Reflective Symmetrization of Images[1]

A. N. Karkishchenko and V. B. Mnukhin

Multiprocessing Computing Systems Scientific-Research Institute, Southern Federal University, ul. Chekhova 2, Taganrog, 347928 Russia

e-mail: karkishalex@gmail.com, mnukhin.valeriy@mail.ru

Abstract—This paper proposes several algebraic methods aimed at refining the positions of characteristic points (that describe some objects on a pattern) based on the *a priori* information about their symmetrical position. These methods are known as the symmetrization of characteristic points. We consider symmetrization of points for the cases of vertical and arbitrary symmetry with known parameters of the symmetry axis, as well as more general case of symmetrization with unknown parameters of axial symmetry. The methods under consideration give a solution to the axial symmetrization problem with minimal variation of characteristic points.

Keywords: symmetrization, reflection symmetry, characteristic points, biometric identification, facial recognition.

DOI: 10.1134/S1054661815010071

1. INTRODUCTION

Symmetry plays a considerable role in appreciation of natural and (to a great extent) artificial objects. Lately, in the analysis of forms of digitized objects, considerable effort has been spent in revealing the symmetry of 2D and 3D objects [1], [2], [3]. The information about symmetry is used effectively in numerous applications, *viz.*, compact description of models [3], processing of scanned images [4], image segmentation [5], forms matching [1], etc. As a rule, on the “atomic” level the methods are reduced to the study of symmetrical properties of the so-called characteristic points, which are problem specific.

Below, we propose a number of methods that are capable of refining the positions of characteristic points (describing some objects on an image) based from the *a priori* information about their symmetrical position. The essence of the problem is that localization of points on an image is always done with some error, whose magnitude depends on many factors: the localization algorithm, the overall image quality, the noise level in the region containing a point to be localized, etc. As a consequence, the coordinates of detected points, from which one knows *a priori* that they are symmetrical, fail to satisfy this condition in reality. Hence, it is expedient to use the known *a priori* information on the symmetric property of points to refine their position. In doing so, the refinement should be performed in the best (in a sense) possible way. For example, the symmetric property should be achieved by minimal variation of their positions.

One class of the most typical problems that are amenable to such methods of refinement in terms of “symmetry” includes the problems of biometric recognition, in which correct determination of the characteristic points is critical for the success of the solution. Here, it suffices to point out that the accuracy of detection and facial recognition substantially depends on the accuracy in the determination of the centers of pupil in a face [6]. To say more, person identification, which is also based on the positions of characteristic points, depends a fortiori on the symmetry factor due to the symmetrical form of the frontal view of a face [7]. In these methods one determines, as a rule, several dozens (depending on a method) of characteristic points, of which the largest part consists of pairs of points that are reflectively symmetric with respect to the vertical axis or points lying directly on the symmetry axis.

2. STATEMENT AND SOLUTION OF THE PROBLEM WITH VERTICAL AXIAL SYMMETRY

Let $P = \{p_1, \ldots, p_n\}$ be the set of all characteristic points and $p_k = (x_k, y_k)$ be the coordinates of the kth point. Assume first that the symmetry axis coincides with the axis of ordinates in the Cartesian system of coordinates. By convention, we label the characteristic points so that the $P_R = \{p_1, \ldots, p_m\}$ will lie on the right half-plane, the points $P_L = \{p_{m+1}, \ldots, p_{2m}\}$ will lie on the left half-plane, and the points $P_O = \{p_{2m+1}, \ldots, p_n\}$ will lie on the symmetry axis. It is also assumed that

[1] This paper uses the materials of a report that was submitted at the 11th International Conference *Pattern Recognition and Image Analysis: New Information Technologies* that was held in Samara, Russia on September 23–28, 2013.

Received March 15, 2014

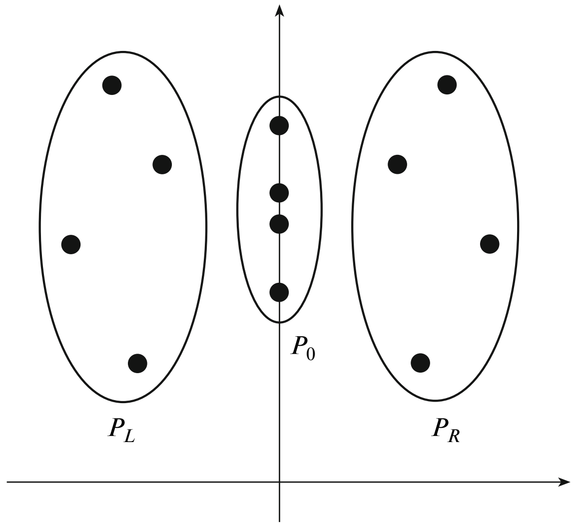

Fig. 1. Symmetry with respect to the vertical axis.

