Algebraic Interpretation of Image Analysis Operations

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Abstract—The study is devoted to mathematical and functional/physical interpretation of image analysis and processing operations used as sets of operations (ring elements) in descriptive image algebras (DIA) with one ring. The main result is the determination and characterization of interpretation domains of DIA operations: image algebras that make it possible to operate with both the main image models and main models of transformation procedures that ensure effective synthesis and realization of the basic procedures involved in the formal description, processing, analysis, and recognition of images. The applicability of DIAs in practice is determined by the realizability—the possibility of interpretation—of its operations. Since DIAs represent an algebraic language for the mathematical description of image processing, analysis, and understanding procedures using image transformation operations and their representations and models, the authors consider an algebraic interpretation. These procedures are formulated and implemented in the form of descriptive algorithmic schemes (DAS), which are correct expressions of the DIA language. The latter are constructed from the processing and transformation of images and other mathematical operations included in the corresponding DIA ring. The mathematical and functional properties of DIA operations are of considerable interest for optimizing procedures of processing and analyzing images and constructing specialized DAS libraries. Since not all mathematical operations have a direct physical equivalent, the construction of an efficient DAS for image analysis involves the problem of interpreting operations for DAS content. Research into this problem leads to the selection and study of interpretation domains of DIA operations. The proposed method for studying the interpretability of DIA operations is based on the establishment of correspondence between the content description of the operation function and its mathematical realization. The main types of interpretability are defined and examples given of the interpretability/uninterpretability of operations of a standard image algebra, which is a restriction of the DIA with one ring.

Keywords: vision, computer vision, descriptive image algebras, algebraic interpretation, physical, semantic, and functional interpretability of image processing operations, interpretation domains of operations, interpretability of operations, image analysis, recognition, and processing automated image-mining

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INTRODUCTION

The article is devoted to mathematical and functional/physical interpretation of image analysis and processing operations used as sets of operations (ring elements) of descriptive image algebras (DIAs) with one ring (DIA1R) [4, 5, 7].

In mathematics, interpretation refers to the assigning of values (meaning) of mathematical expressions (symbols, formulas, etc.). These values are mathematical objects (sets, operations, expressions, etc.). The values themselves are also called interpretations of the corresponding expressions.

DIAs are studied in the framework of developing a mathematical apparatus for analyzing and evaluating information in the form of images. For a structured description of possible algorithms for solving these problems, a formal tool is needed to describe and validate the chosen solution path. For the formalization, an algebraic apparatus was chosen [6, 9] that should ensure the uniformity of procedures for describing image objects and transformations over these image objects.

In the late 1980s and 1990s, I.B. Gurevich [6] specialized a general algebraic approach to solving recognition, classification, and prediction problems [10] (Yu.I. Zhuravlev) in the case of initial data in the form of images (Descriptive Approach to Image Analysis and Understanding—DA).

I.B. Gurevich introduced DIAs in the framework of the DA and continues to develop them in collaboration with his pupils and disciples [4, 5, 7]. In order to construct a DIA, it is necessary to select the operations and operands of the algebra. Some transformations in image processing, analysis, and recognition can formally be used for mathematical description of the algorithm using DIAs; however, they have no physical meaning specific to image processing and

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analysis. At the same time, the practical applicability of DIAs is determined by the practical applicability and realizability of operations by which DIAs are constructed.

In our case, we are talking mainly about algebraic interpretation, since DIAs represent an algebraic language for the mathematical description of procedures of processing, analyzing, recognizing, and understanding images using digital image transformation operations and their representations and models.

These procedures are formed and implemented as descriptive algorithmic schemes (DASs) [6], which are correct (valid) expressions of the DIA language. The latter are constructed from image processing and transformation operations and other mathematical operations included in the corresponding DIA ring.

The mathematical and functional (content/semantic) properties of DIA operations are of considerable interest for optimizing the selection and implementation of image processing and analysis procedures and for constructing specialized DAS libraries.

The choice and optimization of operations included in the DAS are essentially related to the specifics of images as a means of representation, carriers, and sources of information.

The functional interpretation of image transformation operations should ensure the establishment of a relationship between the image analysis task and the DAS specialized for its solution. In essence, this kind of interpretation is reduced to establishing a correspondence between the content division of the decision process into stages and the mathematical operations of the DAS ensuring the realization of these stages.

The functional interpretation is based on the specific information properties of the image discussed below.

The image is by its nature an object with a complex information structure that reproduces information about an original scene using brightness values of discrete image elements—pixels, configurations of image fragments, sets of pixels, and spatial and logical relations between configurations, sets of pixels, and individual pixels. An image differs from other means of data representation by the highly informative, clear, structured, and natural way in which a person perceives its content.

An image is a mixture of initial (raw, "real") data, their realizations, and certain deformations. Realizations (as well as the corresponding descriptions) reflect the informational and physical nature of objects, events, and processes reproduced by the image, and deformations are generated by the technical characteristics of the tools with which the image is captured, recorded, formed, and transformed in the construction of its digital representations and the hierarchy of their descriptions.

In image recognition, mathematical problems arise in relation to the formal description of an image as an object of analysis. Therefore, the internal structure, overall structure, and content of an image are essentially used as a result of operations by which an image can be constructed from primitive elements and objects detected in the image at various stages of working with it.

Image processing and recognition, including the construction of an image model, can be viewed as the realization of an image transformation system.

This means that the efficiency of model synthesis and recognition processes can be achieved by the choice of "content" (function) of image transformation operations, based on what image representation needs to be obtained with the next transformation. Such a choice, in turn, should be based both on the analysis of the mathematical characteristics of the operation and on the analysis of its functional purpose, in other words, the semantic aspects of the operation, i.e., its content, identification of a "physical equivalent," and its underlying functional heuristics.

Since not all mathematical operations have a direct physical equivalent with respect to the construction of effective DASs for image analysis, there is the problem of interpreting operations for filling the DAS. Research into this problem leads to the selection and study of interpretation domains of DIA operations.

Thus, interpretation is considered as a transition from a meaningful description of the operation to its mathematical or algorithmic realization. As a result, the practical applicability of operations is revealed in the context of the more general concept of interpretability.

The following sections of the article present results related to the interpretability of DIA1R operations and examples of domains of interpretability for certain types of operations.

The article consists of the Introduction, four sections, the Conclusion, and References.

In Section 1, "Descriptive Images Algebras with One Ring," the main specifics of DIAs and DIA1R are determined, from which the interpretability of operations is formalized and specified.

Section 2, "Types of Interpretability of Operations of Descriptive Image Algebras," describes the method and tools for formalizing the types of interpretability of image analysis and processing operations. To characterize the interpretability of DIA operations, the following concepts are introduced: (1) physical meaning of the operation, (2) physical interpretability in the context of image analysis and processing, (3) visual interpretability in the context of image analysis and processing, (4) weak physical interpretability, and (5) strong physical interpretation.

Section 3, "Examples of Interpretability/Uninterpretability of Descriptive Image Algebra Operations", provides 18 examples of operands with operations, for which the interpretability is studied.

Section 4 "Interpretation of an Algorithmic Scheme for Solving the Problem of Morphological Analysis of Cell Nuclei in the Lymphatic System" gives an example of constructing a DAS in the DIA1R language to solve an applied problem, demonstrating an algorithmic interpretation in comparison to those described in Section 3.

1. DESCRIPTIVE IMAGE ALGEBRAS WITH ONE RING

In this section, let us briefly recall the basic properties of DIAs.

The algebraization of pattern recognition and image analysis was devoted to creating a universal language for the uniform description of images and transformations over them. In [4, 7], the algebraization stages of pattern recognition and image analysis are described in detail and the basic concepts for defining DIAs and DIA1R are introduced.

