Descriptive Image Analysis: Genesis and Current Trends[1,2]

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Abstract—This paper is devolved to descriptive image analysis, an important, if not a leading, direction in the modern mathematical theory of image analysis. Descriptive image analysis is a logically organized set of descriptive methods and models meant for analyzing and estimating the information represented in the form of images, as well as for automating the extraction (from images) of knowledge and data needed for intelligent decision making about the real-world scenes reflected and represented by images under analysis. The basic idea of descriptive image analysis consists in reducing all processes of analysis (processing, recognition, and understanding) of images to (1) construction of models (representations and formalized descriptions) of images; (2) definition of transformations over image models; (3) construction of models (representations and formalized descriptions) of transformations over models and representations of images; and (4) construction of models (representations and formalized descriptions) of schemes of transformations over models and representations of images that provide the solution to image analysis problems. The main fundamental sources that predetermined the origination and development of descriptive image analysis, or had a significant influence thereon, are considered. In addition, a brief description of the current state of descriptive image analysis that reflects the main results of the descriptive approach to analysis and understanding of images is presented. The opportunities and limitations of algebraic approaches to image analysis are discussed. During recent years, it was accepted that algebraic techniques, particularly, different kinds of image algebras, are the most promising direction of construction of the mathematical theory of image analysis and of the development of a universal algebraic language for representing image analysis transforms, as well as image representations and models. The main goal of the algebraic approaches is designing a unified scheme for representation of objects under recognition and its transforms in the form of certain algebraic structures. This makes it possible to develop the corresponding regular structures ready for analysis by algebraic, geometrical, and topological techniques. The development of this line of image analysis and pattern recognition is of crucial importance for automatic image mining and application problems solving, in particular, for diversification of the classes and types of solvable problems, as well as for significant improvement of the efficiency and quality of solutions. The main subgoals of the paper are (1) to set forth the-state-of-the-art of the mathematical theory of image analysis; (2) to consider the algebraic approaches and techniques suitable for image analysis; and (3) to present a methodology, as well as mathematical and computational techniques, for automation of image mining on the basis of the descriptive approach to image analysis (DAIA). The main trends and problems in the promising basic researches focused on the development of a descriptive theory of image analysis are described.

Keywords: image analysis, descriptive image analysis, descriptive approach, image mining, mathematical theory of image analysis, descriptive models, image representations, pattern recognition

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INTRODUCTION

This paper is devolved to descriptive image analysis, an important, if not a leading, direction in the modern mathematical theory of image analysis. Descriptive image analysis is a logically organized set of descriptive methods and models meant for analyzing and estimating the information represented in the form of images, as well as for automating the extraction (from images) of knowledge and data needed for intelligent decision making about the real-world scenes reflected and represented by images under analysis. The basic idea of descriptive image analysis consists in reducing all processes of analysis (processing, recognition, and understanding) of images to:

(1) construction of models (representations and formalized descriptions) of images;

(2) definition of transformations over image models;

(3) construction of models (representations and formalized descriptions) of transformations over models and representations of images; and

(4) construction of models (representations and formalized descriptions) of schemes of transforma-

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tions over models and representations of images that provide the solution to image analysis problems.

The main fundamental sources that predetermined the origination and development of descriptive image analysis, or had a significant influence thereon, are considered. In addition, a brief description of the current state of descriptive image analysis is presented that reflects the main results of the descriptive approach (DA) to analysis and understanding of images, which is proposed and developed by the authors of this paper [25–28, 36, 39].

Automation of image processing, analysis, estimation, recognition, and understanding is one of the crucial points of theoretical computer science that has a decisive importance for applications, in particular, for diversification and extension of types of solvable applied problem, as well as for improving the efficiency of problem solving and the precision of problem solutions.

The theoretical and methodical basis of automating the processing, analysis, and estimation of experimental data is constructed by the mathematical theory of pattern recognition and mathematical theory of image analysis.

The main purpose of pattern recognition methods is to assign a represented object to one of a number of preset classes by analysis of precedents (computation of similarity metrics) in a multidimensional feature space and problem context through the formulation and solution of the following types of problems:

(1) object identification and classification;

(2) division of a set of given objects into nonintersecting classes (cluster analysis);

(3) estimation of the informativeness of the characteristics (features) of objects under recognition; and

(4) construction of formalized descriptions/representations of objects under recognition (with the use of descriptive methods and models, including descriptive algebras, feature vectors, logical formulae, formal grammars, etc.).

Pattern recognition as a science has evolved from the needs to solve problems of processing, analyzing, and evaluating ill-structured, badly formalized, indistinct, incomplete, contradictory, semantically saturated, and noisy information by using computationally efficient mathematical methods. The initial information in these problems consists of numbers, symbols, and expert data; images; speech; arbitrary signals; texts; documents; diagrams and drawings; and random combinations thereof.

Pattern recognition tools and methods are designed for solving applied problems of intelligent decision making, diagnostics, identification, and prediction.

The role of the image as an object of analysis and estimation is determined by its specific and integral information properties. An image is a certain totality of the initial pictured data and means of its representation, the results of the formation processes of representations of the image and their transformation procedures, the physical and logical aspects, and the models of objects, events, and processes represented in the image.

The specifics and complexity of image analysis and evaluation are due to the need for achieving a certain balance between such contradictory factors as the goals and problems of analysis; the nature of visual perception; the methods and tools for obtaining, forming, and representing images; and mathematical, computational, and technological tools of image analysis.

By now, image analysis and estimation has accrued much experience in applying mathematical methods from various areas of mathematics, computer science, and physics, in particular, algebra, geometry, discrete mathematics, mathematical logic, probability theory, mathematical statistics, mathematical analysis, mathematical theories of pattern recognition and image analysis, digital signal processing, and optics.

On the other hand, with all this diversity of methods used, we still need a firm basis to arrange and choose suitable methods and models of image analysis, represent, in an unified way, data for processing (images), meet the requirements of standard recognition algorithms imposed on initial information, construct mathematical models of images designed for recognition problems, and, on the whole, establish a universal language for unified description and representation of images and transformations over them.

Image analysis and understanding are quite difficult problems for mathematicians because the image is an extremely inconvenient form of information representation for mathematical processing. For a long time, this direction was underdeveloped because mathematicians had no interest in working with such unconventional forms of information. Serious efforts began in the 1950s and have been continuing since. Presently, there is a wide set of mathematical methods allowing one to reduce the image to a form suitable for efficient recognition algorithms. Such formalized image representations (models: images reduced to a recognizable form) form the necessary basis for simulation, recognition, and computation of image characteristics, detection of regularities and properties, and intelligent decision making. These methods have generally been substantiated, developed, and many times tested in practice. In this respect, it is quite correct to speak about the creation of a mathematical theory of image analysis and a certain degree of its maturity; however, it still has to achieve the degree of development of the mathematical theory of image recognition, the essential elements of which are used when working with images.

The Russian school of mathematics has made an extremely significant contribution to the mathematical theories of image recognition and analysis. Its results, particularly, in the field of algebraic methods of image recognition and analysis, define its global level today [41, 123, 124].

In this paper, we give a brief review of the main algebraic methods and features. The paper consists of three main sections (aside from Introduction and Conclusions).

Section 1 describes the current trends in the development of mathematical tools for automation of image analysis, particularly, in image mining. A way to a unified theory follows in path of the works by M. Duff, G. Matheron, J. Serra, J. von Neumann, S. Sternberg, S. Unger, U. Grenander, Yu. Zhuravlev, and others. This section also presents the history of developing the algebraic construction for image analysis and processing: formal grammars, cellular automata, mathematical morphology, image algebras, and multiple classifier algorithms.

Section 2 considers the basic theories of pattern recognition. These theories are Pattern Theory (U. Grenander), Theory of Categories Techniques in Pattern Recognition (M. Pavel), and the Algebraic Approach to Recognition, Classification, and Forecasting Problems (Yu. Zhuravlev). Pattern Theory is techniques for pattern recognition data representation and transformation on the basis of regular combinatorial structures and algebraic and probabilistic means. Theory of Categories Techniques in Pattern Recognition is devoted to the formal description of pattern recognition algorithms by transforms of initial data while preserving their class membership. The Algebraic Approach to Recognition, Classification, and Forecasting Problems includes mathematical set-up of the pattern recognition problem, correctness and regularity conditions, algebras on algorithms, multiple classifiers (representation of a pattern recognition algorithm as an algebraic polynomial), etc.

The Russian mathematical school also obtained some important original results on other algebraic tools for pattern recognition and image analysis, including algebraic multiple classifiers, algebraic committees of algorithms, combinatorial algorithms for recognition of 2D data, descriptive image models, and 2D formal grammars.

The materials on image algebras (IAs) present the main approaches to creating a unified language for concepts and operations used in image processing and analysis. The most famous IAs are the standard IA by G. Ritter and the descriptive IA (DIA) by I. Gurevich. The standard IA is a unified algebraic representation of the image processing and analysis operations. The DIA is a unified algebraic language for describing, performance estimating, and standardizing the representation of algorithms for analysis, recognition, and understanding of image and image models.

Section 3 briefly outlines the DA, specifically, (1) its mathematical foundations and basic elements; (2) the model of the process of image recognition that is fundamental for the DA; and (3) the sources of introducing the descriptive nature into mathematical methods for image analysis (see, for example, the early works by F. Ambler, G. Barrow, R. Burstall, T. Evans, S. Kaneff, R. Kirsh, R. Narasimhan, A. Rosenfeld, A. Shaw, and K.S. Fu). We discuss the goals of the theoretical development in the framework of the DA, image analysis algebraization, and the problem of image reduction to a recognizable form (IRRF). The main results of the DA are also outlined.

In Conclusions, the necessary steps to finalize the DA, its open problems, and the prospects of constructing a descriptive theory of image analysis on the basis of the DA are considered.

1. BREAKTHROUGH LINES OF IMAGE ANALYSIS

1.1. Image Mining

It is one of the breakthrough challenges for theoretical computer science to find automated ways to process, analyze, evaluate, and understand the information represented in the form of images. It is crucial for computer science to develop this branch in terms of solving applied problems, particularly, increasing the diversity of classes of solvable problems and improving the efficiency of the process significantly.

Images are one of the main tools to represent and transfer information needed to automate intelligent decision making in many application domains. Increasing the efficiency, including automatization, of gathering information from images can improve the efficiency of intelligent decision making.

Recently, this area of image analysis, referred to as image mining in English publications, has been set off into a separate line of research.

We list the functions of particular aspects of image processing [36, 42]. Image processing and analysis provides for image mining, which is necessary for decision making, while the very decision making is done by methods of the mathematical theory of pattern recognition. To link these two stages, the information gathered from the image after its analysis is transformed in such a way that standard recognition algorithms can process it. Note that, although this stage seems to have an "intermediate" character, it is a fundamental and necessary condition for the whole recognition to be feasible.

Presently, automated image mining is the main strategic goal of the fundamental research in image analysis, recognition, and understanding, along with the development of the corresponding information technology and algorithmic software systems. Eventually, this automatization is expected to help developers of automated image processing systems, as well as end users, either in an automated or interactive mode, to:

• develop, adapt, and check methods and algorithms of image recognition, understanding, and evaluation;

• choose the optimal or suitable methods and algorithms of image recognition, understanding, and evaluation;

• check the quality of initial data and their usefulness in solving the image recognition problem;

• apply standard algorithmic schemes of image recognition, understanding, evaluation, and search.

To automate image mining, we need an integrated approach to leverage the potential of mathematical apparatus of the main lines in transforming and analyzing the information represented in the form of images, viz., image processing, analysis, recognition, and understanding.