the points p_i and p_{m+i}, $i = 1, \ldots, m$, are *a priori* symmetric relative the axis of ordinates (Fig. 1). With the thus-ordered set of characteristic points P we associate a vector of size $2n$:

$$(p_1, p_2, \ldots, p_n) \longleftrightarrow (x_1, \ldots, x_n, y_1, \ldots, y_n)^T = X.$$

The vectors X may be looked upon as elements of the linear space $\mathbb{R}^{2n}$. Note that if all the characteristic points are precisely determined, then, in view of the partition into the classes P_R, P_L, P_O, the following conditions should be satisfied:

$$x_i = -x_{m+i}, \quad i = 1, \ldots, m;$$

$$y_i = y_{m+1}, \quad i = 1, \ldots, m;$$

$$x_i = 0, \quad i = 2m+1, \ldots, n.$$

The set of vectors that satisfy these conditions forms the n-dimensional subspace R_{Sym} in the space $\mathbb{R}^2$. We note that “symmetrization” of a vector X of characteristic points means finding a vector X_s in R_{Sym} with the least deviation from X in the Euclidean norm:

$$X_s = \arg \min_{Z \in R_{\text{Sym}}} \|Z - X\|.$$

Thus, the problem amounts to finding the orthogonal projection X_s of the vector X to the subspace R_{Sym}.

Let Q be a matrix that corresponds to the operator of the orthogonal projection onto the subspace R_{Sym} and let A be a matrix whose columns are the basis vectors of the subspace R_{Sym}. It is known (see [8], p. 165 of the Russian translation) that

$$Q = AA^+ = A(A^TA)^{-1}A^T,$$

where A^+ is the pseudoinverse of the matrix A. To find Q we note that the matrix A has the following block structure:

$$A^T = \begin{pmatrix} I_m & -I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m & I_m & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n-2m} \end{pmatrix},$$

where I and 0 are, respectively, the unit and the zero matrices of respective sizes. A direct calculation shows that

$$Q = \frac{1}{2}I_{2n} + \frac{1}{2}\begin{pmatrix} -S & 0 \\ 0 & S \end{pmatrix},$$

where S is the following $n \times n$-matrix:

$$S = \begin{pmatrix} 0 & I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & I_{n-2m} \end{pmatrix}.$$

Thus, in general the solution of the problem has the form $X_s = QX$.

It is easily seen that the operation of orthogonal projection of a vector induces fairly simple operations on the coordinates of characteristic points: as an ordinate of symmetrical points one needs to take the arithmetical mean of their ordinates, while the absolute values of abscissas are the arithmetical mean of their absolute values. For “axial” points their ordinates remain unchanged, while the abscissas become zero.

The result of symmetrization by the above method of a system of points with respect to the vertical symmetry axis is shown in Fig. 2. The left part of the figure depicts the front of a face with indicated characteristic points, which may be taken, for example, from some database. The middle part of the figure shows the characteristic points, whose positions are given by some algorithm (for example, using that from [6]). It is easily seen that the symmetry of points is violated, which aggravates the pattern-identification problem. On the right part of the figure, we show the result of symmetrization of the system of characteristic points using the above method.

3. SYMMETRIZATION UNDER ARBITRARY AXIAL SYMMETRY

We assume now that the symmetry axis is given by the equation $y = ax + b$, where $a \neq 0$ (Fig. 3). In this case, the problem is reduced to the above problem by changing the coordinates. Namely, let us construct a new coordinate system ($O'x'y'$) assuming that the $O'y'$-axis coincides with the symmetry axis, the origin of O' coincides with the intersection point of the symmetry axis and the Oy-axis, and the $O'x'$-axis is orthogonal to

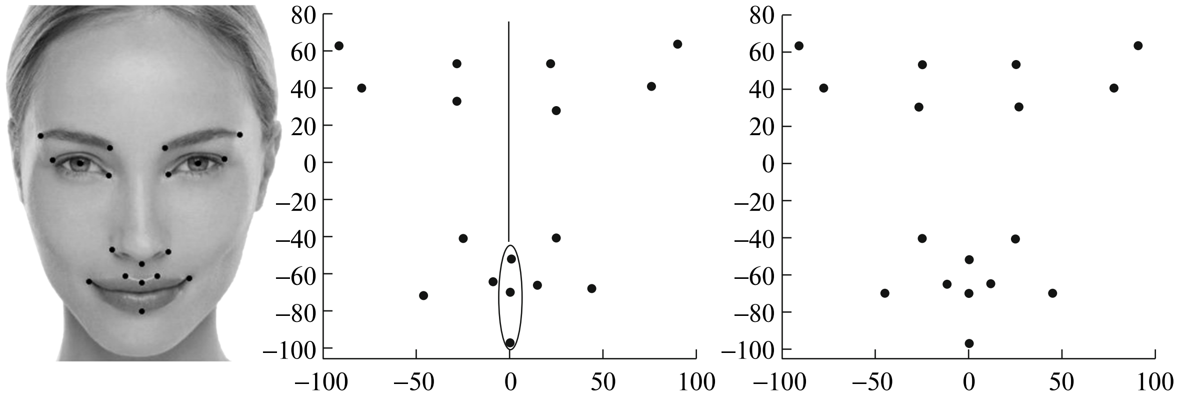

Fig. 2. An example of symmetrization with respect to the vertical axis.