The most significant results of the initial stage of pattern recognition algebraization were Yu.I. Zhuravlev's algebras of algorithms [10] and U. Grenander's image theory [2]; in image analysis, S. Sternberg's image algebra [8] and G. Ritter's standard image algebra.

The classical algebra was developed to generalize operations on numbers; however, direct application of an algebra to information in the form of images is not possible for all problems, and a simple interpretation of the results is not always admissible. There are many natural image transformations that are easily interpreted from the user's viewpoint (e.g., rotation, compression, stretching, color inversion), which are difficult to imagine using standard algebraic operations. It becomes necessary to combine the algebraic apparatus and the set of image analysis and processing transformations.

To solve this problem, DIAs were proposed [4, 7].

The basic definitions and properties of DIAs are as follows:

The purpose of introducing DIAs [7]: DIAs are designed for the combination, standardization, and unification of algorithmic procedures for processing representations and models of images and transformations over them.

DIA Tools [7]:

(a) DIA operands are representations and models of images [6] (including source images) and transformations of images and their representations and models;

(b) DIA operations are transformations of analysis and image processing, standard algebraic operations, algebraic closures, linear combinations, and superpositions of these operations.

DIAs allow the use of image transformation procedures not only as DIA operations, but also as operands for constructing combinations of basic models of transformation procedures.

Definition 1 [4, 7]. *An algebra is called a* ***descriptive image algebra*** *if its operands are either representations and models of images (as well, both the image itself and the set of values and characteristics associated with the image can be selected as a model), or operations on images, or simultaneously both.*

Application of DIAs [7]: DIAs are used to describe the tasks, objects, and transformations considered when information is extracted from images. When constructing DAS descriptions for formal description, processing, analysis, and recognition of images using DIAs, each element of the scheme and any transformation used in the scheme is defined by the structures constructed via the application of DIA operations to the set of DIA operands.

Representation of tasks, objects, and transformations considered when extracting information from images, structures constructed by applying DIA operations to a set of nonderivative tasks, nonderived image elements, and basic transformations provides flexibility and standardization in creating and using DAS to extract information from images. With this approach, it is possible to vary the methods for solving subtasks using image analysis operations as DIA elements while preserving the entire scheme of the technology for extracting information from images.

Restrictions on DIA operations [7]: in order to ensure compliance of the DIA with the requirements that must be met by the mathematical object "algebra," it is necessary to introduce restrictions on the basic DIA operations.

The main research into DIAs was aimed at studying DIA1R (see Definition 2), which is by definition a classical algebra with nonclassical operands. When defining a DIA with several rings, the concept of a graded algebra is assumed, and in the case of two rings, the concept of a superalgebra is used.

The subsequent specifics of DIAs are determined by the properties of the algebras.

Definition 2 [4, 7]. *The ring, which is a finite-dimensional vector space over some field, is a* ***DIA1R*** *if its operands are either representations and models of images, or operations on images and their representations and models.*

The ultimate goal in studying DIA1R is to obtain sets of complete systems of operands and DIA operations to describe image analysis tasks. The use of the algebra concept in defining DIA1R in a strictly classical sense is governed by the fact that in this case, it becomes possible to distinguish the basic DIA operations for various types of operands.

2. TYPES OF INTERPRETABILITY OF DESCRIPTIVE IMAGE ALGEBRAS OPERATIONS

A problem arises in constructing a DAS for solving applied image analysis and recognition problems: the applicability of some classes of DIA to describe the corresponding problem [5]. Evaluating the applicability of the DIA leads to the problem of interpretability of DIA operations. The formulation of the problem and initial results are presented in [4].

Recall [6] that, according to the DA, the source image in recognition tasks is called an ordered set of recorded initial spatial and contextual data, reflecting the form (form and state) of objects, events, and processes of the depicted scene and allowing application of transformations that produce an image convenient for recognition.

Definition 3 [4]. ***Physical meaning of the operation*** *means a content description of the process of transforming the source image(s) into the final image(s), or the description of putting a certain set of characteristics into correspondence with the source image.*

In order to preserve the logic of consideration below, let us recall some notions of the DA associated with description of the image processing and analysis process and leading to the definitions of model/image representation [6].

In image processing and analysis, a certain system of transformations is applied to the source image, ensuring a successive change of "phase states" of the transformed image corresponding to the degree of its current "formalization." The set of valid image representations is defined as the set of phase states of the image.

The system of transformations is given by the DAS image representation (DASIR), written according to DA concepts using DIAs. DASIRs reflect methods of sequential and/or parallel application of transformations from a set of transformations to the initial information from the initial data space. The set of admissible DASIRs is defined as the set of phase states of the DASIR.

To ensure the possibility of applying recognition algorithms to the constructed formal image descriptions, it is necessary to use the constructed DASIRs (to establish specific transformations from fixed DIAs and the parameters included in the transformation schemes) and to apply the implemented schemes to the initial data, i.e., construct image representations and models. In the DA, an image model is a formal (symbolic) description of an image that allows recognition algorithms to be applied to it. An image representation is any element of the set of states of the image in the image formalization space, with the exception of the objects "image model" and "image realization."

A more detailed description of the image formalization space, including both the image phase states and the DASIR phase states, is given in [5].

Definition 4 [4]. *An operation on an image(s) or fragments thereof, or on a model(s) of an image(s), or the representation(s) of an image(s) is called a* ***physically interpretable operation in the context of image analysis and recognition*** *if*

(1) *the result of its use is an image or fragments thereof;*

(2) *the result of its application is an image representation or image model that can be used to reconstruct semantically significant geometric objects, brightness characteristics, and configurations formed due to regular repetitions of geometric objects and brightness characteristics of the source image;*

(3) *the result of its application is a characteristic(s) of the image(s), which can be unambiguously compared to the properties of geometric objects, brightness characteristics, or configurations formed due to regular repetitions of geometric objects and brightness characteristics of the source image.*

Definition 5 [4]. *The operation on some objects is called* ***visually interpretable in the context of image analysis and recognition*** *if as a result of the operation, an image(s) is obtained with which it is possible to reconstruct a one-to-one correspondence between semantically significant geometric objects, brightness characteristics, and configurations formed due to regular repetitions of geometric objects and brightness characteristics in the resultant image(s) and in source objects.*

Statement1 [4]. ***A visually interpretable*** *operation is always a* ***physically interpretable*** *operation.*

Corollary [4]. *If the operation is not a physically interpretable operation, then this operation is also not visually interpretable.*

Physical interpretability can be distinguished in a strong and weak sense.

Definition 6 [4]. *An operation is called* ***strongly physically interpretable*** *if it is also visually interpretable.*

Definition 7 [4]. *An operation is called* ***weakly physically interpretable*** *if it is physically interpretable, but not visually interpretable.*

Visually interpretable operations include, e.g., image rotation, image shift, image contrast enhancement, image brightness enhancement, image noise reduction, image smoothing, image contour selection, and other image processing operations. An example of visually interpretable operations can also be image-constructing operations according to a certain specified rule from a set of original objects, e.g., image reconstruction from equations that define the image type.

Physically interpretable operations include certain operations of constructing image representations and models and such operations with images as calculation of the image histogram or the values of the image's statistical characteristics.

Statement 2 [4]. *An operation is* ***physically uninterpretable in the context of image analysis and recognition*** *if*

(1) *its operands are not images, image models, image representations, or image fragments;*

(2) *as a result of application of an operation to the image*(*s*), *an image model*(*s*) *is constructed*, *with which it is not possible to reconstruct semantically significant geometric objects*, *brightness characteristics*, *or configurations arising due to regular repetition of geometric objects and brightness characteristics of the source image*;

(3) *as a result of application of an image operation*, *characteristics are calculated that cannot be unambiguously compared with the properties of geometric objects*, *brightness characteristics*, *or configurations arising due to regular repetition of geometric objects and brightness characteristics of the source image*;

(4) *an operation is not applicable to images*, *image models*, *image representations*, *or image fragments.*

3. EXAMPLES OF INTERPRETABILITY/UNINTERPRETABILITY OF DESCRIPTIVE IMAGE ALGEBRAS OPERATIONS

3.1. Images as Operands of Descriptive Image Algebras with One Ring

3.1.1. Description of Operands

The DA assumes [6] that an image is described by a set of initial information. Let us define the content of this set.