1.2. Algebraization

Carried out by pattern recognition methods, image mining now tends to multiplicity (multialgorithmic and multimodel) and fusion of the results, i.e., several different algorithms are applied in parallel to process the same model and several different models of the same initial data in order to solve a problem; then, the results are fused to obtain the most accurate solution.

Multialgorithmic classifiers, as well as multimodel and multiple-aspect image representations, are the common tools for implementing this multiplicity and fusion. For the first time, the idea of a combination of qualifiers optimized by algebraic correction was suggested and justified by Zhuravlev [123, 124]. The complex of the mathematical methods related to the synthesis and investigation of such qualifiers is known under the common title of the Algebraic Approach to Tasks of Recognition, Classification, and Prediction. In the English-language literature, qualifiers are referred to as multiple classifiers [117]. Recently, quite interesting results have been obtained in the field of theoretical-informational analysis of combined qualifiers [24], development of specific strategies for merging algorithms [52], and use of methods of the code theory in tomography [121].

Since the 1970s, the majority of image recognition applications and a significant part of researches in artificial intelligence have been dealing with images. As a result, new technical tools emerged that allow the recorded and accumulated data to be represented in the form of images, and the image recognition itself became more popular as a powerful and efficient methodology for mathematical processing and analysis of data, as well as for revealing hidden regularities. Various scientific and technical, as well as economic and social, factors allow the application domain of image recognition to grow steadily.

There are internal scientific problems that have arisen within image recognition. First of all, these are algebraizing the image recognition theory, arranging the image recognition algorithms, estimating the algorithmic complexity of the image recognition problem, automating the synthesis of the corresponding efficient procedures, formalizing the description of the image as an object of recognition, choosing a system of representations of the image in the recognition process, etc. It is these problems that form the mathematical agenda of the DA [25–42], which is developed based on the ideas of the algebraic approach to recognition [123, 124] in order to create a systematized set of methods and tools for data processing in image recognition and analysis problems.

Experience in the development of the mathematical theory of image analysis and its use to solve applied problems shows that, when working with images, it is necessary to solve the problems that are due to three basic issues of image analysis: (1) the description (modeling) of images; (2) the development, exploration, and optimization of the selection of mathematical methods and tools for information processing in image analysis; and (3) the hardware and software implementation of the mathematical methods of image analysis.

What makes the image analysis and recognition problems specific, complex and thus difficult and catching is the necessity to find a compromise between rather contradictory factors. These factors are the requirements imposed on the analysis; the nature of visual perception; the ways to obtain, form, and reproduce images; and the existing mathematical and technical ways to process them. The main contradiction is between the nature of the image and the analysis based on the formal description (essentially a model) of an object; this implies that, to leverage the fact that information is represented in the form of images, it is necessary to make this information non-depictive because the corresponding algorithms can only process certain symbolic descriptions.

Most image processing methods are purely heuristic with their quality being essentially determined by the degree to which they are successful in coping with the "depictive" nature of the image by using "non-depictive" tools, i.e., successful in employing procedures that do not depend on the fact that the information to be processed is organized in the form of images.

When we solve an image recognition problem, it is very important that we can choose a proper recognition algorithm from a great number of available ones, i.e., we need to choose the best (in a certain sense) algorithm for a particular situation. It is obvious that, both in image recognition and in solving recognition problems with standard teaching information [123, 124], to choose the best algorithm systematically, we need to introduce and formalize the corresponding objects of the mathematical theory of image recognition, in particular, the concept of an image recognition algorithm.

It is known that the necessity for setting and solving the problem of choosing an algorithm based on its recognition quality functional led to introducing the concept of a model of recognition algorithm. To choose the optimal or acceptable procedure for solving a particular problem, one needs to somehow fix the class of algorithms. This is the first reason that led to the necessity for synthesizing models of recognition algorithms.

With the concept of a model of recognition algorithm, we can apply strict mathematical methods to study the sets of incorrect recognition procedures (i.e., heuristic procedures that were not justified mathematically but were experimentally tested in solving real recognition problems). By analyzing the totality of incorrect recognition algorithms as they grow in number, we can select and describe particular algorithms, as well as principles of their formation. Acting over subsets of algorithms and being poorly formalized at first, these principles can then become accurate mathematical descriptions. At this stage, the principles are chosen on a heuristic basis, while the algorithms generated according to it can be constructed in a standard way. It is in this way that the formalization of different principles for constructing recognition algorithms leads to models of recognition algorithms.

To construct a model of a recognition algorithm, we need to describe the sets of incorrect procedures that, nevertheless, are efficient in solving practical problems in a uniform way. To obtain such a set, we specify variables, objects, functions, parameters, and their exact variation ranges, thereby introducing the sought-for model of the algorithm. Given a set of the corresponding variables, objects, parameters, and types of functions, we can single out some fixed algorithms from the model we consider.

To construct a model of an image recognition algorithm and determine a proper class of recognition algorithms, it is not enough to simply apply the concept of a model of recognition algorithm, which is developed in the mathematical recognition theory, to the image domain and directly use the formal representations of a number of available recognition models studied in the classical recognition theory [123, 124]. As noted above, the nature and matter of image recognition problems differ from those of the mathematical recognition theory in its classical statement. When we transfer from classical recognition problems to image recognition problems, certain mathematical problems arise due to the formal description of the image as an object to be analyzed.

To obtain formal descriptions of images as objects to be analyzed, as well as to form and choose recognition procedures, we study the internal structure and content of the image as a result of the operations that can be carried out to construct it of sub-images and other objects of a simpler nature, i.e., primitives and objects singled out on the image during different stages of its processing (depending on the aspect, morphological, and/or scale level used to form an image model). Since this way of characterizing the image is operational, we can consider the whole process of image processing and recognition, including the construction of a formal description (model of the image), as a system of transformations carried out on the image and defined on the equivalence classes that represent the ensembles of admissible images [29, 35]. Hence, we deal with the hierarchy of formal descriptions of images, i.e., image models used in recognition depending on different aspects and/or morphological (scale) levels of image representation. Essentially, these are multiple-aspect and/or multilevel models that allow one to choose and change the degree of detail required for description of a recognition object when solving a problem. This approach to formal description of images forms the basis for the multimodel representation of images in recognition problems.

Algebraization of pattern recognition and image analysis has attracted and continues to attract the attention of many researchers. Appreciable attempts to create a formal apparatus ensuring a unified and compact representation for procedures of image processing and image analysis were inspired by the practical requirements for effective implementation of algorithmic tools to process and analyze images on computers with specialized architectures, in particular, cellular and parallel ones.

The idea of constructing a unified language for the concepts and operations used in image processing appeared for the first time in the works by Unger [120], who proposed to parallelize algorithms for processing and image analysis on computers with a cellular architecture.

Mathematical morphology developed by Materon and Serra [64, 103] became a starting point of a new mathematical wave in image processing and analysis. Serra and Sternberg [112–114] were the first to succeed in constructing an integrated algebraic theory of image processing and analysis on the basis of mathematical morphology. It is believed [89] that Sternberg was the one to introduce the term “image algebra” [112] in its current standard meaning. (We noted that Grenander used this concept in the 1970s, but he meant a different algebraic construction [21–23].) In the framework of this direction, many papers are now being written, devoted to the development of specialized algebraic constructions implementing or improving the methods of mathematical morphology.

From that time until the 1990s, the interest in the descriptive and algebraic aspects of image analysis declined. The final view of the idea of the IA was the standard IA by Ritter [89, 90] (algebraic representation of the image analysis and processing operations).

The DIA is created as a new IA that provides a possibility to work with main image models and with the basic models of the procedure of transforms, which lead to the effective synthesis and implementation of the basic procedures for formal image description, processing, analysis, and recognition. The DIA was introduced by Gurevich and is now being developed by him and his followers [28, 31, 32, 34, 37].

The history of algebraization is as follows.

• Von Neumann [80, 81] and Unger [120] (studies of interactive image transformations in a cellular space);

• Duff, Watson, Fountain, and Shaw [8] (a cellular logic array for image processing);

• Rosenfeld [93, 94] (digital topology);

• Minkowski and Hadwiger [43, 74] (pixel neighborhood arithmetics and mathematical morphology);

• Matheron, Serra, Sternberg [64, 103, 113, 114] (a coherent algebraic theory specifically designed for image processing and image analysis: mathematical morphology);

• Zhuravlev [123, 124] (algorithm algebra);

• Sternberg [112] (was the first to use the term "image algebra");

• Grenander [21–23] (pattern theory);

• Maragos [65–68] (introduced a new theory unifying a large class of linear and nonlinear systems under the theory of mathematical morphology);

• Pavel [82, 83] (theory of categories techniques in pattern recognition);

• Davidson [7] (completed the mathematical foundation of mathematical morphology by embedding it into the lattice algebra also known as the minimax algebra);

• Ritter [90, 91] (IA);

• Gurevich [28, 31, 32, 34, 37] (DIA);

• Haralick [43–48], Shapiro [44, 45], Lee [56], Joo Hyoman [47], Zhuang [46], Schafer [66–68], Dougherty [9–12], Sinha [9], Gader [20], Khabou [20], Koldobsky [20], Radunacu [85], Grana [85], Albizuri [85], Sussner [115–117], and Soille [110, 111] (recent papers in mathematical morphology and image algebras).

Only recently its was realized that only an intensive creation of a comprehensive mathematical theory of image analysis and recognition (in addition to the mathematical theory of pattern recognition) can provide a real opportunity to efficiently solve applied problems by extracting the information necessary for intelligent decision making from images.

The most important works on algebraization are discussed in more detail in Section 2.

2. ALGEBRAIC PROTOSOURCES OF DESCRIPTIVE IMAGE ANALYSIS

2.1. Pattern Theory by Grenander

This section presents the most important original results on algebraic tools for pattern recognition and image analysis, including algebras on algorithms, algebraic multiple classifiers, algebraic committees of algorithms, combinatorial algorithms for recognition of 2D data, descriptive image models, and 2D formal grammars.

The most general approach to the algebraic description of information for recognition algorithms is Grenander's general pattern theory [21–23], which unites the metric theory with the probability theory for certain universal algebras of a combinatorial type. The main attention is paid to investigating the structure of recognizing elements. The idea underlying Grenander's theory is that the knowledge about patterns can be expressed in terms of regular structures. Regular structures are structures constructed by certain rules.

The theory is based on three principles, namely, atomism, combinatory, and observability. By atomism, we mean that the structures are composed of certain basic elements. Combinatory means that explicit rules are formulated for definition of the admitted and prohibited structures. The third principle is related to the search for identification rules for determining equivalence classes.

The search for patterns in nature and in the man-made world has generated a huge literature. Grenander tries to formalize the very concept of a pattern in terms of a mathematical framework, a pattern theory.

The subject of Grenander's books [21–23] is order, patterns, and regularity, i.e., the concepts implying that the world we live in has a structure that makes it possible for us to understand it, at least to some extent. Without presupposing such a structure, we would have no hope of comprehending the phenomena we observe and the logical relations between them.

Grenander presents a catalogue of patterns. One extreme is a completely regular pattern, for example, a crystal, which can be explained through simple rules of its generation. Another extreme is complete disorder in terms of pure randomness. He also considers intermediate situations in which phenomena can be partially analyzed through the concept of a typical structure that may be obscured by a high degree of variability as, for example, in many cases of biomedical images. Such patterns not only seem complex but they are complex and, therefore, are essentially different from, e.g., fractals and various patterns in the chaos theory, which seem complicated although they may have been generated by relatively simple rules.