the $O'y'$-axis to form the right-handed coordinate system. In other words, the system $(O'x'y')$ is obtained from (Oxy) by rotating by some angle φ (see Fig. 3). Here, the angular coefficient of the symmetry axis is $a = \tan\left(\frac{\pi}{2} + \varphi\right) = -\cot\varphi$; its equation for $\varphi \neq 0$ may be written as $y = b - x\cot\varphi$.

Let

$$\Re = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

be a matrix that corresponds to a rotation through angle $-\varphi$; its entries $\cos\varphi$ and $\sin\varphi$ are found from the condition $\cot\varphi = -a \neq 0$; that is,

$$\sin\varphi = \frac{1}{\sqrt{a^2 + 1}}, \quad \cos\varphi = -\frac{a}{\sqrt{a^2 + 1}}.$$

Hence, the coordinates (x, y) and (x', y') of an arbitrary points in the old and new systems of coordinates are related by $(x', y')^T = \Re(x, y - b)^T$, or equivalently,

$$x' = x\cos\varphi + (y - b)\sin\varphi,$$
$$y' = -x\sin\varphi + (y - b)\cos\varphi.$$

We consider, as in the previous section, the vector $X = (x_1, ..., x_n, y_1, ..., y_n)^T \in \mathbb{R}^{2n}$ composed of coordinates of characteristic points in the system (Oxy) with due account for the partition into classes P_R, P_L, P_O. We assume that the vector X' consists of the coordinates of the same points in the system $(O'x'y')$. In order to establish the relationship between the vectors X and X', we consider the matrix

$$R^T = \Re \otimes I_n = \begin{pmatrix} \cos\varphi \cdot I_n & \sin\varphi \cdot I_n \\ -\sin\varphi \cdot I_n & \cos\varphi \cdot I_n \end{pmatrix},$$

where $\otimes$ denotes the Kronecker product and I_n is a unit matrix of size $n \times n$. From the properties of the Kronecker product it is readily seen that the matrix R is orthogonal, $R^{-1} = R^T$. Hence,

$$X' = R^T(X - bK), \quad \text{so that } X = RX' + bK,$$

where $K = (0, ..., 0, 1, ..., 1)^T$. This transformation reduces the problem to the one that was just solved. Symmetrizing the vector X' and then changing to the old system of coordinates, we see that

$$X_s = RX'_s + bK = R(QR^T(X - bK)) + bK,$$

and so,

$$X_s = GX + bHK, \quad \text{where } G = RQR^T$$
$$\text{and } H = I - RQR^T.$$

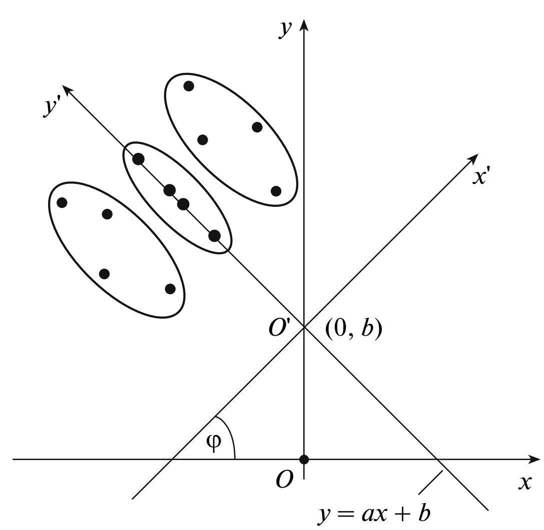

Fig. 3. Symmetry with respect to an arbitrary axis.

Direct calculations show that

$$G = \frac{1}{2}I_{2n} + \frac{1}{2}\begin{pmatrix} -\cos 2\varphi \cdot S & \sin 2\varphi \cdot S \\ \sin 2\varphi \cdot S & \cos 2\varphi \cdot S \end{pmatrix},$$

where the matrix S is defined above. For $b = 0$ and $\varphi = 0$ we get the above solution for the vertical symmetry axis.

4. SYMMETRIZATION WITH UNKNOWN PARAMETERS OF THE SYMMETRY AXIS

The method that was considered above is capable of not only performing symmetrization with the given parameters a and b, but also can determine the parameters of an axis for which the symmetrization will be achieved through minimal variation of the positions of points.