Lemma 1 [6]**:** *The set of initial information* $\{I_0\}$ *consists of two sets* $\{I'\}$ *and* $\{B_0\}$:

(1) *the set of realizations* $I' \in \{I'\}$ *of image I*, *representing the specified object or scene such that* $I' = \{(x, f(x))\}_{x\in D_f}$, *is a set of points x belonging to the definition domain of the realization of image* D_f, *and the set of values f*(*x*) *at each point in the definition domain* D_f;

(2) *semantic and contextual information about the image* $\{B_0\}$.

The definition domain of realization of an image is a subset of an n-dimensional discrete space Z^n. In the case of flat (two-dimensional) images, $n = 2$.

To construct the DIA1R over images, it is necessary to apply some transformations to image realizations in accordance with the contextual and semantic information about the image. Image transformations must satisfy the properties of the algebra [4, 7] and, above all, must be closed with respect to the initial data.

The fact that image processing [6] is the transformation of one image into another using compression, reconstruction, enhancement of visual quality, quantization, and filtering makes it possible to state that image processing operations are closed operations with respect to the set of images, as opposed to the understanding, analysis, classification, and recognition of images. Therefore, to construct DIA1R over images, transformations are applied (as algebra operations) to solve elementary image processing problems [4, 7]. When choosing image processing transformations, an image analysis and processing thesaurus [6] was used to construct DIA1R examples, including, among other things, the classification of image processing methods, operations, and tasks.

3.1.2. Interpretability of Operations of G. Ritter's Standard Image Algebra

G. Ritter's standard image algebra (SIA) is the most detailed of the currently known approaches to describing image operations in terms of algebraic operations on basic operations with a simple structure. It enables the formalization of image processing algorithms at different levels, both for their clear representation and for adaptation to parallel and distributed architectures.

The creation of a new specialized DIA1R has resulted in the study of the existing SIA to determine whether it is possible to use a detailed description of SIA operations to generate a DIA1R.

G. Ritter [9] has introduced SIA operations in our sense for image realizations; i.e. any image I is described by its realization $I' = \{(x, f(x))\}_{x\in D_f}$—a set of points x belonging to the definition domain of image realization D_f and a set of values $f(x)$ at each point in the definition domain D_f. Semantic and contextual information about the image is taken as an empty set: $\{\tilde{B}\} = \varnothing$.

Let us consider a subset of operands and SIA operations (described explicitly; see [9]).

Example 1.

Let F be some field. The elements of ring W are images I assigned by functions of the type $\{(x, a(x)), x \in X, a(x) \in F\}$, where F is a set of values and X is a set of points ($I \in F^X$). Over the elements of ring W, it is possible to introduce operations ⊕, ⊗ and multiplication by a field element. The operations belong to the set of SIA operations.

As an operation of multiplication by a field element, we consider two operations generated by operations in algebraic system F [9]:

For $k \in F$ and $a \in F^X$,

$$k\gamma a = \{(x, c(x)) : c(x) = k\gamma a(x), x \in X\} \quad (1)$$

and

$$A\gamma k = \{(x, c(x)) : c(x) = a(x)\gamma k, x \in X\}. \quad (2)$$

Physical meaning of the operation: operations correspond to multiplication of the image by a number.

Statement 3. *Operations of type* (1) *and* (2) *of multiplication of the image by a number are* ***strongly physically interpretable operations.***

Proof:

1. Physical interpretability of an operation: By Definition 4, an operation is physically interpretable if its application results in an image.

2. Visual interpretability of an operation: this operation is visually interpretable, since when applied to an image, its brightness changes visually (Definition 5).

3. By Definition 6, a physically and visually interpretable operation is a strongly physically interpretable operation.

Q.E.D.

Binary operations on images are generated by operations introduced in algebraic system F. If, e.g., γ is a binary operation on set F; $a, b \in F^X$, then [9]

$$A\gamma b = \{(x, c(x)) : c(x) = a(x)\gamma b(x), x \in X\} \quad (3)$$

Let us consider narrower sets of operands.

Example 2

Let $I_1, I_2 \in (R^X)$.

Replacing γ for specific operations $+, \cdot, \vee$ (operation of taking the maximum), $\wedge$ (operation of taking the minimum), we obtain the following binary operations on real-valued images [9]:

$$I_1 + I_2 = \{(x, c(x)) : c(x) = a(x) + b(x), x \in X\} \quad (4)$$

Physical meaning of the operation: the operation corresponds to the pointwise addition of two images.

$$I_1 \cdot I_2 = \{(x, c(x)) : c(x) = a(x) \cdot b(x), x \in X\} \quad (5)$$

Physical meaning of the operation: the operation corresponds to the pointwise multiplication of two images.

$$I_1 \vee I_2 = \{(x, c(x)) : c(x) = a(x) \vee b(x), x \in X\} \quad (6)$$

Physical meaning of the operation: the operation corresponds to the pointwise taking of the maximum of two images.

$$I_1 \wedge I_2 = \{(x, c(x)) : c(x) = a(x) \wedge b(x), x \in X\} \quad (7)$$

Physical meaning of the operation: the operation corresponds to the pointwise taking of the minimum of two images.

Statement 4. *The operations of pointwise addition* (4), *multiplication* (5), *taking the maximum* (6), *and taking the minimum* (7) *of two images are* ***weakly physically interpretable operations***.

Proof:

1. Physical interpretability of operations: By Definition 4, an operation is physically interpretable if its application results in an image.

2. Visual interpretability of operations: this operation is not visually interpretable, since when applied to arbitrary images, the visual result is unpredictable (Definition 5).

3. By Definition 6, a physically interpretable, but not visually interpretable operation is a weakly physically interpretable operation.

Q.E.D.

Example 3

Let $I_1, I_2 \in (2^F)^X$.

Let be 2^X be a power set, i.e., the set of all subsets of set X. Let also image I be such that $I: X \to 2^F$. In this case, the following binary operations can be introduced [9]:

$$I_1 \cup I_2 = \{(x, c(x)) : c(x) = a(x) \cup b(x), x \in X\} \quad (8)$$

Physical meaning of the operation: the operation corresponds to the operation of the pointwise union of two images.

$$I_1 \cap I_2 = \{(x, c(x)) : c(x) = a(x) \cap b(x), x \in X\} \quad (9)$$

Physical meaning of the operation: the operation corresponds to the pointwise intersection of two images.

Statement 5. *The operations of pointwise union* (8) *and intersection* (9) *of two images are* ***strongly physically interpretable operations***.

Proof:

1. Physical interpretability of operations: by Definition 4, an operation is physically interpretable if its application results in an image.

2. Visual interpretability of operations: this operation is visually interpretable, since at each point there is a union or intersection of the sets that define the images (Definition 5).

3. By Definition 6, a physically and visually interpretable operation is a strongly physically interpretable operation.

Q.E.D.

Example 4.

Let $I_1, I_2 \in (R^{\geq 0})^X$.

Introduce the operation of taking the exponent [9]:

$$I_1^{I_2} = \{(x, c(x)) : c(x) = a(x)^{b(x)}, x \in X\} \quad (10)$$

Physical meaning of the operation: the operation corresponds to the pointwise operation of taking the exponent of one image over another.

Further assertions are given without proof, since they are similar to the previous ones.

Statement 6. *The operation of taking the exponent* (10) *of one image over another is* ***a weakly physically interpretable operation***.