Pattern theory is a way to approach patterns through a mathematical formalism, a way of reasoning about patterns. This approach is demonstrated by analytical tools and by employing computational methods.

Patterns are divided into two groups: open patterns and closed patterns.

Open patterns are patterns whose internal structure is maximally simple. By "simple," Grenander means patterns whose logical architecture (its connectivity) does not involve recurrences (loops) but is straightforward. Such patterns are "ornaments," "language patterns," "a motion pattern," "time recordings," "tracks," and "behavior." Thus, their structure shares a one-dimensional flavor, not in the sense of dimension in geometry, but in terms of dependencies (logical couplings) in their structure. They are linear arrangements where one part follows another. They are not closed loops in their topology, which will later on be described by graphs: they are open, meaning the absence of cycles.

In contrast to open patterns, closed ones can possess intricately woven systems of dependencies. This induces a topology, an information architecture, that is often hidden from the observer. The pattern analyst faces a serious challenge in finding meaningful representations for closed patterns because their surface may not give a clear indication of a deep (regular) structure that supports them. Their mathematical processing requires more thought. Such patterns are "difficult ornaments," "weaving," "textures," "shapes," "inside structure," "connections," "internal patterns," "multiple object patterns," "pattern interference," and "pattern of speculation."

To understand patterns and analyze their structure, it is necessary to introduce a mathematical pattern formalism: a pattern algebra.

Representations of patterns are built from simple building blocks, which are referred to as generators. Then, after gluing generators together, a configuration appears, and the bonds of the configuration tell us what combinations hold together. A configuration space is the first of the regular structures that we build. The next one consists of images: a concept that formalizes the idea of observables. In other words, a configuration is a mathematical abstraction that typically cannot be observed directly, but the image can. An ideal observer with the perfect instrumentation such that the sensor used has no observation errors will be able to see a certain object, called the image, that may carry less information than the configuration being observed. The loss of information is not caused by noise in the sensors but is more fundamental, and it takes some efforts to formalize the concept of the image in order to get a suitable algebraic structure.

Then, from the concept "image" and the properties of the configuration spaces, we could define patterns.

Once the representations of a pattern are constructed in the form of regular structures, we can use them for many purposes. Actually, constructing the representations will turn out to be the hardest part in the endeavor; once they are built, their use will be derived by applying general mathematical and computational principles. The tasks are divided into two categories: synthesis (or simulation) and analysis (or inference).

2.2. Theory of Categories Techniques in Pattern Recognition by Pavel

Theory of Categories Techniques in Pattern Recognition (Pavel [82, 83]) is a formal description of pattern recognition algorithms in terms of transforms of initial data that preserves its class membership.

Recognizing patterns consists in associating a name, or canonical pattern, or prototype, with a given image. The aim of the categories techniques is to give general mathematical meanings to the terms "patterns" and "recognition," as well as to unify different possible approaches.

Patterns can be described by their primitive components and their composition, and/or be defined axiomatically by their invariant properties. Recognizing a pattern generally means detecting a certain equivalence between two images, given a collection of images and a certain rule of isomorphism or equivalence, deciding whether or not a given image and a certain prototype of a set of canonical patterns (i.e., representatives from the equivalence classes) are equivalent. This is done by a recognition function, and the standard way of solving the problem is to establish this equivalence by computing and comparing probes relative to a number of transformation-invariant features that are shared by the image and the pattern and which no other figure possesses.

The algebraic formalism leaves open the problem of intrinsic characterization of signs images and the problem of synonymy.

To answer these questions, a discrete topology and a discrete analog of topological homeomorphism can be introduced. The first is obtained by using the homeomorphism of Euclidean spaces with finitely generated torsion-free abelian groups. With a suitable topologization of a lattice of points in an n-space, connected sets are sets of consecutive integers.

The discrete analog of topological homeomorphism can be defined in such a way that, for the spaces of dimension 2, one can establish classes of equivalence using the local connectivity properties of images. By comparing and computing only connectivity and the order of connectivity, one can obtain algorithms that assign an arbitrary image an equivalent one displaying certain features of regularity.

Other definitions (e.g., reducibility) of topological homeomorphism are the stronger equivalence relations in the sense that they preserve more topological invariants of figures homotopy, homology, and cohomology groups. These different approaches coincide for the connected figures in the discrete spaces of dimension 2.

In this formalism, isomorphic images are homeomorphic ones, while the local connectivity properties are the invariants that allow one to check whether or not two objects of the category are isomorphic. In this setting, the invariants of the category forget all except for connectivity equivalence. If one considers reducibility as a defining isomorphism, then one obtains "group valued" invariants (homotopy and homology groups).

The algebraic formalism of pattern generation and recognition is needed and has partially been completed in the recent years by the topological intrinsic definitions of the concepts involved, as well as by a certain number of results. However, the link between the two points of view is missed. Pavel was first to present a general formalism of pattern definition and recognition using the category theory, which unifies the two (algebraic and topological) convergent themes of this field. It states and proves the condition under which a recognition category can be associated with a category of images and also defines algebraic and topological invariants and recognition functions in their general meaning.

2.3. The Algebraic Approach to Recognition, Classification, and Forecasting Problems by Zhuravlev

The Algebraic Approach to Recognition, Classification, and Forecasting Problems (Zhuravlev [123, 124]) is a mathematical set-up for a pattern recognition problem, correctness and regularity conditions, and multiple classifiers.

One of the topical problems in image recognition is searching for an algorithm that would provide a correct classification of an image by its description (i.e., an algorithm that has zero errors on a control set of objects). For Zhuravlev, the relational approach implies that there are no accurate mathematical models for weakly formalized fields such as geology, biology, medicine, and sociology. However, in many cases, inexact methods based on heuristic considerations are effective in practice. Therefore, it is sufficient to construct a family of such heuristic algorithms for solving the corresponding problems and, then, construct an algebraic closure of this family. An existence theorem has been proved, stating that any problem in the set of problems associated with the study of poorly formalized situations is solvable in this closure.

Suppose that we have a certain set of admissible patterns described by n-dimensional feature vectors. The set of admissible patterns is covered by a finite number of subsets called classes. Suppose that we have l classes $K_1, \ldots, K_l$. There is a recognition algorithm A that constructs an l-dimensional information vector by an n-dimensional description vector. Recall that the information vector is a vector of membership of an object in the classes where the values of the elements of the information vector 0, 1, Δ are interpreted, according to [123], as follows: "the object does not belong to the class," "the object belongs to the class," and "the algorithm cannot determine whether or not the object belongs to the class." We assume that each recognition algorithm $A \in \{A\}$ can be represented as a sequential execution of algorithms B and C, where B is a recognition operator that transforms the learning information and the description of an admissible object into a numerical vector (called the estimate vector) and C is a decision rule that transforms an arbitrary numerical vector into an information vector.

The operation of a recognition algorithm can be schematically represented as follows.

Feature description of an object $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$

$\downarrow$ Recognition algorithm B

Vector of estimates for a class $\beta = (\beta_1, \beta_2, \ldots, \beta_l)$

$\downarrow$ Decision rule C

Information vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_l)$

Thus, during the solution of a recognition problem, an object of recognition, i.e., an image, is described by three different vectors: the n-dimensional feature vector, the l-dimensional vector of estimates for a class, and the l-dimensional information vector.

Let us briefly recall the pattern recognition problem in the standard formulation by Zhuravlev.

$Z(I_0, S_1, \ldots, S_q, P_1, \ldots, P_l)$ is a recognition problem, where I_0 is admissible initial information; $S_1, \ldots, S_q$ is a set of admissible objects described by feature vectors; $K_1, \ldots, K_l$ is a set of classes; and $P_1, \ldots, P_l$ is a set of predicates on the admissible objects, $P_i = P_i(S)$, $i = 1, 2, \ldots, l$. Problem Z consists in evaluating the predicates $P_1, \ldots, P_l$.

Definition 1. An algorithm is said to be correct for problem Z if the following equality holds:

$$A(I, S_1, \ldots, S_q, P_1, \ldots, P_l) = \|\alpha_{ij}\|_{q\times l}, \text{ where } \alpha ij = Pj(Si).$$

One of the main tasks of pattern recognition is searching for an algorithm that correctly solves the image recognition problem. Zhuravlev proved the existence theorem for such an algorithm, stating that the algebraic closure of the algorithms for estimate calculation (AECs) for the image recognition problem is correct. AECs are based on the formalization of the concepts of precedence or partial precedence: an algorithm analyzes the proximity between the parts of the descriptions of the previously classified objects and the object to be recognized.

Suppose that we have some standard descriptions of objects $\{\tilde{S}\}$, $\tilde{S} \in K_j$ and $\{S'\}$, $S' \in K_j$, as well as a method for determining the degree of proximity between certain parts of the description of S and the corresponding parts of the descriptions $\{I(\tilde{S})\}, \{I(S')\}$;

here, $S, j = 1, 2, ..., l$, is an object of recognition. When calculating estimates for the proximity between the parts of the descriptions $\{I(\tilde{S})\}$ and $\{I(S')\}$ and, correspondingly, between $I(S)$ and $I(S')$, one can construct a generalized estimate for the proximity between S and the sets of objects $\{\tilde{S}\}$, $\{S'\}$ (in the simplest case, the generalized estimate is a sum of estimates for the proximity between the parts of the descriptions). Then, using the set of estimates, a general estimate of an object over a class is formed, which is precisely the value of the membership function of the object in the class.

For the algebraic closure of AECs, the following existence theorem for an AEC is proved that correctly solves the recognition problem Z.

Theorem 1. *Suppose that the natural assumptions on the difference between the descriptions of classes and recognition objects hold for the vectors of features in the recognition problem Z. Then the algebraic closure of the class of AECs is correct for problem Z.*

The image recognition problem is one of the classical examples of problems with incompletely formalized and partially contradictory data. This implies that the application of the algebraic approach to image recognition may lead to important results; hence, the "algebraization" of this field is the most promising approach to development of the mathematical apparatus required for the analysis and estimation of the information represented as images.

In the case where recognition objects are images, this theorem cannot be applied directly. This is due to several reasons. First, the representation of an image by a feature vector (as in the case of a standard recognition object) often causes the loss of a considerable portion of the information about the image and, therefore, to misclassification. Second, the existence of equivalence classes is an essential difference between the image recognition problem and the recognition problem in the classical formulation.

Transition from the algebra of pattern recognition algorithms to the algebra of image recognition algorithms requires choosing, first, the algorithms used as elements of the algebra and, second, the algebraic representations of images that make it possible to formalize the selection of descriptors. In this connection, Schlesinger's work [109] on two-dimensional grammars should be noted: based on the representations of images by two-dimensional grammars, a unified formulation was proposed for such image processing and recognition problems that previously seemed to be significantly different; in addition, the computational complexity of the formulated problem in its general statement was analyzed. It is expedient to select representations taking into account the possibility of combining the initial information and algorithms of different types.

Image analysis and understanding have a certain peculiarity due to which the use of Zhuravlev's algebraic approach in its general form is inconvenient. This is because:

• the nature of the problem under consideration is not taken into account if algebraic methods are applied to the information represented in the form of images;

• the results of applying the theory are sometimes difficult to interpret;

• there are many natural transformations of images that are easy to interpreted from the user's point of view (for instance, rotation, contraction, stretching, color inversion, etc.) but are hardly representable by standard algebraic operations.

The necessity arises for using algebraic tools to record natural transformations of images. Moreover, the algebraization of image analysis and understanding must include the construction of algebraic descriptions of both the images themselves and the algorithms for their processing, analysis, and recognition.