We assume, as before, that $P = \{p_1, \ldots, p_n\}$ be the set of all characteristic points with coordinates $p_k = (x_k, y_k)$. We assume that from some *a priori* information this set may be partitioned into the disjoint classes

$$P_R = \{p_1, \ldots, p_m\}, \quad P_L = \{p_{m+1}, \ldots, p_{2m}\},$$
$$\text{and } P_O = \{p_{2m+1}, \ldots, p_n\},$$

so that the corresponding points $p_i \in P_R$ and $p_{m+1} \in P_L$ of the first two classes are symmetrical relative to some *a priory* unknown symmetry axis, while the points of the class P_O lie on this axis. Having this partition at our disposal, we associated, as above, the vector $X = (x_1, \ldots, x_n, y_1, \ldots, y_n)^T \in \mathbb{R}^{2n}$ of the original coordinates of characteristic points with the set $P = P_R \cup P_L \cup P_O$. As well, it is convenient to adopt the following notation:

$$\sigma = (\underbrace{1\ldots1}_{n}\ \underbrace{1\ldots1}_{n})^T, \quad \sigma_0^1 = (\underbrace{1\ldots1}_{n}\ \underbrace{0\ldots0}_{n})^T,$$

$$\sigma_1^0 = (\underbrace{0\ldots0}_{n}\ \underbrace{1\ldots1}_{n})^T.$$

We shall assume that the symmetry axis is given by the equation $y = ax + b$ (the parameters a and b are as yet unknown), where $a = -\cot\varphi \neq 0$. We consider the vector

$$X'(\varphi, b) = R^T(\varphi)(X - n\sigma_1^0),$$

where the rotation matrix $R^T(\varphi)$, which depends on the rotation angle φ, is defined in the previous section. Now, the optimization problem may be stated as follows:

$$\|AY - R^T(\varphi)(X - b\sigma_1^0)\|^2 \xrightarrow{Y, \varphi, b} \min;$$

where the matrix A is defined in the first section, $Y \in \mathbb{R}^2$ is a variable vector, and $AY = Z$ is the sought-for symmetrized vector of coordinates of characteristic points. In other words, finding the optimal symmetrization requires finding φ_s, b_s, and a vector Y_s to minimize the left-hand side of the previous expression, and next to return the thus-obtained vector Y_s in the original system of coordinates: $X_s = R^T(\varphi_s)Y_s + b_s\sigma_1^0$. The so-constructed vector X_s will give the optimal symmetrization of a given set of characteristic points.

For convenience, we denote the expression to be minimized by

$$F(Y, \varphi, b) = \|AY - R^T(\varphi)(X - b\sigma_1^0)\|^2$$

and express it in detail in the matrix form

$$\begin{aligned} F(Y, \varphi, b) &= (AY - R^T(\varphi)X + bR^T(\varphi)\sigma_1^0, AY \\ &\quad - R^T(\varphi)X + bR^T(\varphi)\sigma_1^0) \\ &= (AY - R^T(\varphi)X + bR^T(\varphi)\sigma_1^0)^T \\ &\quad \times (AY - R^T(\varphi)X + bR^T(\varphi)\sigma_1^0) \\ &= Y^TA^TAY - 2Y^TA^TR^T(\varphi)X + 2bY^TA^TR^T(\varphi)\sigma_1^0 \\ &\quad + X^TX - 2bX^T\sigma_1^0 + b^2(\sigma_1^0)^T\sigma_1^0. \end{aligned}$$

To solve the optimization problem, one needs to solve the system of equations

$$\frac{\partial F(Y, \varphi, b)}{\partial Y} = 0, \quad \frac{\partial F(Y, \varphi, b)}{\partial \varphi} = 0, \quad \frac{\partial F(Y, \varphi, b)}{\partial b} = 0.$$

After calculating the derivatives and making the necessary transformations, we obtain

$$\begin{cases} AY = QR^T(\varphi)(X - b\sigma_1^0), \\ (X - b\sigma_1^0)^T \dfrac{dR(\varphi)}{d\varphi} AY = 0, \\ (\sigma_1^0)^T R(\varphi)AY = (\sigma_1^0)^T (X - b\sigma_1^0). \end{cases}$$

From the first (matrix) equation we obtain the expression for the symmetrized vector Y as a function of φ and b. The second and third equations are scalar and may be used to find the unknown parameters of the symmetry axis φ and b. We write these equations separately, expressing AY in terms of φ and b using the first equation

$$\begin{cases} (X - b\sigma_1^0)^T \dfrac{dR(\varphi)}{d\varphi} A(A^TA)^{-1}A^TR^T(\varphi)(X - b\sigma_1^0) = 0, \\ (\sigma_1^0)^T R(\varphi)A(A^TA)^{-1}A^TR^T(\varphi)(X - b\sigma_1^0) \\ = (\sigma_1^0)^T (X - b\sigma_1^0). \end{cases}$$