Example 5.

Let $I_1, I_2 \in (R^+)^X$.

The operation of taking the logarithm is introduced [12]:

$$\log_{I_2} I_1 = \{(x, c(x)) : c(x) = \log_{b(x)} a(x), x \in X\} \quad (11)$$

Physical meaning of the operation: the operation corresponds to the pointwise operation of taking the logarithm of one image over another.

Statement 7. *The operation of taking the logarithm* (11) *of one image over another is* ***a weakly physically interpretable operation.***

Example 6.

Let $I_1 \in (F)^X$, $I_2 \in (F)^Y$, X, Y be subsets of the topological space.

We call extension of the image $a \in F^X$ with image $b \in F^Y$ $A/^b(x)$ on the set Y, where X and Y, subsets of some topological space [9]:

$$A/^b(x) = \begin{cases} a(x), & if\_x \in X \\ b(x), & if\_x \in Y \backslash X \end{cases} \quad (12)$$

Concatenation operations for a series of images $a \in F^{Z_m} x^{Z_k}$ and $b \in F^{Z_m} x^{Z_n}$:

$$(a/b) \equiv a\Big|^{b+(0,k)} \quad (13)$$

Using the notion of matrix transposition, concatenation in a column is introduced in the same way [9]:

$$\begin{pmatrix} a \\ b \end{pmatrix} = (a/b)' \quad (14)$$

Physical meaning of the operation: operations correspond to concatenation of images by row and column, respectively.

Statement 8. *Concatenation of images by row and column* (13) *and* (14) *of one image over another are* ***weakly physically interpretable operations***.

Example 7.

Let $I_1, I_2 \in (R^n)^X$.

The following binary operations are introduced:

$$I_1 + I_2 = (I_1^1 + I_2^1, \ldots, I_1^n + I_2^n) \quad (15)$$

Physical meaning of the operation: the operation corresponds to the pointwise addition of images.

$$I_1 \cdot I_2 = (I_1^1 \cdot I_2^1, \ldots, I_1^n \cdot I_2^n) \quad (16)$$

Physical meaning of the operation: the operation corresponds to the pointwise multiplication of images.

$$I_1 \vee I_2 = (I_1^1 \vee I_2^1, \ldots, I_1^n \vee I_2^n) \quad (17)$$

Physical meaning of the operation: the operation corresponds to the pointwise taking of the maximum of images.

$$I_1 \wedge I_2 = (I_1^1 \wedge I_2^1, \ldots, I_1^n \wedge I_2^n) \quad (18)$$

Physical meaning of the operation: the operation corresponds to the pointwise taking of the minimum of images.

Statement 9. *The operations of pointwise addition* (15), *multiplication* (16), *taking the maximum* (17), *and taking the minimum* (18) *of two images are* ***weakly physically interpretable operations***.

For any index j it is possible to introduce the following operations of taking the maximum and minimum of the j component of the image representation:

$$\begin{aligned} I_1 \vee /_j I_2 = \{(x, c(x)) : c(x) = a(x) \\ if \quad a_j(x) \ge b_j(x), \quad else \quad c(x) = b(x)\} \end{aligned} \quad (19)$$

Physical meaning of the operation: the operation corresponds to taking the maximum of the j component of images.

$$\begin{aligned} I_1 \wedge /_j I_2 = \{(x, c(x)) : c(x) = a(x) \\ if \quad a_j(x) \le b_j(x), \quad else \quad c(x) = b(x)\} \end{aligned} \quad (20)$$

Physical meaning of the operation: the operation corresponds to taking the minimum of the j component of images.

Statement 10. *The operations of taking the maximum and minimum of the j component of type* (19), (20) *of two images are* ***weakly physically interpretable operations.***

Let the binary operation γ be such that γ_j: $R^n x R^n \to R$, where $j = 1, \ldots, n$, and is defined as

$$I_1 \gamma I_2 = (I_1 \gamma_1 I_2, \ldots, I_1 \gamma_n I_2) \quad (21)$$

If, e.g. [9], γ_j: $R^n x R^n \to R$ such that $(x_1, \ldots x_n)\gamma(y_1, \ldots, y_n) = \max\{x_i \vee y_j : 1 \le i \le j\}$, then for $I_1, I_2 \in (R^n)^X$ and $I_3 = I_1 \gamma I_2$, the components $c(x) = (c_1(x), \ldots, c_n(x))$ have the values

$$\begin{aligned} c_j(x) = a(x)\gamma_j b(x) = \max\{a_i(x) \vee a_j(x) : 1 \le i \le j\} \\ \text{for} \quad j = 1, \ldots, n. \end{aligned} \quad (22)$$

Statement 11. *A multidimensional binary operation of type* (21), *in particular, of type* (22), *for two images is* ***a weakly physically interpretable operation.***

Let us consider another example of a binary operation γ [9]: let γ_1 and γ_2 be binary operations $R^2 x R^2 \to R$ defined as (23) and (24).

$$(x_1, x_2)\gamma_1(y_1, y_2) = x_1 y_1 - x_2 y_2, \quad (23)$$

$$(x_1, x_2)\gamma_2(y_1, y_2) = x_1 y_2 + x_2 y_1, \quad (24)$$

and if $I_1, I_2 \in (R^2)^X$ are two complex-valued images, then the product $I_3 = I_1 \gamma I_2$ is the complex product

$$\begin{aligned} c(x) = (a_1(x)b_1(x) - a_2(x)b_2(x), \\ a_1(x)b_2(x) + a_2(x)b_1(x)) \end{aligned} \quad (25)$$

Statement 12. *The operation of multiplication of type* (26) *of two images is* ***a weakly physically interpretable operation.***

Example 8.

Let $I_1, I_2 \in (X)^X$.

We introduce the operation of superposition

$$I_1(I_2) = \{(x, a(b(x))), x \in X\} \quad (26)$$

Physical meaning of the operation: definition of one image on a set specified by another image.

Statement 13. *The operation of superposition of two images of type* (26) *is* ***a weakly physically interpretable operation.***

3.1.3. Interpretability of Image Processing Operations

Image processing operations are mainly applied to one image, rather than two; therefore, the choice of binary addition and multiplication operations used to construct the DIA1R is limited.

Example 9.

Let $I_1 = \{((x, y), (r_1(x, y), g_1(x, y), b_1(x, y)))\}_{(x, y) \in X}$, $I_2 = \{((x, y), (r_2(x, y), g_2(x, y), b_2(x, y)))\}_{(x, y) \in X}$, where the functions $r_1(x, y), g_1(x, y), b_1(x, y), r_2(x, y), g_2(x, y), b_2(x, y) \in [0...M - 1]$ take integer values; $M = 256$ is the maximum brightness value of the color components; X is the set of pixels on which images are assigned; let $\alpha \in R$.

On the set of selected operands, the following image processing operations are assigned from the set of arithmetic operations on images.

$$I_1 + I_2 = \{((x, y), ((r_1(x, y) + r_2(x, y)) \bmod M, (g(x, y) + g_2(x, y)) \bmod M, (b_1(x, y) + b_2(x, y)) \bmod M)\}_{(x,y)\in X} \quad (27)$$

Physical meaning of the operation: the operation corresponds to pointwise modulo M addition of the color components of two color images.

$$I_1 \cdot I_2 = \{((x, y), ((r_1(x, y) \cdot r_2(x, y)) \bmod M, (g_1(x, y) \cdot g_2(x, y)) \bmod M, (b_1(x, y) \cdot b_2(x, y)) \bmod M)\}_{(x,y)\in X} \quad (28)$$

Physical meaning of the operation: the operation corresponds to the pointwise modulo M multiplication of the color components of two color images.

$$\alpha \cdot I_1 = \{((x, y), ([\alpha r_1(x, y) \bmod M], [\alpha g_1(x, y) \bmod M], [\alpha b_1(x, y) \bmod M]))\}_{x,y)\in X} \quad (29)$$

Physical meaning of the operation: operation of modulo multiplication M of a field element by the color image.