Having analyzed the publications devoted to the applications of algebraic methods to image analysis and understanding, we can distinguish the following advantages of a unified representation of images and algorithms for their processing and analysis:

• construction of unified representations for descriptions of images;

• efficiency of transition from the input data in the form of images to different formal models of the images;

• naturalness of uniting the algebraic representation of information with the algebraic tools for pattern recognition that have been successfully employed;

• the possibility of using the methods of mathematical modeling employed in the application domains to which the processed images belong;

• the possibility of using image descriptions in the form of group-theoretic representations;

• naturalness of uniting the methods for structural analysis of images with the tools for probabilistic analysis;

• the possibility of formalized description of parallelization problems with due regard for the specifics of particular computational architectures.

2.4. Contribution of the Russian Mathematical School

Since the 1960s, Zhuravlev's school has been developing the algebraic approach to solving the problems of classification and/or pattern recognition as a means for building correct algorithms over specified sets of features. Within the framework of the algebraic approach, the algorithms are built as compositions of type, where A is the entire algorithm, B is a "base classifier" operator that maps the feature space into a matrix of estimates for the assignments of the objects' classes, and C is a "decision rule" operator that maps the matrix of estimates into a binary matrix of the answers of the entire algorithm A.

Zhuravlev's scientific school obtained several essential results in algebraic direction by Matrosov [60–63] and by Rudakov [95–102].

Within the category-theoretic approach to the algebraic approach developed by Rudakov [95–102], the composition scheme of the entire algorithm is complemented by corrective operations ("aggregative function") that are built over the space of the cartesian products of the answers of the B-operators. The aggregative functions provide more flexibility in achieving the correctness of algorithms over arbitrary selection of the training/control sets of objects.

The application of the category-theoretic apparatus to the constructions of the algebraic approach made it possible to demonstrate the universal nature of the constructions of the type B, which thus guarantees the existence of a correct solution for any non-contradictory sets of objects (B_1, ..., B_p are base classifiers).

This approach significantly improves the accuracy of classification and was applied in a number of fields: monitoring trade markets at the Moscow Interbank Currency Exchange (and other problems of analysis and prognosis of time series), text analysis problems (Antiplagiat system), problems of bioinformatics, etc. The purpose of the Antiplagiat system is detecting citations in documents: (1) protecting intellectual property from unauthorized copying and (2) finding duplicates and similar documents in vast storages. A problem-oriented formalism for describing the problem of protein secondary structure recognition was developed. Experiments were based on 165 000 precedents found in the Protein Data Bank (www.rcsb.org). The most informative feature values were effectively selected by solvability analysis, which ensured more than 95% solvability for the recognition problem on an arbitrary set of objects of a sufficient size.

Mazurov (and his followers) [49, 50, 70–73] began with this simple definition (see definition 1) and developed an elegant mathematical theory of discrete approximations for infeasible systems of constraints and collective learning algorithms in pattern recognition. Now this theory is known as the method of committees.

Recognition and forecasting are the fundamental concepts of mathematical modeling in a wide range of economical, social, and natural (e.g., geophysical) phenomena. These concepts lie in the basis of decision making. Committee constructions represent a class of discrete generalizations of the concept of solution for the problems that can be both feasible and infeasible (contradictory). The first result in this area belongs to Ablow and Kaylor [1], who formulated the committee solution concept for a system of linear inequalities in explicit terms.

A finite sequence (q is a tuple of vectors in $\mathbb{R}^n$) $Q = (x_1, \ldots, x_q)$ is called *a committee (generalized) solution* (or just a committee) of the system

$$a_j^T < b_j \quad (j = 1, 2, \ldots, m) \tag{1}$$

if, for any j, the majority of elements of Q satisfy the jth inequality. The number q is called *a number of the elements (length)* of the committee Q. Finally, the committee Q with the minimum length q for system (1) is called *a minimum committee.*

Theorem 2 [70]. *System* (1) *has a committee solution if and only if any of its subsystems of two inequalities is feasible*.

Theorem 3 [70]. *Let any subsystem of rank k (of system* (1)) *has a committee solution with no more than q elements. Then, the system itself also has a committee* solution of no more than $2q\left\lceil\frac{\lfloor (m-1)/2\rfloor}{k}\right\rceil + 1$ elements.

Theorem 4 [70]. *If any* (k + 1)-*subsystem of system* (1) *is feasible then this system has a committee solution of no more than* $2\left\lceil\frac{\lfloor (m-1)/2\rfloor}{k}\right\rceil + 1$ *elements.*

These results can be extended to infinite-dimensional spaces. Suppose that f_1, ..., f_m are real-valued functionals over a Banach space B. Consider a system of inequalities

$$f_l(x) > 0 \ (j = 1, 2, \ldots, m). \tag{2}$$

Theorem 5 [70]. *Suppose that* $f_1, \ldots, f_m$ *is a Frechet differentiable at the point* $x_0 = 0$ *such that*

(1) $f_j(x_0) = 0, j = 1, 2, \ldots, m$,

(2) *rank r of the system of linear functionals* $f_g'(0)$ is *positive*,

(3) *given the system*

$$f_g'(0)x > 0 \ (j = 1, 2, \ldots, m),$$

any (k + 1)-*subsystem is feasible for* $0 \le k < r$.

Then, system (2) has a committee solution.

Suppose that X is a real vector space (e.g., $\mathbb{R}^n$), which is called the *feature space* and can be interpreted as a space of measurements over the objects to be recognized, and $Y = \{0, 1, \ldots, K - 1\}$ is a finite set of *patterns.* Any function $f: X \to Y$ is called *a decision rule (or classifier)* and can be used to classify an object o by using a vector $x = x(o)$ obtained during certain measurements over it. Denote a family of feasible decision rules by $\mathcal{F}$.

A decision rule $F[f_1, \ldots, f_q], f_i \in \mathcal{F}$ is called *committee decision rule* if, for any $x \in X$, $F(x) = y$ if and only if

y is the maximum element of Y for which the most part of $f_i(x) = y$.

Learning in the class of committee decision rules is simply searching for the most admissible number q and functions $f_1, \ldots, f_q$ for a training sample and is closely related to the concept of a *separating committee*. Introduce it for the simplest case where $K = 2$.

A finite sequence $Q = (f_1, \ldots, f_q)$ is called *a separating committee* for finite subsets A, $B \subset X$ if the major part of $f_i(a) = 1$ for any $a \in A$ and, conversely, the major part of $f_j(b) = 0$ for any $b \in B$.

Most of the results were obtained in the field of learning in the class of *affine* decision rules, where $f_i(x) = w_i^T x + b_i$ for a vector w_i and bias b_i.

Theorem 6 [70]. *Suppose that A, $B \subset X$. An affine separating committee for the sets A and B exists if and only if $A \cap B = \varnothing$.*

Theorem 7 [70]. *Suppose that $\mathcal{F}_q$ is a family of linear committee decision rules over $\mathbb{R}^n$; then, $VCD(\mathcal{F}_q) = 0(nq)$.*

As stated in Theorem 7, any time during the learning procedure (due to the well-known Vapnik–Chervonenkis theory), it is important to construct a committee decision rule with the least possible q (for a given training sample). This fact motivates the study of the following combinatorial optimization problem and related problems.

Minimum affine separating committee (MASC) problem. For finite A, $B \subset \mathbb{R}^n$, it is required to find an affine separating committee $Q = (f_1, \ldots, f_q)$ with the minimum possible q.

Below is a list of the selected results.

1. The MASC problem is strongly NP-hard and remains intractable even under the following additional constraints:

(a) dimensionality $n > 1$ is fixed;

(b) A, $B \subset \{-1, 0, 1\}^n$;

(c) A, B are in the general position.

2. The MASC problem is solvable in a polynomial time if

(a) $n = 1$;

(b) the sets A and B are induced by the sets uniformly distributed (in terms of D. Gale) on a unit sphere.

3. The MASC problem is poorly approximable. In particular, it does not belong to the APX approximability class (unless $P = NP$) and is MaxSNP-hard for any fixed $n > 1$.

4. Several polynomial approximation algorithms were developed. The best know approximation guarantee is $O(n)$.

There are some other results.

1. The concept of a *hypergraph of maximal* (*by inclusion*) *feasible subsystems* (*HMFS*) for a system in question; a characterization theorem for HMFSs; a classification of the minimal committee generalized solutions in terms of their HMFS; and existence theorems of committee solutions in terms of the HMFS.

2. An antagonistic game against nature based on the committee solutions existence was studied; the equilibrium conditions were obtained.

3. The computational complexity and approximability of several combinatorial optimization problems related to affine separating committees were investigated.

Aside from the basic researches by Zhuravlev's scientific school, there are a significant number of papers concerned with algebraic methods for analysis and estimation of the information represented as signals, particularly, Labunetc [55], Pytiev [84], Sinicyn [106], Furman [16–19], and Chernov [4, 5, 15].

Furman [16–19] considers the methodology of the vector signal processing theory: the basis of information, the signal and its mathematical model, the mathematical apparatus of the signal processing theory, the vector-geometrical representation of signals, random vector signals, scalar multiplication in problems of vector signal processing, the vector product of vectors, and the cartesian reference system. Furman introduces new types of signals: complex and quaternionic; these signals are used for recognition of boundary points (contours) of an image, as well as for image analysis. The properties of scalar multiplication, the orthogonal basis, the questions of spectral and correlation analysis, and the questions of matched filtering are described for each signal.

Furman considered complex discrete signals: the assignment of complex signals, complex numbers as elements of a complex linear space, the spectral analysis of complex digital signals, correlation functions of complex digital signals, and contour matched filtering.

The most interesting part of Furman's research is devoted to discrete quaternion signals. Hypercomplex numbers, the association of quaternions with complex numbers, the scalar product of a quaternion, the rotation of vectors in the three-dimensional space, quaternion discrete signals, the orthogonal basis in a quaternionic space, the spectral representation of discrete quaternion signals, the decomposition of discrete quaternion signals, correlation functions of discrete quaternion signals, the matched filtering of discrete quaternion signals, and the conjugate-matched filtering of discrete quaternion signals were considered.

2.5. Image Algebras

Mathematical morphology [43, 74] proposed by Minkowski and Hadwiger and developed by Matheron [64] and Serra [103–105] seems to be the first attempt to create a theoretical apparatus that allows one to describe many widespread operations of image processing in the composition of quite a small set of standard simple local operations. Such representations allow one to formalize the choice of procedures for image processing and can easily be implemented on parallel architectures. It might have been a success of mathematical morphology, which initiated numerous attempts of algebraization both in the domain of algorithm representations and in closed domains. Mathematical morphology is an effective tool for uniform representation of local operations of image processing, analysis, and understanding in terms of algebras over sets. It describes algorithms for image transformations in terms of four basic local operations, namely, erosion, dilatation, opening, and closing; moreover, any two of these operations form a basis in terms of which the other two operations can be easily expressed. This is very convenient for the development of software systems, in which the user can quickly design particular algorithms from basic blocks. It is also convenient to use morphological operations with other theories, particularly, with the graph theory [6], etc. [119]. Mathematical morphology is widely used for solving applied problems of image analysis [52].

On the basis of mathematical morphology, Sternberg [112–114] introduced the concept of the IA.

The IA makes it possible to represent algorithms for image processing in the form of algebraic expressions, where variables are images, while operations are geometrical and logical transformations of images. The capabilities of mathematical morphology are known to be very limited. In particular, many important and widely used operations of image processing (feature extraction based on the convolution operation, Fourier transforms, use of the chain code, equalization of histograms, rotations, recording, and nose elimination), except for the simplest cases, can hardly (if ever) be implemented in the class of morphological operations.

