:

- *there exists a general solution of equation* (2.7);
- *the function f on the right-hand side of equation* (2.8) *satisfies the assumptions of Filippov's theorem.*

Then there exists a solution of equation (1.3).

Theorem 1 allows us to find simple sufficient conditions for existence of solutions of Stratonovich and Itô equations.

Corollary 1. *If the conditions of Theorem* 2 *are satisfied with probability* 1*, where σ is a deterministic Borel function and B is a $\mathcal{P} \times B(\mathbb{R}^2)$-measurable function, then there exists a strong solution of equation* (1.2).

Indeed, neither equation (2.7) nor the function φ^* is random and solutions of equation (1.2) have the form $\xi(t) = \varphi^*\big(t, W(t), C(t)\big)$, where C is a random function that agrees with (F_t).

If σ is a smooth function then it is possible to rewrite the corresponding Itô equation in the form (1.2). Hence, the following assertion is valid.

Corollary 2. *Assume that the conditions of Corollary* 1 *of Theorem* 2 *are satisfied, the function σ is continuous, and the partial derivatives σ'_u and σ'_φ are continuous too. Then there exists a strong solution of equation* (1.1).

Remark 1. Assume that, in equation (1.3), the function σ is of the (simpler) form $\sigma(t, \varphi)$. Then equation (2.7) becomes a separable ODE. Its general solution has the form $\varphi^* = \varphi^*\big(t, v + C(t)\big)$; hence, solutions of equation (1.3) are of the form

$$\xi(t) = \varphi^*\big(t, X(t) + C(t)\big). \tag{2.9}$$

Remark 2. As is shown above, for SDEs (1.1) and (1.2) with a deterministic coefficient $\sigma = \sigma(t, v, x)$, strong solutions of the equations are of the form

$$\xi(t) = \varphi^*\big(t, W(t), C(t)\big), \tag{2.10}$$

i.e., each solution is a deterministic function that depends on the Wiener process W and a smooth random function C. In the case of a Markov process, solutions assume the (simpler) form (2.9). Therefore, we may call the function $\varphi = \varphi^*(t, v, u)$ the *structure of a solution* of SDE (1.2). Thus, each solution of equation (1.2) is a deterministic function of a Wiener process with a random drift $C = C(t, \omega)$ (we call it a *small drift*).

4. The structure of a solution of SDE (1.1) allows us to clarify the connection (see [2, 4] for more detail) between solutions of SDEs and parabolic equations. In the simplest case, this connection can be formulated as follows: The mathematical expectation $u = u(t, x) = \mathbb{E}_x\xi(t)$ is a solution of the parabolic equation $u_t = \sigma^2 u_{xx} + bu_x$. The following assertion shows that solutions (2.10) of equation (1.1) themselves satisfy a parabolic equation along trajectories of the Wiener process W.

Theorem 3. *Let the conditions of Corollary* 2 *of Theorem* 2 *hold for equation* (1.1). *Then there exist a strong solution* ξ *of equation* (1.1) *with* $\xi(t) = \varphi^*\big(t, W(t), C(t)\big)$ *and the random function* ϕ*, where*

$$\phi\big(t, W(t)\big) \equiv \varphi^*\big(t, W(t), C(t)\big),$$

satisfies the condition

$$\phi'_t\big(t, W(t)\big) = -\frac{1}{2}\phi''_{uu}\big(t, W(t)\big) + b\Big(t, W(t), \phi\big(t, W(t)\big)\Big), \quad t \in [t_0, T], \tag{2.11}$$

with probability 1.

Proof. We rewrite equation (1.1) with the Stratonovich integral. We obtain equation (1.2). Only the second summand is changed; namely, we have

$$B(s, v, x, \omega) = b(s, v, x, \omega) - \frac{1}{2}\big[\sigma'_x(s, v, x)\sigma(s, v, x) + \sigma'_v(s, v, x)\big]. \tag{2.12}$$

Formula (2.12) is a consequence of the formula that connects the Itô and Stratonovich integrals:

$$\int_0^t f\big(s, W(s)\big) * dW(s) = \int_0^t f\big(s, W(s)\big)\, dW(s) + \frac{1}{2}\int_0^t f'_v\big(s, W(s)\big)\, ds,$$

where $f = f(\omega, s, v)$ is a $\mathcal{P} \times B(\mathbb{R}^2)$-measurable function such that the derivative f'_v is continuous with probability 1 and the Itô and Stratonovich integrals make sense for f.

We differentiate the equality $\phi'_v(t, v) = \sigma\big(t, v, \phi(t, v)\big)$ with respect to v. Taking (2.5) into account, we obtain

$$\phi''_{vv}(t, v) = \sigma'_\phi\big(t, v, \phi(s, v)\big)\sigma\big(s, v, \phi(s, v)\big) + \sigma'_v\big(t, v, \phi(s, v)\big).$$

In view of the latter equality and formula (2.6), we may rewrite relation (2.12) in the form of equation (2.11). □

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