Statement 14. *The introduced operations of pointwise addition and multiplication of two color images* (27), (28) *are* ***weakly physically interpretable operations.***

Statement 15. *The introduced operation of pointwise multiplication of a color image* (29) *by an element of the f.*

Statement *yield of real numbers is* ***a strongly physically interpretable operation***.

Example 10.

Let grayscale images $J_1 = \{((x, y), gray_1(x, y))\}_{(x, y) \in X}$, $J_2 = \{((x, y), gray_2(x, y))\}_{(x, y) \in X}$, where the functions $gray_1(x, y), gray_2(x, y) \in [0...M - 1]$ take integer values $M = 256$ is the maximum brightness value of the color components; X is the set of pixels on which images are assigned; let $\alpha \in R$.

On the set of selected operands, the following image processing operations are assigned from the set of arithmetic operations over images.

$$J_1 + J_2 = \{((x, y), (gray_1(x, y) + gray_2(x, y)) \bmod M)\}_{(x,y)\in X} \quad (30)$$

Physical meaning of the operation: the operation corresponds to pointwise modulo M addition of two grayscale images.

$$J_1 \cdot J_2 = \{((x, y), ((gray_1(x, y) \cdot gray_2(x, y)) \bmod M)\}_{(x,y)\in X} \quad (31)$$

Physical meaning of the operation: the operation corresponds to the pointwise modulo M multiplication of two grayscale images.

$$\alpha \cdot J_1 = \{((x, y), [\alpha \cdot gray_1(x, y) \bmod M])\}_{x,y)\in X} \quad (32)$$

Physical meaning of the operation: the operation corresponds to the pointwise modulo M multiplication of the grayscale image by the field element.

Statement 16. *The introduced operations of pointwise addition and multiplication of two grayscale images* (30), (31) *are* ***weakly physically interpretable operations.***

Statement 17. *The introduced operation of pointwise multiplication of a grayscale image* (32) *by an element of the field of real numbers is* ***a strongly physically interpretable operation.***

Example 11.

Let $I_1 = \{((x, y), f_1(x, y))\}_{(x, y) \in X}$, $I_2 = \{((x, y), f_2(x, y))\}_{(x, y) \in X}$, where the functions $f_1(x, y)$, $f_2(x, y)$ take any valid values; $X = \{1, ..., n; 1, ..., n\}$ is the set of pixels on which the images are specified. Let square matrices M_1, M_2 describe the whole set of image values I_1, I_2, respectively, given on the square $\{1, ..., n; 1, ..., n\}$ (the matrix elements are functions that define the image at each point of the set X). Let $\alpha \in$ R.

We introduce the following matrix operations on the set of selected operands:

$$I_1 + I_2 = \{((x, y), (M_1 + M_2)(x, y))\}_{(x,y)\in X} \quad (33)$$

Physical meaning of the operation: the operation corresponds to the matrix addition of two images.

$$I_1 \cdot I_2 = \{((x, y), (M_1 \cdot M_2)(x, y))\}_{(x,y)\in X} \quad (34)$$

Physical meaning of the operation: the operation corresponds to the matrix multiplication of two images

$$\alpha \cdot I = \{((x, y), \alpha \cdot M(x, y))\}_{x,y)\in X} \quad (35)$$

Physical meaning of the operation: the operation of multiplication of an image given by a matrix by a real number.

Statement 18. *The introduced operations of matrix addition and multiplication of two images* (33), (34) *are* ***weakly physically interpretable operations***.

Statement 19. *The introduced operation of multiplication of an image* (35) *by an element of the field of real numbers is* ***a strongly physically interpretable operation***.

Example 12.

Let the sets of binary masks of color images be given as follows: $I_{b1} = \{((x, y), (bool_1(x, y), bool_1(x, y), bool_1(x, y)))\}_{(x, y) \in X}$, $I_{b2} = \{((x, y), (bool_2(x, y), bool_2(x,$

$y), bool_2(x, y)))\}_{(x, y) \in X}$, where $bool_1(x, y), bool_2(x, y) \in \{0, 1\}$; let $\alpha \in R$.

On the set of selected operands, the following image processing operations from a set of logical operations over images are defined.

$$I_{b1} + I_{b2} = \{((x, y), ((bool_1(x, y) + bool_2(x, y)) \bmod 2, (bool_1(x, y) + bool_2(x, y)) \bmod 2, (bool_1(x, y) + bool_2(x, y) \bmod 2)))\}_{(x,y)\in X} \quad (36)$$

Physical meaning of the operation: the operation corresponds to the addition modulo two of each component of the binary masks of the color images.

$$I_{b1} \cdot I_{b2} = \{((x, y), (bool_1(x, y)\ OR\ bool_2(x, y), bool_1(x, y)\ OR\ bool_2(x, y), bool_1(x, y)\ OR\ bool_2(x, y)))\}_{(x,y)\in X} \quad (37)$$

Physical meaning of the operation: the operation corresponds to modulo two of each component of the binary masks of color images.

$$\alpha \cdot I_b = \{((x, y), ([\alpha \cdot bool(x, y) \bmod 2], [\alpha \cdot bool(x, y) \bmod 2], [\alpha \cdot bool(x, y) \bmod 2])\}_{(x,y)\in X} \quad (38)$$

Physical meaning of the operation: the operation corresponds to multiplication of the binary mask of a color image by an element of the field of real numbers.

Statement 20. *The introduced addition operations and binary masks of two color images* (36), (37) *are* ***weakly physically interpretable operations***.

Statement 21. *The introduced operation of multiplication of a binary mask of a color image* (38) *by an element of the field of real numbers is* ***physically interpretable operation***.

3.2. Operations on Images as Operands of Descriptive Image Algebras with One Ring

3.2.1. Types of Transformations over Images

The DA considers three classes of allowable image transformations: procedural transformations, parametric transformations, and generating transformations [6].

Below, we recall the definitions of the main classes of image transformations (procedural, parametric, and generating).

Definition 5 [6]. ***A procedural transformation*** $\boldsymbol{O_T} \in \{\boldsymbol{O_T}(\Im)\}$ *on an image or images is an operation whose application transforms them either into some other set of images, into some image, or into fragments thereof.*

Note that the procedural transformation in fact applies to realizations of the source image(s).

Definition 6 [6]. ***A parametric transformation*** $\boldsymbol{O_P} \in \{\boldsymbol{O_P}(\Im)\}$ *over the image is an operation whose application converts the image into a numerical characteristic p, which can be used to compare the properties of geometric objects, brightness characteristics, or configurations formed due to repetitions of geometric objects and brightness characteristics of the source image.*

To construct the numerical characteristic p of an image, the set of realizations of the image and semantic or contextual information about the image can also be used.

Definition 7 [6]. ***A generating transformation*** $O_G \in \{\tilde{O}_G\}$ *above an image is an operation, the application of which to it converts it into a particular representation representing the specific features of the analyzed image.*

Examples of such transformations are functions describing curves, a conjunction function, disjunction function, and code functions for images.

3.2.2. Operations on Procedural Transforms

For clarity, let us again turn to Ritter's image algebra operations [9]. Note that in the DA, Ritter's SIA operations are considered procedural transformations.

Example 13.

Let the images be given on a fixed set X with a range of values X: $A = \{(x, a(x)), x \in X, a(x) \in X\}$. Let $B = \{(x, b(x)), x \in X, b(x) \in X\}$, $C = \{(x, c(x)), x \in X, c(x) \in X\}$. Let $\alpha \in R$.