The impossibility of constructing a universal algebra for image processing tasks on the basis of the morphological algebra can be explained by the limitations of the basis consisting of the set-theoretical operations of addition and subtraction in terms of Minkowski.

It is well known that this basis has the following drawbacks [89]:

• difficult implementation of widely used operations of image processing;

• impossibility of establishing a correspondence between the operations of mathematical morphology and linear algebra;

• impossibility of using mathematical morphology for transformations between different algebraic structures, in particular, sets including real and complex numbers and vector quantities.

These problems were solved in the standard IA by Ritter [89, 90] on the basis of a more general algebraic representation of the image processing and analysis operations. The standard IA is a unified algebraic representation of the image processing and analysis operations. The IA generalizes the well-known local methods for image analysis, particularly, mathematical morphology, and has the following advantages over mathematical morphology:

• it allows one to work with both real and complex quantities;

• it allows both scalar and vector data to be included into the input information;

• it makes image-algebra structures consistent with linear structures;

• it provides a more accurate and complete description of its operations and operands;

• with the help of a special structure “template,” composite operations of image processing are divided into a number of the simplest parallel operations.

A bottleneck in the applications of IA methods to image recognition is the choice of a sequence of algebraic operations and templates for representation of composite operations of image processing.

Presently, this choice is usually made based on the general representations of the character of images and tasks. Deficiencies of this approach are obvious: first, it is subjective and its success depends heavily on the user experience; second, it is intended to solve problems of a specific narrow class. The IA generalizes the well-known local methods for image analysis, particularly, mathematical morphology.

Investigations in the area of algebraization and image analysis carried out in the 1970–1980s represent a source of development of the DIA by Gurevich [28, 31, 32, 34, 37]. The DIA is a unified algebraic language for describing, performance estimating, and standardizing the representation of algorithms for analysis, recognition, and understanding of images and image models.

An object that is most closely related to the developed mathematical object is the image algebra proposed and developed by Ritter [90]. Ritter’s main goal in developing the image algebra was to design a standardized language for description of image processing algorithms intended for parallel execution of operations. A key difference between the new image algebra and Ritter’s standard IA is that the DIA is developed as a descriptive tool, i.e., as a language for description of algorithms and images, rather than as a language for algorithm parallelization.

The conceptual difference between the DIA and the standard IA is that the objects of this algebra (along with its algorithms) are descriptions of input information. The DIA generalizes the standard IA and allows one to use (as ring elements) the basic models of images and operations on images or the models and operations simultaneously. In the general case, the DIA is a direct sum of rings whose elements can be images, image models, operations on images, and morphisms. As operations, we can use both standard algebraic operations and specialized operations of image processing and transformations represented in an algebraic form. For a wide use of the DIA, it is necessary to investigate its capabilities and try to unite all possible algebraic approaches, e.g., use the standard image algebra as a convenient tool for recording certain algorithms for image processing and understanding or use Grenander's concepts [21–23] for representation of input information.

In the 1980s, Sternberg [112] formalized the concept "image algebra" and introduced the following definition.

The IA is a representation of algorithms for image processing on a cellular computer in the form of algebraic expressions whose variables are images and whose operations are procedures for constructing logical and geometrical combinations of images.

This IA is described based on mathematical morphology and is identified by the author with mathematical morphology. In 1985, Sternberg [113, 114] noted that the languages for image processing were developed for each processor architecture, and none of them was created for one computer and run on another. However, there are explicit language structures that satisfy the same principles. The image algebra (or mathematical morphology) was developed precisely for the description of these structures. Ritter's IA generalizes mathematical morphology, unites the apparatus of local methods for image analysis with linear algebra, and generates more complex structures. Examples of such structures are templates and morphological algorithms. In [90], various operations and operands of the standard image algebra, as well as applications of these structures to actual problems, are described. Since the standard IA does not simply generalize mathematical morphology but is a wider and more convenient structure, the image algebra language enables both the implementation of well-known algorithms and the design of new ones. The structure of the standard image algebra can be extended by introducing new operations. Hence, it can be successfully applied in the cases where morphology and linear algebra fail to provide a satisfactory result.

The standard IA [89, 90] is a heterogeneous (or multivalued) algebra [3] with a complex structure of operands and operations if the basic operands are images (sets of points), as well as values and characteristics related to these images (sets of values related to these points).

When analyzing the existing algebraic apparatus [3, 54, 57–59], we came to the formulation of the following requirements on the language designed to record the algorithms for solving the problems of image processing and understanding [34]:

- the new algebra must enable the processing of images as objects of analysis and recognition;
- the new algebra must enable operations on image models, i.e., arbitrary formal representations of images, which are objects and, sometimes, results of analysis and recognition; the introduction of image models is a step in formalizing the initial data of algorithms;
- the new algebra must enable operations on the main models of procedures for image transformations;
- it is convenient to use procedures for image modifications both as operations of the new algebra and as its operands to construct compositions of the basic models of the procedures.

An algebra [31] is called a DIA if its operands are either image models (for instance, as a model, we can take the image itself or a collection of values and characteristics related to it), or operations on images, or models and operations simultaneously.

It should be noted that, due to the variety of "algebras," we need to specify what kind of algebra is meant in the definition of the DIA. For the generality of the results and extension of the application domain of the new algebra, to define a DIA with one ring, we use the definition of the classical algebra of Van der Waerden [121].

Thus, the DIA with one ring must satisfy the properties of classical algebras. The DIA with one ring is the basic DIA as it contains a ring of elements of the same nature, i.e., either a ring of image models or a ring of operations on images.

Now we specify the place chosen for the DIA in the structure of algebra. Figure 1 presents a classification that reflects the authors' point of view on the contemporary hierarchy of algebras and the place of the DIA in this hierarchy.

To design efficient algorithmic schemes for image analysis and understanding, it is necessary to investigate different types of operands and different types of operations applicable to the chosen operands that generate the DIA.

3. THE DESCRIPTIVE APPROACH TO IMAGE ANALYSIS AND UNDERSTANDING

The current state and development trends in descriptive image analysis are determined by the methods, models, and results of the DA obtained during its elaboration [25–40].

This section presents a methodology, as well as mathematical and computational techniques, for automation of image mining based on the DA. These researches on the mathematical fundamentals of the image analysis and recognition procedures were conducted at the Federal Research Center Computer Science and Control of the Russian Academy of Sciences, Moscow, Russia.

During the development and implementation of the DA, a new class of image algebras—DIA—was introduced, defined, and investigated; the main types of image models were introduced, classified, and investigated; axioms of the descriptive theory of image

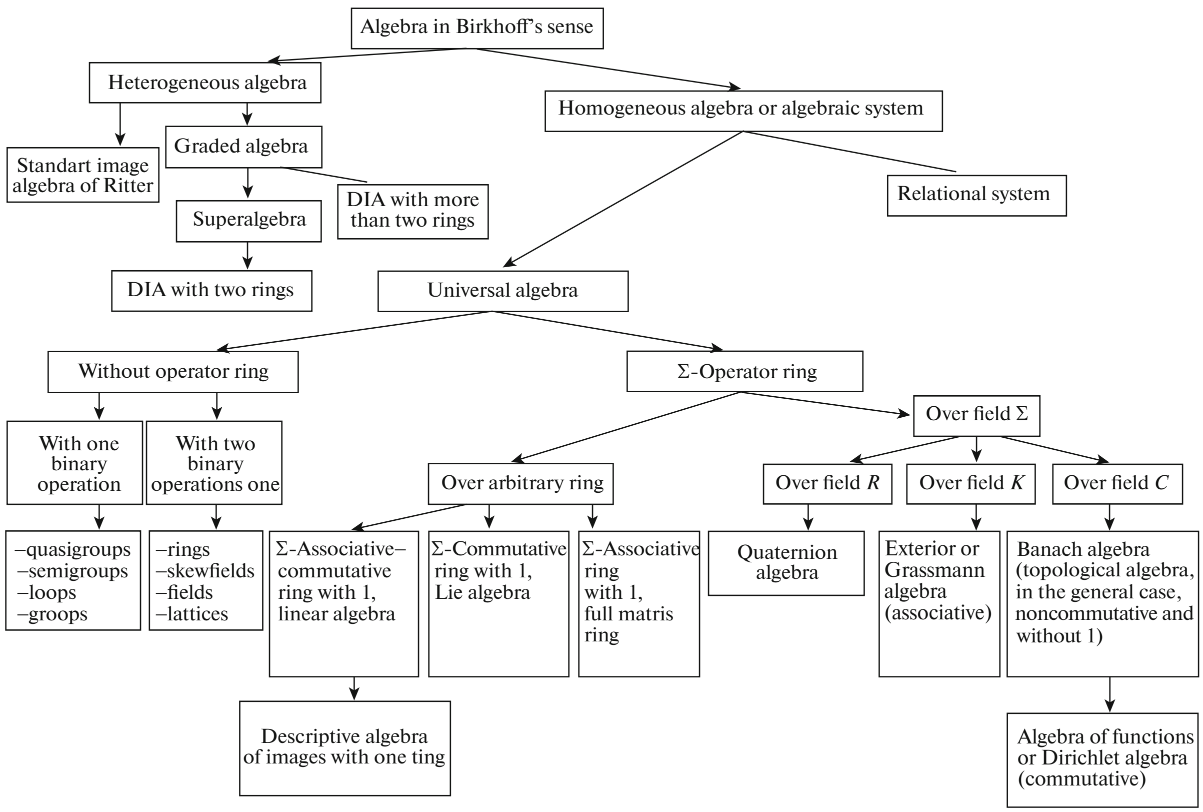

Fig. 1. Algebraic scheme.

analysis were introduced; the general model of the image recognition process was defined and investigated; new statements of image analysis and recognition problems were introduced; a concept of image equivalence in the recognition problem was introduced and investigated; new classes of image recognition algorithms were defined and investigated; and an image formalization space was introduced, defined, and analyzed.

In applied mathematics and computer science, constructing and applying the mathematical and simulation models of objects, as well as procedures to transform them, is a conventional method of standardization. It is the necessity for solving complex recognition problems and developing structural recognition methods and specialized image languages that generated the interest in formal description models of initial data and in the formalization of the descriptions of the procedures for their transformation in the area of pattern recognition (and especially in image recognition in the 1960s).

As for significant achievements in this "descriptive" line of study, we mention here the publications by Rosenfeld [93, 94], Evans [13, 14], Narasimhan [76–79], Kirsh [51], Shaw [107, 108], Barrow et al. [2], and Kaneff [48].

In the 1970s, Zhuravlev proposed the so-called algebraic approach to recognition and classification problems [123, 124], where he defined formalization methods for describing heuristic algorithms of pattern recognition and proposed a universal structure of recognition algorithms. This approach forms the basis of the modern mathematical image recognition theory. In the same years, Grenander formulated his pattern theory [21–23], where he considered methods of data representation and transformation in recognition problems in terms of regular combinatorial structures, leveraging the algebraic and probabilistic apparatus. Both the approaches addressed the recognition problem in its classical statement and did not consider the representation of initial data in the form of images.

Then, until the mid-1990s, the interest in the descriptive and algebraic aspects of pattern recognition and image analysis slightly dropped.