We have $\frac{dR(\varphi)}{d\varphi} = R\left(\frac{\pi}{2}\right)R(\varphi)$ and note that $R(\varphi)QR^T(\varphi) = G = \frac{1}{2}I + \tilde{G}$ using our notation. Hence, we may rewrite the system in a simpler form

$$\begin{cases} (X - b\sigma_1^0)^T R\left(\frac{\pi}{2}\right)\left(\frac{1}{2}I + \tilde{G}\right)(X - b\sigma_1^0) = 0, \\ (\sigma_1^0)^T \tilde{G}(X - b\sigma_1^0) = \frac{1}{2}(\sigma_1^0)^T(X - b\sigma_1^0). \end{cases}$$

It is easily verified that $(X - b\sigma_1^0)^T R\left(\frac{\pi}{2}\right)(X - b\sigma_1^0) = 0$; hence

$$\begin{cases} (X - b\sigma_1^0)^T R\left(\frac{\pi}{2}\right)\tilde{G}(X - b\sigma_1^0) = 0, \\ (\sigma_1^0)^T \tilde{G}(X - b\sigma_1^0) = \frac{1}{2}(\sigma_1^0)^T(X - b\sigma_1^0). \end{cases}$$

To determine φ and b we first consider the second equation of the system. We shall need the quantities

$$x_{av} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad y_{av} = \frac{1}{n}\sum_{i=1}^{n} y_i,$$

which are, respectively, the averaged abscissa and ordinate of the already obtained characteristic points. Hence, as one may easily verify,

$$(\sigma_1^0)^T G(X - b\sigma_1^0) = n(y_{av} - b).$$

Considering the left-hand side of the second equation, we see that

$$2(\sigma_1^0)^T \tilde{G}(X - b\sigma_1^0) = \cos 2\varphi \cdot \underbrace{(\sigma_1^0)^T X}_{ny_{av}}$$

$$- \sin 2\varphi \cdot \underbrace{(\sigma_0^1)^T X}_{nx_{av}} + b\sin 2\varphi \cdot \underbrace{(\sigma_0^1)^T(\sigma_1^0)}_{n}$$

$$- \cos 2\varphi \cdot \underbrace{(\sigma_1^0)^T(\sigma_1^0)}_{n}$$

$$= n[(y_{av} - b)\cos 2\varphi - x_{av}\sin 2\varphi].$$

From the above expressions, we find that

$$b = y_{av} + x_{av}\cot\varphi.$$

The meaning of this relation is quite transparent from Fig. 4.

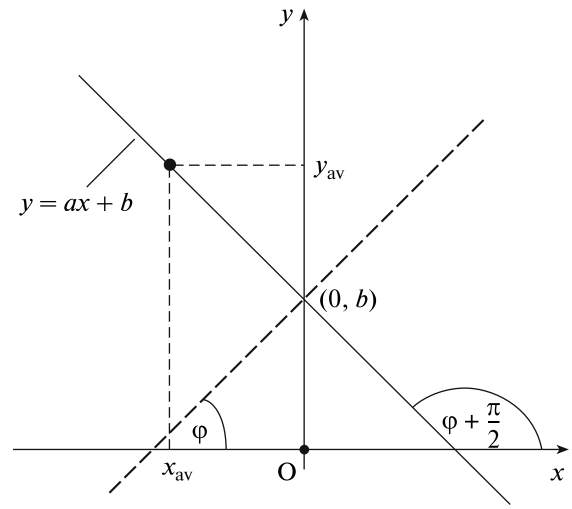

Fig. 4. Finding the parameter.

Now, we consider the first equation of the last system. To do so, we write this equation in more detail

$$\underbrace{X^T R\left(\frac{\pi}{2}\right)\tilde{G}X}_{1} - b\underbrace{\left[X^T R\left(\frac{\pi}{2}\right)\tilde{G}(\sigma_1^0) + (\sigma_1^0)^T R\left(\frac{\pi}{2}\right)\tilde{G}X\right]}_{2}$$

$$+ b^2\underbrace{(\sigma_1^0)^T R\left(\frac{\pi}{2}\right)\tilde{G}(\sigma_1^0)}_{3} = 0.$$

and calculate each term in this expression in sequence.