Let the following image operations be introduced:

$$A + B = \{(x, c(x)) : c(x) = a(x) + b(x), x \in X\}; \quad (39)$$

$$A \cdot B = \{(x, c(x)) : c(x) = a(x) \cdot b(x), x \in X\}; \quad (40)$$

$$A \vee B = \{(x, c(x)) : c(x) = a(x) \vee b(x), x \in X\}; \quad (41)$$

$$A \wedge B = \{(x, c(x)) : c(x) = a(x) \wedge b(x), x \in X\}; \quad (42)$$

$$\frac{A}{B} = \left\{(x, c(x)) : c(x) = \frac{a(x)}{b(x)}, \text{ if } b(x) \neq 0, \text{ else } c(x) = 0; x \in X\right\}; \quad (43)$$

$$A^B = \{(x, c(x)) : c(x) = a(x)^{b(x)} \text{ if } a(x) > 0, \text{ else } c(x) = 0, x \in X\}; \quad (44)$$

$$A - B = \{(x, c(x)) : c(x) = a(x) - b(x), x \in X\}. \quad (45)$$

Denote $r_1, r_2, \ldots \in \{+, *, \vee, \wedge, -, \backslash, A^B \ldots\}$ are standard binary operations over two images. $r(A, B)$ is the image after operation r to images A and B.

The following operations are defined on the set of selected operands:

$$(r_1 \oplus r_2)(A, B) = r_1(A, B) + r_2(A, B); \quad (46)$$

Physical meaning of the operation: the operation corresponds to the sequential application of operations r_1 and r_2 with the subsequent addition of images constructed as a result of operations r_1, r_2 to the source images.

$$(r_1 \otimes r_2)(A, B) = r_1(r_2(A, B), r_2(A, B)); \tag{47}$$

Physical meaning of the operation: the operation corresponds to application of the first binary transform to the result of application of the second binary transform of images, taken twice, to the source images.

$$(\alpha r)(A, B) = \alpha r(A, B). \tag{48}$$

Physical meaning of the operation: the operation corresponds to multiplication of the result of the binary image transformation to the element of the field of real numbers.

Statement 22. *Operation* (46) *of sequential application of operations* r_1 *and* r_2 *with subsequent addition of images constructed as a result of operations* r_1, r_2 *to the source images* ***is not physically and visually interpretable***.

Proof:

1. Physical interpretability of the operation: according to Assertion 2, an operation is not physically interpretable in the context of image analysis and recognition if its operands are not images, image representations, and image fragments.

2. Visual interpretability of the operation: according to Assertion 1, if the operation is not physically interpretable, then it is also not visually interpretable.

Q.E.D.

Example of application: this operation can be used to combine operations over images.

Statement 23. *Operation* (47) *of sequential application of the first binary transform* r_1 *to the result of the second binary transform* r_2 *of images*, *taken twice*, *to source images* ***is not physically and visually interpretable.***

The proof is done similarly to the previous **Statement**.

Example of application: this operation can be used to combine operations on images.

Statement 24. *Operation* (48) *of multiplication of the result of application of a binary image transformation by an element of the field of real numbers* ***is not physically and visually interpretable.***

The proof is done similarly to **Statement** 22.

Example of application: this operation can be used to correct image brightness and color, after application of operation *r*.

***Example 14*.**

Now let operations $f(U \to V) \in F$ of conversion of elements of the set of color images described in Example 9 to the elements of the set of grayscale images described in Example 10. Let $I = \{((x, y), (r(x, y), g(x, y), b(x, y))\}_{(x, y) \in X}$, where the functions $r(x, y)$, $g(x, y)$, $b(x, y)$ take integer values from the set $[0...M-1]$. Let $J = \{((x, y), gray(x, y))\}_{(x, y) \in X}$, $J_1 = \{((x, y), gray_1(x, y))\}_{(x, y) \in X}$, $J_2 = \{((x, y), gray_2(x, y))\}_{(x, y) \in X}$ where the functions $gray(x, y)$, $gray_1(x, y)$, $gray_2(x, y)$ take integer values from the set $[0...M-1]$. $M = 256$ is the value of the maximum intensity of the color components; X is the set of pixels on which the images are given. Let $f_1(I) = J_1$, $f_2(I) = J_2$, $f(i) = J$ be elements of a set of a given type. Let $\alpha \in R$.

The following operations are defined on the set of selected operands:

$$f_1(I) + f_2(I) = J_1 + J_2; \tag{49}$$

Physical meaning of the operation: the operation corresponds to addition of the results of application of the two operations of conversion of a color source image into grayscale images. Note that addition of the obtained grayscale images is carried out according to the rules defined in Example 10.

$$f_1(I) \cdot f_2(I) = J_1 \cdot J_2; \tag{50}$$

Physical meaning of the operation: the operation corresponds to multiplication of the results of application of the two operations of converting a color source image into grayscale images. Note that multiplication of two grayscale images is carried out according to the rules defined in Example 10.

$$\alpha \cdot f(I) = \alpha \cdot J. \tag{51}$$

Physical meaning of the operation: the operation corresponds to multiplication of the result of application of the operation of converting a color source image into a grayscale image by a real number. Note that multiplication of the grayscale image by a real number is carried out according to the rules defined in Example 10.

Statement 25. *Operations* (49), (50), (51) *of addition*, *multiplication*, *and multiplication by a real number*, *respectively*, *of the results of application of operations of converting a color source image into grayscale images* **are not physically and visually interpretable.**

The proof is carried out similarly to **Statement** 22.

***Example 15*.**

Let $I = \{((x, y), (r(x, y), g(x, y), b(x, y))\}_{(x, y) \in X}$, where the functions $r(x, y)$, $g(x, y)$, $b(x, y)$ take integer values from the set $[0...M-1]$ (the set of all such images U). X is the set of pixels on which the images are given. Let there be carried out operations $sb((U, C) \to B)$ for obtaining binary masks corresponding to selected objects (nuclei of lymphatic cells), C be information on the contours of the selected nuclei, set B be a subset of set U such that if an image point (x, y) belongs to the selected kernel, then the components $r(x, y) = g(x, y) = b(x, y) = 1$ if the point (x, y) belongs to the background of the kernel, then $r(x, y) = g(x, y) = b(x, y) = 0$. Denote the elements of this set as $I_{b1} = \{((x, y), (bool_1(x, y), bool_1(x, y), bool_1(x, y)))\}_{(x, y) \in X}$, $I_{b2} = \{((x, y), (bool_2(x, y), bool_2(x, y), bool_2(x, y)))\}_{(x, y) \in X}$, where functions $bool_1(x, y)$, $bool_2(x, y) \in \{0, 1\}$. Let $sb_1(I, C) = I_{b1}$, $sb_2(I, C) = I_{b2}$ be elements of the set of a given type; let $\alpha \in R$.

The following operations are defined on the set of selected operands:

$$sb_1(I,C) + sb_2(I,C) = I_{b1} + I_{b2}; \quad (52)$$

Physical meaning of the operation: the operation corresponds to addition of the results of application of the two operations for obtaining binary masks corresponding to the selected objects. Note that addition of the two binary masks is carried out according to the rules defined in Example 12.

$$sb_1(I,C) \cdot sb_2(I,C) = I_{b1} \cdot I_{b2}; \quad (53)$$

Physical meaning of the operation: the operation corresponds to the multiplication of the results of applying the two operations of obtaining binary masks corresponding to the selected objects. Note that the multiplication of two binary masks is carried out according to the rules defined in Example 12.

$$\alpha \cdot sb_1(I,C) = \alpha \cdot I_{b1}. \quad (54)$$

Physical meaning of the operation: the operation corresponds to multiplication, by a real number, of the result of applying the operation of obtaining a binary mask corresponding to the selected object. Note that multiplication of a binary mask by a real number is carried out according to the rules defined in Example 12.