By the mid-1990s, it became apparent that, for the development of image analysis and recognition, it is crucial to:

• understand the nature of initial information, i.e., images;

• find the methods for image representation and description that make it possible to construct image models for recognition problems;

• establish a mathematical language for unified description of image models and their transformations that make it possible to construct image models and solve recognition problems;

• construct models to solve recognition problems in the form of standard algorithmic schemes that make it possible, in the general case, to transfer from an original image to its model and from the model to a sought-for solution.

The DA solves the fundamental problems of formalization and systematization of the methods and forms of information representation in the problems of image analysis, recognition, and understanding. In particular, the problems arise in connection with the automation of information extraction from images for intelligent decision making (diagnostics, prediction, detection, evaluation, and identification of patterns). The final goal of this research is automated image mining: (1) automated design, testing, and adaptation of techniques and algorithms for image recognition, estimation, and understanding; (2) automated selection of techniques and algorithms for image recognition, estimation, and understanding; and (3) automated testing for raw data quality and suitability to solve the image recognition problem.

The axiomatics and formal structures of the DA provide methods and tools to represent and describe images for their subsequent analysis and evaluation. The theoretical basis of the research is the DA; general algebraic methods; and methods of the mathematical theories of image processing, image analysis, and pattern recognition.

It is established that the overall success and effectiveness of the analysis and evaluation of the information represented in the form of images are determined by the capabilities of the IRRF.

The IRRF processes are crucial for solving applied image analysis problems and, particularly, for making intelligent decisions based on the information extracted from images. The DA makes it possible to solve both the problems associated with the construction of formal descriptions of images as objects of recognition and the problems of synthesis of the procedures for pattern recognition and image understanding. An operational approach to characterization of images requires that the processes of analyzing and evaluating the information represented in the form of images (the trajectory of problem solving), as a whole, be viewed as a sequence/combination of transformations and computations of a set of intermediate and final (defining the solution) evaluations. These transformations are defined on the equivalence classes of images and their representations. The latter are defined descriptively, i.e., by using the basic set of prototypes and the corresponding generative transformations that are functionally complete with respect to the equivalence class of admissible transformations.

This section contains a brief description of the principal features of the DA that are needed to understand the meaning of the introduction of the conceptual apparatus and schemes of the IRRF, which are proposed to formalize and systematize the methods and forms of image representation.

The main intention of the DA is to structure different techniques, operations, and representations used in image analysis and recognition. The axiomatics and formal constructions of the DA establish the conceptual and mathematical basis for representing and describing images, as well as for its analysis and estimation. The DA provides a methodology and a theoretical basis for solving the problems associated with the development of formal descriptions for the image as an object of recognition, as well as for solving the problems associated with the synthesis of transformation procedures for image recognition and understanding. The analysis of the problems is based on the investigation of the inner structure and content of an image as a result of the procedures "constructing" it from its primitives, objects, descriptors, features, tokens, and relations between them.

The automated extraction of information from images includes (1) the automation of the development, testing, and adaptation of methods and algorithms for analysis and evaluation of images; (2) the automation of the selection of methods and algorithms for analysis and evaluation of images; (3) the automation of the evaluation of quality and adequacy of initial data to solve the image recognition problem; and (4) the development of standard technological schemes for detecting, assessing, understanding, and retrieving images.

The automation of information extraction from images requires a complex use of all features of the mathematical apparatus used or potentially suitable for use in determining the transformations of the information represented in the form of images, specificaly, in the problems of processing, analysis, recognition, and understanding of images.

The main purpose of the DA is to structure and standardize the variety of methods, processes, and concepts used for analysis and recognition of images.

The DA is proposed and developed as a conceptual and logical basis for extraction of information from images. It includes the following basic tools for image analysis and recognition: a set of methods for analysis and recognition of images, IRRF techniques, a conceptual system of image analysis and recognition, descriptive image models [36–38], classes, a DIA language [28, 31, 32, 34, 37], statements of image analysis and recognition problems, and a basic model of image recognition.

The main areas of research within the DA are (1) the creation of axiomatics for analysis and recognition of images; (2) the development and implementation of a common language to describe the processes of analysis and recognition of images (the study of the DIA); and (3) the introduction of formal systems based on certain regular structures to determine the processes of analysis and recognition of images.

The mathematical foundations of the DA are (1) the algebraization of information extraction from images; (2) the specialization of Zhuravlev's algebra [123, 124] to the case of representation of the recognition source data in the form of images; (3) a standard language for describing the procedures for analysis and recognition of images (DIA) [28, 31, 32, 34, 37]; (4) the mathematical formulation of the problem of image recognition; (5) mathematical theories of image analysis and pattern recognition; and (6) a model of the process for solving the standard problem of image recognition.

The main objects and means of the DA are (1) images; (2) a universal language (DIA) [28, 31, 32, 34, 37]; (3) two types of descriptive models, namely, an image model [36–38] and a model for solving the procedures of problems of image recognition and their implementation [32]; (4) descriptive algebraic schemes of image representation [36–38]; and (5) multimodel and multiaspect representations of images that are based on generating descriptive trees [33].

The basic methodological principles of the DA are (1) the algebraization of image analysis; (2) the standardization of the representation of problems of analysis and recognition of images; (3) the conceptualization and formalization of the phases of transforming the image while solving the recognition problem; (4) the classification and specification of admissible models of images (descriptive image models); (5) the IRRF; (6) the use of the standard algebraic language of the DIA for describing models of images and procedures for their construction and transformation; (7) the combination of algorithms in multialgorithmic schemes; (8) the use of multimodel and multiaspect representations of images; (9) the construction and use of the basic model of the solution process for the standard problem of image recognition; and (10) the definition and use of a nonclassical mathematical theory for the recognition of new formulations of problems of analyzing and recognizing images.

Note that the construction and use of the mathematical and simulation models of objects and procedures used for their transformation is an accepted method of standardization in the applied mathematics and computer science.

The creation of the DA was significantly influenced by the following basic theories of pattern recognition: (1) the algebraic approach to pattern recognition by Zhuravlev [123, 124] and his algorithmic algebra; (2) the theory of images by Grenander [21–23], particularly, algebraic methods for representation of source data in image recognition problems developed in it; and (3) the theory of vision by Marr [69].

As noted above, in the DA, it is proposed to carry out the algebraization of the analysis and recognition of images by using the DIA. The DIA was developed based on the studies in the field of the algebraization of pattern recognition and image analysis that have been carried out since the 1970s. The creation of a new algebra was directly influenced by Zhuravlev's algorithms [123, 124] and the research by Sternberg [112] and Ritter [86–92], who identified classic versions of image algebras.

A more detailed description of the DA methods and tools obtained in the development of its results can be found in [25–40].

By now, in the framework of the DA, the following main results were obtained.

1. Algebraization of image analysis:

• the DAIA was characterized;

• DAIA axioms were introduced and substantiated;

• a mathematical object "DIA" was introduced and defined;

• new statements of image recognition problems were introduced and substantiated;

• descriptive image algebras with one ring (DIA1Rs) were introduced and investigated;

• the definition, method, and necessary and sufficient conditions for the construction of DIA1Rs were proposed;

• specialized versions of DIA1Rs over images, over image models, and over image transformations were defined;

• a set of operations for the standard IA that enable the construction of DIA1Rs was defined;

• the DIA1R classes generating the classes of image models were defined.

2. Effective methods and tools for the description and representation of images in recognition problems:

• an image formalization space was introduced, defined, and investigated;

• the structure of the image formalization space was defined;

• the topological properties of the image formalization space were investigated;

• an object "descriptive algorithmic image transformation scheme" (DAITS) was introduced, defined, and investigated;

• a classification of descriptive algorithmic image transformation schemes was constructed;

• the standardization of representation was carried out and some examples of descriptive algorithmic image transformation schemes were constructed for solving applied problems of analysis of biomedical images;

• a mathematical object "image representation" was introduced and defined;

• types of image representations were introduced and defined;

• a mathematical object "descriptive image model" was introduced and defined;

• types of descriptive image models were introduced, defined, and investigated;

• a model for solving the image recognition problem based on descriptive image models was proposed and implemented;

• a method for selection of image transformations that takes into account the information nature of images was proposed and implemented in software;

• criteria for classification of the features used for description of images were formulated;

• a classification of image features as a tool for formal description of image representations was constructed;

• a method for constructing multiple-aspect image descriptions based on classifications of image features was proposed.

3. Effective algorithms based on estimate calculation for image recognition problems:

• a multilevel model for combining algorithms and initial data in image recognition that is based on the combined use of multi-algorithmic classifiers and dual representations of initial data of the type "combinatorial structures of local neighborhoods/formalized models" was introduced and substantiated;

• a class of effective recognition algorithms based on estimate calculation (RAECs) that allow for spatial data was defined and investigated;

• a class of RAECs with square support sets was defined and investigated;

• a method for effective implementation of RAECs with systems of rectangular support sets and their compositions that is based on two-step procedures for finding rectangles was proposed;

• methods for effective implementation of RAECs with systems of rectangular support sets and proximity functions on pairs of support sets were proposed;

• a method was proposed for constructing and estimating the computational complexity of RAECs with systems of support sets generated by arbitrary etalons that are based on multistep procedures for finding etalons on the raster.

4. Linguistic and knowledge-oriented tools for supporting the automation of image analysis:

• an informational web resource (its concept, architecture, and functional scheme) for processing, analysis, and recognition of images was proposed and implemented;

• a classification of problems of image processing, analysis, and recognition was constructed;

• an Automated Knowledge Base System for Processing, Analysis, and Recognition of Images was developed;

• an Information Retrieval Thesaurus for Image Analysis was developed;

• a Thesaurus of Cytohematological Terms for a Aystem of Automating the Analysis of Blood Cell Images was developed;

• an experimental version of an image analysis ontology in the OWL language was developed;

• a Knowledge Base for Processing, Analysis, and Recognition of Images was developed.

5. Developments:

• Open System for Automation of Processing, Analysis, and Recognition of Images;

• Automated System of Analysis of Biomedical Microimages for Detection and Characterization of Informative Objects of a Given Form on a Heterogeneous Background;

• an image feature calculation library for a "Chernyi kvadrat" (Black Square) software tool complex for automation of scientific research and training in the field of processing, analysis, recognition, and understanding of images.

CONCLUSIONS

The critical points of solving the image analysis problem are as follows.

1. Precise setting of the problem.

2. Correct and "computable" representation of the raw and processed data for each algorithm at each stage of processing.

3. Automated selection of an algorithm:

(a) decomposition of the solution process for basic stages;

(b) indication of points of potential improvement of the solution ("branching points");

(c) collection and application of the problem solving experience;

(d) selection of basic algorithms, basic operations, and basic models (operands) for each problem solution stage;

(e) classification of the basic elements.

4. Performance estimation at each step of processing and solution:

(a) analysis, estimation, and use of the raw data specificity;

(b) diversification of mathematical tools used for performance estimation;

(c) reduction of raw data to the requirements of the algorithms selected.

The DA, as demonstrated by the results of its development and application, is a promising basis for creating a descriptive theory of image analysis. The way to its creation involves research and development in the following directions.

1. Mathematical settings of new image recognition problem.

2. Descriptive image algebras.

3. Descriptive image models and image features classification.

4. Image formalization space, including its topologies, image representations, and problem solving trajectories.

5. Descriptive algorithmic schemes.

6. Generating descriptive trees and multiple model representations of images.

7. Linguistic and knowledge-oriented tools.

8. Image equivalence.

9. Image metrics.

10. Pattern recognition algorithms accepting 2D and 3D data.

11. Combined use of the multiplicity of image representations and the hierarchical model of multi-algorithmic classifiers.

We hope that, after following the steps above mentioned, we will be able to formulate the axiomatic and basic statements of the descriptive theory of image analysis.