The first term. We represent the vector X in the block form

$$X = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) = (\mathbf{x}_1^T \mathbf{x}_2^T \mathbf{x}_3^T \mathbf{y}_1^T \mathbf{y}_2^T \mathbf{y}_3^T),$$

where

$$\mathbf{x}_1 = (x_1, x_2, \ldots, x_m), \quad \mathbf{x}_2 = (x_{m+1}, x_{m+2}, \ldots, x_{2m}),$$
$$\mathbf{x}_3 = (x_{2m+1}, x_{2m+2}, \ldots, x_n),$$

$$\mathbf{y}_1 = (y_1, y_2, \ldots, y_m), \quad \mathbf{y}_2 = (y_{m+1}, y_{m+2}, \ldots, y_{2m}),$$
$$\mathbf{y}_3 = (y_{2m+1}, y_{2m+2}, \ldots, y_n),$$

are the vectors in the x- and y-coordinates of the characteristic points that lie, respectively, to the right of, to the left of, and on the symmetry axis. Taking this into account, this gives:

$$\underbrace{X^T R\left(\frac{\pi}{2}\right)\tilde{G}X}_{1} = \underbrace{(\mathbf{y}_1^T\mathbf{y}_2^T\mathbf{y}_3^T - \mathbf{x}_1^T - \mathbf{x}_2^T - \mathbf{x}_3^T)\cdot\frac{1}{2}}_{X^T R(\pi/2)}$$

$$\times\begin{pmatrix} -\mathbf{x}_2\cos 2\varphi - \mathbf{y}_2\sin 2\varphi \\ -\mathbf{x}_1\cos 2\varphi - \mathbf{y}_1\sin 2\varphi \\ -\mathbf{x}_3\cos 2\varphi - \mathbf{y}_3\sin 2\varphi \\ -\mathbf{x}_2\sin 2\varphi + \mathbf{y}_2\cos 2\varphi \\ -\mathbf{x}_1\sin 2\varphi + \mathbf{y}_1\cos 2\varphi \\ -\mathbf{x}_3\sin 2\varphi + \mathbf{y}_3\cos 2\varphi \end{pmatrix}$$

$$= -[(\mathbf{x}_1, \mathbf{y}_2) + (\mathbf{x}_2, \mathbf{y}_1) + (\mathbf{x}_3, \mathbf{y}_3)]\cos 2\varphi$$

$$-\left[(\mathbf{y}_1, \mathbf{y}_2) - (\mathbf{x}_1, \mathbf{x}_2) - \frac{1}{2}(\mathbf{x}_3, \mathbf{x}_3) + \frac{1}{2}(\mathbf{y}_3, \mathbf{y}_3)\right]\sin 2\varphi.$$

The second term. Since the matrix $\tilde{G}$ is symmetric and $R^T(\frac{\pi}{2}) = -R(\frac{\pi}{2})$, we write the second term in the form $(\sigma_1^0)^T\left[R\left(\frac{\pi}{2}\right)\tilde{G} - \tilde{G}R\left(\frac{\pi}{2}\right)\right]X$. Next, direct calculation shows that

$$[\cos 2\varphi\cdot(\sigma_0^1)^T + \sin 2\varphi\cdot(\sigma_1^0)^T]X$$
$$= nx_{\text{av}}\cos 2\varphi + ny_{\text{av}}\sin 2\varphi.$$

The third term. Direct calculation shows

$$\underbrace{2(\sigma_1^0)^T R\left(\frac{\pi}{2}\right)\tilde{G}(\sigma_1^0)}_{3}$$

$$= \underbrace{(\sigma_0^1)^T}_{(\sigma_1^0)^T R\left(\frac{\pi}{2}\right)}\cdot\underbrace{\cos 2\varphi\cdot\sigma_1^0 - \sin 2\varphi\cdot\sigma_0^1}_{\tilde{G}\sigma_1^0} = -n\sin 2\varphi.$$

Thus, the second equation assumes the form:

$$[(\mathbf{x}_1, \mathbf{y}_2) + (\mathbf{x}_2, \mathbf{y}_1) + (\mathbf{x}_3, \mathbf{y}_3)]\cos 2\varphi$$
$$+\left[(\mathbf{y}_1, \mathbf{y}_2) + (\mathbf{x}_1, \mathbf{x}_2) - \frac{1}{2}(\mathbf{x}_3, \mathbf{x}_3) - \frac{1}{2}(\mathbf{y}_3, \mathbf{y}_3)\right]\sin 2\varphi$$
$$-bn(x_{\text{av}}\cos 2\varphi + y_{\text{av}}\sin 2\varphi) + \frac{n}{2}b^2\sin 2\varphi = 0.$$