Statement 26. *Operations* (52), (53), (54) *of addition, multiplication, and multiplication by a real number, respectively, of the results of obtaining binary masks corresponding to the selected objects* ***are not physically and visually interpretable.***

The proof is carried out similarly to **Statement** 22.

Note that in this case, the operations of addition, multiplication, and multiplication by a field element may differ from those proposed in Example 12. On the set of selected operands, the following operations can be assigned:

$$sb_1(I,C) + sb_2(I,C) = \{\{(bool_1(x,y) \vee bool_2(x,y), \\ bool_1(x,y) \vee bool_2(x,y),\, bool_1(x,y) \vee bool_2(x,y)), \\ bool_1(x,y),\, bool_2(x,y) \in \{0,1\}\},\, (x,y) \in X\}; \quad (55)$$

Physical meaning of the operation: the operation corresponds to the union of the results of applying two operations of obtaining binary masks corresponding to the selected objects.

$$sb_1(I,C) \cdot sb_2(I,C) = \; = \{\{(bool_1(x,y) \wedge bool_2(x,y), \\ bool_1(x,y) \wedge bool_2(x,y),\, bool_1(x,y) \wedge bool_2(x,y)), \\ bool_1(x,y),\, bool_2(x,y) \in \{0,1\}\},\, (x,y) \in X\}; \quad (56)$$

Physical meaning of the operation: the operation corresponds to multiplication of the results of applying the two operations of obtaining binary masks corresponding to the selected objects. Note that multiplication of two binary masks is carried out according to the rules defined in Example 12.

Statement 27. *Operations* (*55*), (*56*) *of addition and multiplication, respectively, of the results of obtaining binary masks corresponding to the selected objects*, **are not physically and visually interpretable**.

The proof is carried out similarly to **Statement** 22.

3.2.3. Operations on Parametric Transformations

Below is an example of operations on parametric image transformations.

Example 16.

Let operations $g\ (V \rightarrow P_1)$ of calculating the values of features of grayscale images described in Example 10 be elements of the set G; P_1 is the set of parametric models of images described below in Example 17; $J = \{((x,\ y),\ gray(x,\ y))\}_{(x,\ y)\ \in\ X}$, where function $gray(x, y)$ takes integer values from the set $[0...M-1]$; $M = 256$ is the value of the maximum intensity of the color components; X is the set of pixels on which the images are assigned; $g_1(J) = p_1$, $g_2(J) = p_2$, $g(J) = p$; let $\alpha \in R$.

The following operations on the assigned operands are introduced:

$$g_1(I) + g_2(I) = p_1 + p_2; \quad (57)$$

Physical meaning of the operation: the operation corresponds to the addition of two parametric models of images p_1 and p_2 obtained after applying operations of calculating the values of grayscale image features g_1, g_2.

$$g_1(I) \cdot g_2(I) = p_1 \cdot P_2; \quad (58)$$

Physical meaning of the operation: the operation corresponds to the operation of multiplication of two parametric models of images p_1 and p_2 obtained after applying operations of calculating the values of grayscale image features g_1, g_2.

$$\alpha \cdot g(I) = \alpha \cdot p. \quad (59)$$

Physical meaning of the operation: the operation corresponds to the operation of multiplication of a parametric model of images p obtained after applying the operation of calculating the values of grayscale image features g.

Statement 28. *Operations* (57), (58), (59) *of addition, multiplication, and multiplication by a real number, respectively, of the results of applying the operations of calculating the values of grayscale image features* ***are not physically and visually interpretable.***

The proof is carried out similarly to **Statement** 22.

3.3. Image Models/Representations as Descriptive Image Algebras Operands

Recall that if application of procedural, parametric, and generating transformations to the source image or to its representation leads to the construction of a model, then accordingly, a procedural, parametric, or generating model will be constructed.

Example 17.

Let P_1 be a set of parametric models of images such that any element of $\overline{p}$ set P_1 assigned by the following vector $\overline{p} = (p^1,\ldots,p^n)$, where $n = 1, 2, \ldots$ is the number of features describing the parametric model of the image; $p^i = (p_x^i, p_y^i)$ are vector elements ($i = 1, \ldots, n$); $p_x^i = 1, 2, \ldots$ is the unique number of the feature in the feature space assigning the set P_1; p_y^i is the value of the feature from the set of real numbers, and: (1) $p_x^i < p_x^j$ if a $i < j$; (2) $p_{1y}^i = p_{2y}^j$, $i, j = 1, 2, \ldots$ if $p_{1x}^i = p_{2x}^j$; $\overline{p}_1 = (p_1^1,\ldots,p_1^{n_1})$, $\overline{p}_2 = (p_2^1,\ldots,p_2^{n_2})$ are any elements of the set P_1; let $\alpha \in R$.

We introduce the following operations on parametric image models:

$$\begin{gathered}\overline{p}_1 + \overline{p}_2 = (p_1^1,\ldots,p_1^{n_1}) + (p_2^1,\ldots,p_2^{n_2}) = (p_3^1,\ldots,p_3^{n_3}): \\ p_{3x}^i \subset [p_{1x}^1,\ldots p_{1x}^{n_1}] \cap [p_{2x}^1,\ldots,p_{2x}^{n_2}], \\ n_3 = \left\|[p_{1x}^1,\ldots p_{1x}^{n_1}] \cap [p_{2x}^1,\ldots,p_{2x}^{n_2}]\right\| \\ p_{3y}^i = p_{1y}^j, \quad if \quad p_{3x}^i = p_{1x}^j \\ p_{3y}^i = p_{2y}^j, \quad if \quad p_{3x}^i = p_{2x}^j\end{gathered} \tag{60}$$

Physical meaning of the operation: the operation corresponds to the addition of two parametric models of images $\overline{p}_1$, $\overline{p}_2$, as a result of which the parametric models are combined and contain all the unique features from both models.

$$\begin{gathered}\overline{p}_1 \cdot \overline{p}_2 = (p_1^1,\ldots,p_1^{n_1}) + (p_2^1,\ldots,p_2^{n_2}) = (p_4^1,\ldots,p_4^{n_4}): \\ n_3 = \left\|[p_{1x}^1,\ldots p_{1x}^{n_1}] \cap [p_{2x}^1,\ldots,p_{2x}^{n_2}]\right\|, \quad n_4 < n_3 \\ (p_4^1,\ldots,p_4^{n_4}) - essential\ features\end{gathered} \tag{61}$$

Physical meaning of the operation: the operation corresponds to the operation of multiplication of two parametric models of images $\overline{p}_1$, $\overline{p}_2$, as a result of which the most essential features describing the object are distinguished by analysis of the original parametric models.

$$\begin{aligned}\alpha \cdot \overline{p} &= (\alpha \cdot p^1,\ldots,p^n) \\ &= ((p_x^1, \alpha \cdot p_y^1),\ldots,(p_x^n, \alpha \cdot p_y^n)).\end{aligned} \tag{62}$$

Physical meaning of the operation: the operation corresponds to the operation of multiplication of a parametric model of images $\overline{p}$ by a real number, as a result of which the values of features $p_{1y}^i = p_{2y}^j$, $i, j = 1, 2, \ldots$ described above are multiplied by a real number.

Statement 29. *Operations* (60), (61), (62) *of addition*, *multiplication*, *and multiplication by a real number*, *respectively*, *of parametric models are* ***weakly physically interpretable***.

The proof is carried out similarly to the previous.

Statements

Example 18.

Let P_2 be a set of parametric models of images such that any element $\overline{p}$ of set P_2 is assigned by the following vector $\overline{p} = (p^1,\ldots,p^n)$, where $n = 1, 2, \ldots$ *is* the number of features describing the parametric model of the image; n is a fixed number of features; p^i are vector elements ($i = 1, \ldots, n$) from the set of real numbers; $\overline{p}_1 = (p_1^1,\ldots,p_1^n)$, $\overline{p}_2 = (p_2^1,\ldots,p_2^n)$ are any elements of set P_2; let $\alpha \in R$.