The main results of the DA will be detailed and interpreted in a series of future papers.

The development of the DA should result in the formation of a descriptive theory of image analysis.

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REFERENCES

1. C. M. Ablow and D. J. Kaylor, "Inconsistent homogeneous linear inequalities, Bull. Am. Math. Soc. **71** (5), 724 (1965).
2. H. G. Barrow, A. P. Ambler, and R. M. Burstall, "Some techniques for recognizing structures in pictures," in *Proc. Int. Conf. on Frontiers of Pattern Recognition,* Ed. by Satosi Watanabe (Academ. Press, New York, London, 1972), pp. 1–30.
3. G. Birkhoff and J. D. Lipson, "Heterogeneous algebras," J. Combinat. Theory **8**, 115–133 (1970).
4. V. M. Chernov, "Clifford algebras are group algebras projections," in *Advanches in Geometric Algebra with Applications in Science and Engineering,* Ed. by E. Bayro-Corrochano and G. Sobczyk (Birkhauser, Boston, 2001), pp. 467–482.
5. V. M. Chernov, "On defining equations for the elements of associative and commutative algebras," in *Space-Time Structure. Algebra and Geometry,* Ed. by D. Pavlov, Gh. Atanasiu, and V. Balan (Lilia Print, 2007), pp. 182–188.
6. J. Crespo, J. Serra, and R. W. Schaffer, "Graph-based morphological filtering and segmentation," in *Proc. 6th Symp. on Pattern Recognition and Image Analysis* (Cordoba, 1995), pp. 80–87.
7. J. L. Davidson, "Classification of lattice transformations in image processing," Comput. Vision, Graph., Image Processing: Image Understand. **57** (3), 283–306 (1993).
8. M. J. B. Duff, D. M. Watson, T. J. Fountain, and G. K. Shaw, "A cellular logic array for image processing," Pattern Recogn. **5** (3), 229–247 (1973).
9. E. R. Dougherty and D.Sinha, *Computational Gray-Scale Mathematical Morphology on Lattices (A Comparator-Based Image Algebra),* Part 1: *Architecture. Real-Time Imaging* (Acad. Press, 1995), Vol. 1, pp. 69–85.
10. E. R. Dougherty and D. Sinha, *Computational Gray-Scale Mathematical Morphology on Lattices (A Comparator-based Image Algebra),* Part 2: *Image Operators. Real-Time Imaging* (Acad. Press, 1995), Vol. 1, pp. 283–295.
11. E. R. Dougherty, "A homogeneous unification of image algebra. Part I: The homogenous algebra," Imaging Sci. **33** (4), 136–143 (1989).
12. E. R. Dougherty, "A homogeneous unification of image algebra. Part II: Unification of image algebra," Image Sci. **33** (4), 144–149 (1989).
13. T. G. Evans, "A formalism for the description of complex objects and ist implementation," in *Proc. 5th Int. Conf. on Cybernetics* (Namur, Sept. 1967).
14. T. G. Evans, "Descriptive pattern analysis techniques: Potentialities and problems," in *Proc. Int. Conf. on Methodologies of Pattern Recognition* (Acad. Press, New York, London, 1969), pp. 149–157.
15. M. Felsberg, Th. Bulov, G. Sommer, and V. M. Chernov, "Fast algorithms of hypercomplex Fourier transforms," in *Geometric Computing with Clifford Algebras,* Ed. by G. Sommer (Springer Verlag, 2000), pp. 231–254.
16. Ya. A. Furman, "Parallel recognition of different classes of patterns," Pattern Recogn. Image Anal. **19** (3), 380–393 (2009).
17. Ya. A. Furman, "Recognition of vector signals represented as a linear combination," J. Commun. Technol. Electron. **55** (6), 627–638 (2010).
18. Ya. A. Furman and I. L.Egoshina, "Inverse problem of rotation of three-dimensional vector signals," Optoelectron. Instrumentat. Data Processing **46** (1), 37–45 (2010).
19. Ya. A. Furman, R. V. Eruslanov, and I. L. Egoshina, "Recognition of images and recognition of polyhedral

objects," Pattern Recogn. Image Anal. **22** (1), 196–209 (2012).

20. P. D. Gader, M. A. Khabou, and A. Koldobsky, "Morphological regularization neural networks," Pattern Recogn. **33**, 935–944 (2000).
21. U. Grenander, *Lectures in Pattern Theory* (Sprinder-Verlag, New York, 1976–1981).
22. U. Grenander, *General Pattern Theory. A Mathematical Study of Regular Structure* (Clarendon Press, Oxford, 1993).
23. U. Grenander, *Elements of Pattern Theory* (Johns Hopkins Univ. Press, 1996).
24. J. Grin, J. Kittler, P. Pudil, and P. Somol, "Information analysis of multiple classifier fusion," in *Proc. 2nd Int. Workshop on Multiple Classifier Systems, MCS 2001, Cambridge, UK, July 2001* (Springer-Verlag, 2001), pp. 168–177.
25. I. B. Gurevich, "Algebraic approach for pattern recognition and analysis," in *Proc. All-Union Conf. Mathematical Methods for Pattern Recognition (Dilizhan, May 16–21, 1985)* (Armenian SSR Acad. Sci., Erevan, 1985), p. 55 [in Russian].
26. I. B. Gurevich, "The descriptive framework for an image recognition problem," in *Proc. 6th Scandinavian Conf. on Image Analysis* (Pattern Recognition Soc. of Finland, 1989), Vol. 1, pp. 220–227.
27. I. B. Gurevich, "Descriptive technique for image description, representation and recognition," Pattern Recogn. Image Anal.: Adv. Math. Theory Appl. USSR **1**, 50–53 (1991).
28. I. B. Gurevich, "The descriptive approach to image analysis. Current state and prospects," in *Proc. 14th Scandinavian Conf. on Image Analysis* (Springer-Verlag, Berlin, Heidelberg, 2005), pp. 214–223.
29. I. B.Gurevich and I. A. Jernova, "The joint use of image equivalents and image invariants in image recognition," Pattern Recogn. Image Anal.: Adv. Math. Theory Appl. **13** (4), 570–578 (2003).
30. I. B. Gurevich and I. V. Koryabkina, "Comparative analysis and classification of features for image models," Pattern Recogn. Image Anal.: Adv. Math. Theory Appl. **16** (3), 265–297 (2006).
31. I. B. Gurevich and V. V. Yashina, "Descriptive image algebras with one ring," Pattern Recogn. Image Anal.: Adv. Math. Theory Appl. **13** (4), 579–599 (2003).
32. I. B. Gurevich and V. V. Yashina, "Algorithmic scheme based on a descriptive image algebra with one ring: image analysis example," Pattern Recogn. Image Anal.: Adv. Math. Theory Appl. **15** (1), 192–194 (2005).
33. I. B. Gurevich and V. V. Yashina, "Generating descriptive trees," in *Proc. 10th Int. Fall Workshop on Vision, Modeling, and Visualization* (2005), pp. 367–374.
34. I. B. Gurevich and V. V. Yashina, "Operations of descriptive image algebras with one ring," Pattern Recogn. Image Anal.: Adv. Math. Theory Appl. **16** (3), 298–328 (2006).
35. I. B. Gurevich and V. V. Yashina, "Computer-aided image analysis based on the concepts of invariance and equivalence," Pattern Recogn. Image Anal.: Adv. Math. Theory Appl. **16** (4), 564–589 (2006).
36. I. B. Gurevich and V. V. Yashina, "Descriptive approach to image analysis: image models," Pattern Recogn. Image Anal.: Adv. Math. Theory Appl. **18** (4), 518–541 (2008).
37. I. Gurevich and V. Yashina, "Image formalization via descriptive image algebras," in *Proc. 3rd Int. Workshop on Image Mining Theory and Applications—IMTA 2010 (in Conjunction with VISIGRAPP 2010), Angers, France, May 2010,* Ed. by I. Gurevich, H. Niemann, and O. Salvetti (INSTICC Press, 2010), pp. 19–28.
38. I. B. Gurevich and V. V. Yashina, "Image formalization space: Formulation of tasks, structural properties, and elements," Pattern Recogn. Image Anal.: Adv. Math. Theory Appl. **21** (2), 134–139 (2011).
39. I. B. Gurevich and V. V. Yashina, "Descriptive approach to image analysis: Image formalization space," Pattern Recogn. Image Anal. **22** (4), 495–518 (2012).
40. I. Gurevich, Yu. Trusova, and V. Yashina, "Current trends in the algebraic image analysis. A survey," in *Proc. 18th Iberoamerican Congress on Progress in Pattern Recognition, Image Analysis, Computer Vision, and Applications. CIARP 2013, Havana, Cuba, Nov. 20–23, 2013,* Ed. by J. Ruiz-Shulcloper and G. Sanniti di Baja (Springer-Verlag, Berlin, Heidelberg, 2013), Ch. 1, pp. 423–430.
41. I. B. Gurevich and Yu. I. Zhuravlev, "Computer science: subject, fundamental research problems, methodology, structure, and applied problems," Pattern Recogn. Image Anal.: Adv. Math. Theory Appl. **24** (3), 333–346 (2014).
42. I. B. Gurevich, Yu. I. Zhuravlev, A. A. Myagkov, Yu. O. Trusova, and V. V. Yashina, "On basic problems of image recognition in neurosciences and heuristic methods for their solution," Pattern Recogn. Image Anal.: Adv. Math. Theory Appl. **25** (1), 132–160 (2015).
43. H. Hadwiger, "Uber Treffanzahlen bei translationsgleichen Eikorpern," Arch. Math. **8**, 212–213 (1957).
44. R. Haralick and L. Shapiro, "Image segmentation techniques," Comput. Vision, Graph., Image Processing **29**, 100–132 (1985).
45. R. Haralick, L. Shapiro, and J. Lee, "Morphological edge detection," IEEE J. Robot. Automat. **RA-3** (1), 142–157 (1987).
46. R. M. Haralick, S. R. Sternberg, and X. Zhuang, "Image analysis using mathematical morphology," IEEE Trans. Pattern Anal. Machine Intellig. **PAMI-9** (4), 532–550 (1987).
47. Joo Hyoman, R. N. Haralick, and L. G. Shapiro, "Toward the automatic generation of mathematical morphology procedures using predicate logic," in *Proc. 3rd Int. Conf. on Computer Vision* (Osaka, 1990), pp. 156–165.
48. S. Kaneff, "Pattern cognition and the organization of information," in *Proc. Int. Conf. on Frontiers of Pattern Recognition,* Ed. by Satosi Watanabe (Acad. Press, New York, London, 1972), pp. 193–222.
49. M. Yu. Khachai, "On the computational complexity of the minimum committee problem," J. Math. Model. Algor. **6** (4), 547–561 (2007).

50. M. Yu. Khachai, "Computational complexity of recognition learning procedures in the class of piecewise-linear committee decision rules," Automat. Remote Control **71** (3), 528–539 (2010).

51. R. Kirsh, "Computer interpretation of english text and picture patterns," IEEE-TEC **EC-13** (4) (1964).

52. J. Kittler and F. M. Alkoot, "Relationship of sum and vote fusion strategies," in *Proc. 2nd Int. Workshop on Multiple Classifier Systems MCS 2001 Cambridge, UK, July 2001* (Springer-Verlag, 2001), pp. 339–348.

53. *Applications of Discrete Geometry and Mathematical Morphology,* Ed. by U. Köthe, A. Montanvert, and P. Soille (Lecture Notes in Computer Science, 2012), Vol. 7346.