Since $b = y_{\text{av}} + x_{\text{av}}\cot\varphi$, we have, after the transformation,

$$[(\mathbf{x}_1, \mathbf{y}_2) + (\mathbf{x}_2, \mathbf{y}_1) + (\mathbf{x}_3, \mathbf{y}_3) - nx_{\text{av}}y_{\text{av}}]\cos 2\varphi$$
$$+\left[(\mathbf{y}_1, \mathbf{y}_2) - (\mathbf{x}_1, \mathbf{x}_2) - \frac{1}{2}(\mathbf{x}_3, \mathbf{x}_3) + \frac{1}{2}(\mathbf{y}_3, \mathbf{y}_3)\right.$$
$$\left.+\frac{1}{2}n(x_{\text{av}}^2 + y_{\text{av}}^2)\right]\sin 2\varphi = 0,$$

hence,

$$\tan 2\varphi = \frac{(\mathbf{x}_1, \mathbf{y}_2) + (\mathbf{x}_2, \mathbf{y}_1) + (\mathbf{x}_3, \mathbf{y}_3) - nx_{\text{av}}y_{\text{av}}}{(\mathbf{x}_1, \mathbf{x}_2) - (\mathbf{y}_1, \mathbf{y}_2) + \frac{1}{2}(\mathbf{x}_3, \mathbf{x}_3) - \frac{1}{2}(\mathbf{y}_3, \mathbf{y}_3) - \frac{1}{2}n(x_{\text{av}}^2 + y_{\text{av}}^2)}.$$

This expression may be written more conveniently in centered coordinates:

$$\dot{\mathbf{x}}_i = \mathbf{x}_i - x_{\text{av}}\mathbf{e},\quad \dot{\mathbf{y}}_i = \mathbf{y}_i - y_{\text{av}}\mathbf{e},\quad i = 1, 2, 3,$$

where **e** is a vector of corresponding dimension that is composed of units. It is easily seen that this transformation consists in moving the origin to the point with coordinates $(x_{\text{av}}, y_{\text{av}})$. After transformations, we obtain the solution of the optimization problem:

$$\tan 2\varphi_s = \frac{(\dot{\mathbf{x}}_1, \dot{\mathbf{y}}_2) + (\dot{\mathbf{x}}_2, \dot{\mathbf{y}}_1) + (\dot{\mathbf{x}}_3, \dot{\mathbf{y}}_3)}{(\dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2) - (\dot{\mathbf{y}}_1, \dot{\mathbf{y}}_2) + \frac{1}{2}(\dot{\mathbf{x}}_3, \dot{\mathbf{x}}_3) - \frac{1}{2}(\dot{\mathbf{y}}_3, \dot{\mathbf{y}}_3)}.$$

$$b_s = y_{\text{av}} + x_{\text{av}}\cot\varphi_s.$$

Figure 5 shows the result of symmetrization that was obtained by the above method. In the upper left part we show the system of characteristic points that are symmetric with respect to the marked symmetry axis. In the upper right part we show the same points after noise, which resulted in a loss of symmetry. The bottom left part depicts the points after symmetrization. The original and recovered symmetry axes are superimposed in the bottom right part of the figure.

5. CONCLUSIONS

The problem of the symmetrization of characteristic points relative to the axial symmetry is known to have numerous applications, as it features the simplest and most widespread form of symmetry. At the same time, problems of image processing may involve different types of symmetry (rotational symmetry, dihedral symmetry, translation symmetry, etc.). Symmetrization under conditions of affine distortions, which are found in the majority of actual designs is of partic-