We introduce the following operations on parametric image models:

$$\begin{aligned}\overline{p}_1 + \overline{p}_2 &= (p_1^1,\ldots,p_1^n) + (p_2^1,\ldots,p_2^n) \\ &= (p_1^1 + p_2^1,\ldots,p_1^n + p_2^n)\end{aligned} \tag{63}$$

Physical meaning of the operation: the operation corresponds to the addition of two parametric models of images $\overline{p}_1$, $\overline{p}_2$, as a result of which the values of the same features in the parametric models are added.

$$\begin{aligned}\overline{p}_1 \cdot \overline{p}_2 &= (p_1^1,\ldots,p_1^n) \cdot (p_2^1,\ldots,p_2^n) \\ &= (p_1^1 \cdot p_2^1,\ldots,p_1^n \cdot p_2^n)\end{aligned} \tag{64}$$

Physical meaning of the operation: the operation corresponds to the operation of multiplication of two parametric models of images $\overline{p}_1$, $\overline{p}_2$, as a result of which the values of the same features in the parametric models are multiplied.

$$\alpha \cdot \overline{p} = (\alpha \cdot p^1,\ldots,\alpha \cdot p^n). \tag{65}$$

Physical meaning of the operation: the operation corresponds to the operation of multiplication of each feature of a parametric image model $\overline{p}$ by a real number.

Statement 29. *Operations* (63), (64), (65) *of addition*, *multiplication*, *and multiplication by a real number*, *respectively*, *of parametric models are* ***weakly physically interpretable.***

4. INTERPRETATION OF AN ALGORITHMIC SCHEME FOR SOLVING THE PROBLEM OF MORPHOLOGICAL ANALYSIS OF CELL NUCLEI IN THE LYMPHATIC SYSTEM

4.1. Formulation of the Problem and Applied Information Technology

Sometimes the interpretation of algorithmic procedures for image processing and analysis is understood as the construction of a DAS in the DIA1R language for solving applied problems. This section illustrates this interpretation method with the example of a DAS proposed by the authors for automated morphological analysis of cytological slides from patients with lymphatic tumors [5].

INITIAL DATA (1) Micrographs of imprints of lymphatic organs of patients with three diagnoses: B-cell chronic lymphocytic leukemia (CLL); sarcomatous transformation of B-cell chronic lymphocytic leukemia (TCLL), and primary B-cell lymphosarcoma (LS). (2) Contours of diagnostically valuable lymphoid cell nuclei indicated by expert morphologists.

A mathematical information technology model was constructed to refer new images of the lymphatic cell nuclei to one of the classes of images corresponding to the three diagnoses of the patients: malignant tumors (LC, TCLL), benign tumors (CLL).

The technology is based on image processing and analysis methods and image recognition methods. Image processing and analysis methods are used in solving problems of obtaining images of cytological slides, cell nuclei segmentation, and necessary data for constructing a feature-based description of nuclei. Pattern recognition methods are used to construct a feature-based description of nuclei, patients, and to classification of patients.

The information technology includes the following steps:

(1) creation of an image archive of slides of lymphatic organs with isolated lymphocyte nuclei for patients with benign and malignant lymphatic tumors;

(2) image processing to eliminate differences in the light and color of slides;

(3) selection and calculation of features reflecting the morphological characteristics of the lymphoid cell nuclei used in the diagnosis;

(4) qualitative and statistical analysis of the calculated features and evaluation of their informativeness;

(5) factor analysis of traits;

(6) formation of a feature-based description of a patient based on the results of cluster analysis of his/her nuclei;

(7) experiments on classifying (diagnosing) patients.

Study [5] presents the results of testing the elements of the described information technology.

4.2. Descriptive Image Algebras for Constructing an Algorithmic Scheme

Each step of this technology was described by a DIA1R.

All algebras are used in the construction of a unified DASIR, which yields a solution to the problem for solving the task.

Let us describe the operands and operations (and their functions) of DIAs and descriptive image groups (DIGs) necessary for constructing an algebraic model for the morphological analysis of lymphatic cell nuclei.

DIA 1 is a set of color images (Example 9) with algebraic operations (27–29). DIA 1 is used to describe the source images, and the operation of DIA 1 is used to describe the step of segmenting diagnostically important nuclei in images.

DGI 1 is a set of operations of obtaining binary masks corresponding to the selected lymphatic cell nuclei (Example 15) with algebraic operations (52–53). DIG 1 is used to describe the segmentation process.

DIG 2 is a set of binary masks corresponding to the selected lymphatic cell nuclei (Example 15) with algebraic operations (54), (55), (53). DIG 2 is used to describe binary masks for color images.

DIA 2 is a set of grayscale images (Example 10) with algebraic operations (30–32). *DIA 2* is used to describe the selected nuclei in the images.

DIA 3 is a set of operations for converting elements of a set of color images into elements of a set of grayscale images (Example 14) with algebraic operations (49–51). *DIA 3* is used for smoothing the difference in light and in the color range of cell images.

DIA 4 is a set of operations for calculating the features of grayscale images (Example 15) with algebraic operations (52–54). *DIA 4* is used to calculate the values of features.

DIA 5 is a set of parametric models (Example 17) with algebraic operations (58–60). *DIA 5* is used to select the most informative features. Addition is used to construct a combined parametric image model. Multiplication is used to bring a set of image features to the set of essential features. Multiplication by an element of the field of real numbers is used to normalize the feature vector.

***DIA 6* is** a set of parametric models (Example 18) with algebraic operations (61–63). *DIA 6* is used to describe images that are converted to a form suitable for recognition.

CONCLUSION

So the interpretation is considered as a transition from a meaningful description of an operation to its mathematical or algorithmic realization. As a result, the practical applicability of operations is revealed in the context of the more general concept of interpretability.

The article presents results related to the interpretability of DIA1R operations and examples of interpretability domains for some types of operations:

(1) The main specifics of DIAs and DIA1R are defined, from which the formalization and specification of the concept of interpretability of operations follow.

(2) The method and tools for formalizing types of interpretability of image analysis and processing operations are considered.

(3) To characterize the interpretability of DIA operations, the following concepts were introduced: (a) the physical meaning of an operation, (b) physical

interpretability in the context of image analysis and processing, (c) visual interpretability in the context of image analysis and processing, (d) weak physical interpretability, and (e) strong physical interpretability.

This study is significant because, as far as the authors know, it is the first to be formulated and illustrated by specific examples of a mathematical approach to regularizing the choice of transformations in image analysis based on the type, content, and technical characteristics of the image being processed and establishment the correspondence between the objective function and mathematical description of an operation.

In the future, the authors intend to continue studying the formal aspects of the interpretability of image transformations in the following directions:

(1) interpretability for all basic DIA operations and their basis sets;

(2) establishing a relationship between the interpretability of operations and image equivalence classes [3];

(3) establishing a relationship between interpretability and invariance of image transformations;

(4) the interpretability of operations of new types of universal and specialized DIAs;

(5) study of the relationship between the computational efficiency of DASs and the interpretability of their operations.

The results of this study are essential for developing the descriptive theory of image analysis and can also be used to develop the linguistic apparatus of automation systems for processing and analyzing images and knowledge bases for image analysis (navigation, ontologies, thesauri), as well as for expanding the range of effectively solvable especially complex and difficult applied image analysis and recognition tasks.

These results are also very significant for studies of the linguistic and psychosomatic aspects of image analysis and understanding and, accordingly, studies of the human brain activity.

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Listed results were used in development of software kits for image analysis and recognition and for solution of important and difficult applied problems of automated biomedical image analysis.

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