54. E. N. Kuzmin, "On the Nagata-Higman theorem," in *Proc. Dedicated to the 60th Birthday of Academician L. Iliev Mathematical Structures—Computational Mathematics—Mathematical Modeling* (Sofia, 1975), pp. 101–107.

55. V. G. Labunetc, *Algebraic Theory of Signals and Systems (Digital Signal Processing)* (Krasnoyarsk Univ., Krasnoyarsk, 1984).

56. J. S. J. Lee, R. M. Haralick, and L. G. Shapiro, "Morphologic edge detection," IEEE J. Robot. Automat. **RA-3** (2), 142–156 (IEEE, 1987).

57. D. A. Leites, "Introduction to the theory of supermanifolds," Russ. Math. Survey **35** (1), 1–64 (1980).

58. Yu. N. Maltsev and E. N. Kuzmin, "A basis for identities of the algebra of second order matrices over a nite eld," Algebra Logic **17**, 17–21 (1978).

59. A. I. Malcev, *Algebraic Systems* (Nauka, Moscow, 1970; Springer-Verlag, Berlin, 1973).

60. V. L. Matrosov, "Pair isomorphism of permissible objects in recognition problems," USSR Comput. Math. Math. Phys. **23** (1), 123–127 (1983).

61. V. L. Matrosov, "On the incompleteness of a model of algorithms for computing estimates," USSR Comput. Math. Math. Phys. **23** (2), 128–136 (1983).

62. V. L. Matrosov, "Lower bounds of the capacity of L-dimensional algebras of estimatecomputing algorithms," USSR, Comput. Math. Math. Phys. **24** (6), 182–188 (1984).

63. V. L. Matrosov, "The capacity of polynomial expansions of a set of algorithms for calculating estimates," USSR Comput. Math. Math. Phys. **25** (1), 79–87 (1985).

64. G. Matheron, *Random Sets and Integral Geometry* (Wiley, New York, 1975).

65. P. Maragos, "Algebraic and PDE approaches for lattice scale-spaces with global constraints," Int. J. Comput. Vision **52** (2/3), 121–137 (2003).

66. P. Maragos and R. Schafer, "Morphological skeletons representation and coding of binary images," IEEE Trans. Acoustics, Speech, Signal Processing **34** (5), 1228–1244 (1986).

67. P. Maragos and R. W. Schafer, "Morphological filters. Part I: their set-theoretic analysis and relations to linear shift-invariant filters," IEEE Trans. Acoustics, Speech, Signal Processing **ASSP-35**, 1153–1169 (1987).

68. P. Maragos and R. W. Schafer, "Morphological filters. Part II: Their relations to median, order-statistic, and stack filters," IEEE Trans. Acoustics, Speech, Signal Processing **ASSP-35**, 1170–1184 (1987).

69. D. Marr, *Vision* (Freeman, New York, 1982).

70. V. D. Mazurov, "A committee of a system of convex inequalities," Siberian Math. J. **9** (2), 354–357 (1998).

71. V. D. Mazurov, "Committees of inequalities systems and the recognition problem," Cybernetics **7** (3), 559–567 (1971).

72. V. D. Mazurov and M. Yu. Khachai, "Committees of systems of linear inequalities," Automat. Remote Control **65** (2), 193–203 (2004).

73. V. D. Mazurov and M. Yu. Khachai, "Parallel computations and committee constructions," Automat. Remote Control **68** (5), 912–921 (2007).

74. P. Miller, "Development of a mathematical structure for image processing," Optical Division Tech. Report (Perkin-Elmer, 1983).

75. H. Minkowski, *Geometrie der Zahlen* (Teubner, Leipzig, 1896/1910).

76. R. Narasimhan, "Syntax-directed interpretation of classes of pictures," Commun. ACM **9** (3) (1966).

77. R. Narasimhan, "Labeling schemata and syntactic descriptions of pictures," Inf. Control **7** (2) (1967).

78. R. Narasimhan, "On the description, generalization and recognition of classes of pictures," in *Proc. NATO Summer School on Automatic Interpretation and Classification of Images* (Pisa, Aug. 26–Sept. 7, 1968).

79. R. Narasimhan, "Picture languages," in *Picture Language Machines,* Ed. by S. Kaneff (Acad. Press, London, New York, 1970), pp. 1–30.

80. J. von Neumann, "The general logical theory of automata," in *Proc. Hixon Symp. Celebral Mechenism in Behavior* (John Wiley and Sons, New York, 1951).

81. J. von Neumann, *Theory of Self-Reproducing Automata* (Univ. of Illinois Press, Urbana, IL, 1966).

82. M. Pavel, "Pattern recognition categories," Pattern Recogn. **8** (3), 115–118 (1976).

83. M. Pavel, *Fundamentals of Pattern Recognition* (Marcell Dekker, New York, 1989).

84. Yu. P. Pytiev, *Method of Mathematical Modeling of Measuring and Computing Systems,* 2nd ed. (MAIK Nauka, Moscow, 2004) [in Russian].

85. B. Radunacu, M. Grana, and F. X. Albizuri, "Morphological scale spaces and associative morphological memories: results on robustness and practical applications," J. Math. Imag. Vision **19**, 113–131 (2003).

86. G. X. Ritter, P. Sussner, and J. L. Diaz-de-Leon, "Morphological associative memories," IEEE Trans. Neural Networks **9** (2), 281–292 (1998).

87. G. X. Ritter and P. Sussner, "Introduction to morphological neural networks," in *Proc. ICPR* 1996 (IEEE, 1996), pp. 709–716.

88. G. X. Ritter, J. L. Diaz-de-Leon, and P. Sussner, "Morphological bidirectional associative memories," Neural Networks **12**, 851–867 (1999).

89. G. X. Ritter and J. N. Wilson, *Handbook of Computer Vision Algorithms in Image Algebra,* 2d ed. (CRC Press, 2001).

90. G. X. Ritter, *Image Algebra* (Center for Computer Vision and Visualization, Dep. Computer and Information Science and Engineering, Univ. of Florida, Gainesville, FL, 2001).

91. G. X. Ritter and P. D. Gader, "Image algebra techniques for parallel image processing," Parallel Distribut. Comput. **4** (5), 7–44 (1987).

92. G. X. Ritter, J. N. Wilson, and J. L. Davidson, "Image algebra: an overview," Comput. Vision, Graph., Image Processing **49**, 297–331 (1990).

93. A. Rosenfeld, *Picture Languages. Formal Models for Picture Recognition* (Acad. Press, New York, San Francisco, London, 1979).

94. A. Rosenfeld, "Digital topology," Am. Math. Monthly **86**, 621–630 (1979).

95. K. V. Rudakov, "Universal and local constraints in the problem of correction of heuristic algorithms," Cybernetics **23** (2), 181–186 (1987).

96. K. V. Rudakov, "Completeness and universal constraints in the correction problem for heuristic classification algorithms," Cybernetics **23** (3), 414–418 (1987).

97. K. V. Rudakov, "Symmetric and functional constraints in the correction problem of heuristic classification algorithms," Cybernetics **23** (4), 528–533 (1987).

98. K. V. Rudakov, "Application of universal constraints in the analysis of classification algorithms," Cybernetics **24** (1), 1–6 (1988).

99. K. V. Rudakov and K. V. Vorontsov, "Methods of optimization and monotone correction in the algebraic approach to the recognition problem," Dokl. Math. **60**, 139–142 (1999).

100. K. V. Rudakov and Yu. V. Chekhovich, "Completeness criteria for classification problems with set-theoretic constraints," Comput. Math. Math. Phys. **45** (2), 329–337 (2005).

101. K. V. Rudakov, A. A. Cherepnin, and Yu. V. Chekhovich, "On metric properties of spaces in classification problems," Dokl. Math. **76** (2), 790–793 (2007).

102. K. V. Rudakov and I. Yu. Torshin, "Selection of informative feature values on the basis of solvability criteria in the problem of protein secondary structure recognition," Dokl. Math. **84** (3), 871–874 (2011).

103. J. Serra, *Image Analysis and Mathematical Morphology* (Acad. Press, London, 1982).

104. J. Serra, "Morphological filtering: an overview," Signal Processing **38** (1), 3–11 (1994).

105. J. Serra, "Introduction to morphological filters," in *Image Analysis and Mathematical Morphology,* Vol. 2: *Theoretical Advances,* Ed. by J. Serra (Acad. Press 1998), Ch. 5, pp. 101–114.

106. I. N. Sinicyn, *Calman and Pugachev Filters,* 2nd ed. (Logos, Moscow, 2007) [in Russian].

107. A. Shaw, "A proposed language for the formal description of pictures," CGS Memo (Stanford Univ., 1967), no. 28.

108. A. Shaw, "The formal description and parsing of pictures," Ph.D. Thesis (Computer Sciences Department, Stanford Univ., Dec. 1967); Tech. Rept CS94 (Apr. 1968).

109. M. Schlesinger and V. Hlavac, *Ten Lectures on Statistical and Structural Pattern Recognition,* Vol. 24: *Computational Imaging and Vision* (Kluwer Acad. Publ., Dordrecht, Boston, London, 2002).

110. P. Soille, "Morphological partitioning of multispectral images," J. Electron. Imag. **5** (3), 252–265 (1996).

111. P. Soille, *Morphological Image Analysis. Principles and Applications,* 2nd ed. (Springer-Verlag, Berlin, Heidelberg, New York, 2003, 2004).

112. S. R. Sternberg, "Language and architecture for parallel image processing," in *Proc. Conf. on Pattern Recognition in Practice* (Amsterdam, 1980).

113. S. R. Sternberg, "An overview of image algebra and related architectures," in *Integrated Technology for Parallel Image Processing,* Ed. by S. Levialdi (Academic Press, London, 1985).

114. S. R. Sternberg, "Grayscale morphology," Comput. Vision, Graphics Image Processing **35**, (3), 333–355 (1986).

115. P. Sussner, "Observations on morphological associative memories and the kernel method," Neurocomputing **31**, 167–183 (2000).

116. P. Sussner and G. X. Ritter, "Rank-based decompositions of morphological templates," IEEE Trans. Image Processing **09** (8), 1420–1430 (2000).

117. P. Sussner, "Generalizing operations of binary autoassociative morphological memories using fuzzy set theory," J. Math. Imag. Vision **19**, 81–93 (2003).

118. D. M. J. Tax and R. P. W. Duin, "Combining one-class classifiers," in *Proc. 2nd Int. Workshop on Multiple Classifier Systems MCS 2001, Cambridge, UK, July 2001* (Springer-Verlag, 2001).

119. A. Toet, "A morphological pyramidal image decomposition," Pattern Recogn. Lett. **9**, 255–261 (1989).

120. S. H. Unger, "A computer oriented toward spatial problems," Proce. IRE **46**, 1744–1750 (1958).

121. B. L. Van Der Waerden, *Algebra I, Algebra II* (Springer-Verlag, Berlin, Heidelberg, New York, 1971).

122. D. Winbridge and J. Kittler, "Classifier combination as a tomographic process," in *Proc. 2nd Int. Workshop on Multiple Classifier Systems, MCS 2001, Cambridge, UK, July 2001* (Springer-Verlag, 2001), pp. 248–258.

123. Yu. I. Zhuravlev, "On algebraic approach for solving the recognition and classification problems," in *Problems of Cybernetics* (Nauka, Moscow, 1978), Issue 33 [in Russian].

124. Yu. I. Zhuravlev, "An algebraic approach to recognition and classification problems," Pattern Recogn. Image Anal.: Adv. Math. Theory Appl. **8**, 59–100 (1998).

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