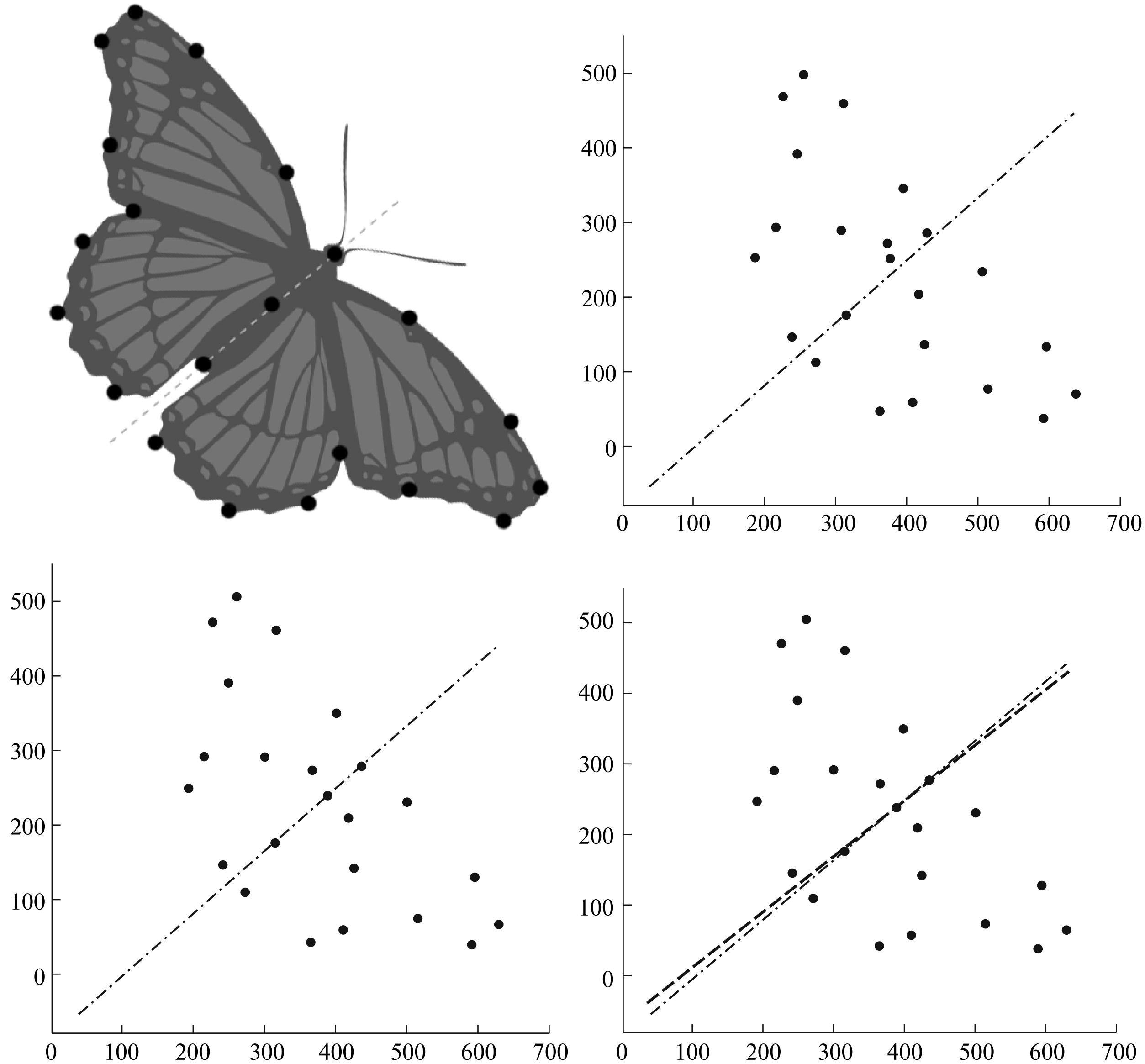

Fig. 5. An example of symmetrization with the determination of the symmetry axis.

ular interest. Although the strict mathematical solution of these problems may present a certain challenge, it provides additional possibilities for qualitative image processing.

ACKNOWLEDGMENTS

This work was supported by RFBR, project nos. 11-07-00591 and 13-07-13112-ofi_m-RZhD.

REFERENCES

1. J. Podolak, P. Shilane, J. Giesen, M. Gross, and L. Guibas, "Example-based 3d scan completion," in *Proc. Symp. on Geometry Processing* (Vienna, 2005), pp. 23–32.
2. A. Martinet, C. Soler, N. Holzschuch, and F. Sillion, "Accurate detection of symmetries in 3d shapes," ACM Trans. Graph **25** (2), 439–464 (2006).
3. N. J. Mitra, L. J. Guibas, and M. Pauly, "Partial and approximate symmetry detection for 3d geometry," ACM Trans. Graph **25** (3), 560–568 (2006).
4. S. Thrun and B. Wegbreit, "Shape from symmetry," in *Proc. Int. Conf. on Computer Vision* (Beijing, 2005).
5. P. Simari, E. Kalogerakis, and K. Singh, "Folding meshes: hierarchical mesh segmentation based on planar symmetry," in *Proc. Symp. on Geometry Processing* (Cagliari, 2006).
6. A. N. Karkishchenko and I. A. Grechukhin, "Statistical face recognition according to characteristic points geometry for transport safety systems. Great systems

control," in *Collection of Scientific Works* (Institute of Control Sciences V. A. Trapeznikov Acad. Sci., Moscow, 2012), Issue 38, pp. 78–90 [in Russian].

7. A. N. Karkishchenko and I. A. Grechukhin, "The way to localize characteristic points on the base of images nature symmetry," in *Proc. 1st Sci.-Techn. Conf. "Intelligence Control Systems for Railways"* (Nauch.-Issl. Inst. Avtomatiz. Svyazi, Moscow, 2012), pp. 261–265.
8. G. Strang, *Linear Algebra and Its Applications* (Acad. Press, 1980).

Translated by A. Alimov

Alexander N. Karkishchenko. Born 1956. Graduated from the Taganrog Radio Engineering Institute in 1978. Received Candidate's degree in 1983 and Doctor's in 1997. Scientific interests: data mining and pattern recognition, artificial intelligence and decision making, processing and analysis of images. Author of more than 170 publications.

Valeriy B. Mnukhin. Born 1958. Graduated from the Taganrog Radio Engineering Institute in 1979. Received Candidate's degree in 1985. At present he is an associate professor at the Chair of Higher Mathematics at the Academy of Engineering and Technology of Southern Federal University. Scientific interests: mathematical methods of pattern recognition, algebraic and topological combinatorics, problems of graphs recovery, spectral theory of graphs. Author of more than 60 publications.