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Micromechanical Damage Model for Plain Concrete Considering Propagation of Matrix Microcracks

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Abstract—Based on the tenets of continuum micromechanics, a damage model is developed in the present work to investigate the effect of microcracking on the constitutive relations of cement based materials such as concrete. The model considers concrete as a two phase particulate composite consisting of coarse aggregates and mortar matrix. The microcracks are assumed to be present in the matrix material. Making use of Eshelby's solution for equivalent inclusion, the stress and strain fields are evaluated at the mesoscale. A two step homogenization scheme is adopted to obtain the effective response of the composite. The crack density parameter is used as a damage variable in the formulation. Strain energy release rate, obtained from the micromechanical analysis, is used as the criterion for describing the propagation of microcracks. The effect of various mesoscopic parameters, such as aggregate content, elastic properties of the phases, microcrack density and fracture resistance of the matrix, on the overall behavior of concrete is demonstrated through a parametric study.

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1. INTRODUCTION

Damage induced in different engineering materials can be attributed to energy dissipative, permanent changes in their microstructure. The internal structure of these materials is marked by the presence of various flaws or defects which, typically, act as precursors to damage. Nucleation and growth of microcracks have been identified as the major dissipative mechanism occurring in quasi-brittle materials such as concrete, rocks and ceramics. They are also responsible for various complex phenomena at the macroscopic scale, such as degradation of elastic properties, load induced anisotropy, etc. [1]. In plain concrete, microcracks may exist as bond cracks (at the interface between mortar matrix and coarse aggregates) or as cracks randomly distributed in the mortar matrix. The present work aims at analyzing the effect of microcracks present in mortar on the mechanical behavior of concrete under tensile loads.

Microcracks develop in concrete during the early stages, primarily due to shrinkage, bleeding, cement hydration, etc. When external loads are applied, these cracks propagate resulting in inelastic strains, thus imparting nonlinearity to the resultant macroscopic stress–strain response [2, 3]. The propagation of microcracks followed by their coalescence are important mechanisms that deserve much attention in predicting the response of concrete at the macroscopic scale. Significant advances have been made to numerically model the behavior of concrete by considering damage caused by microcracking. Continuum damage mechanics is a convenient tool to describe the behavior of materials in which distributed or smeared microcracking occurs. However, the choice of an arbitrary damage variable and heuristic nature of the damage evolution law restrict the applicability of such models. The actual damage phenomena occurring in the material are described only qualitatively. Moreover, for a material like concrete, the microstructural attributes play a significant role in deciding its macroscopic response and cannot be neglected. Alternately, micromechanics based damage models account for the inherent heterogeneity in composites and are best suited to model the behavior of such materials.

Micromechanical analysis of damage in concrete is an area of active research to describe the response of cementitious composites by modeling the specific damage mechanisms at the microscale. Considering distribution of penny shaped microcracks in the matrix, constitutive relations for quasibrittle materials have been derived by several authors [1, 4–8]. The interaction between microcracks and the unilateral effect are described by homogenization based models. Principles of irreversible thermodynamics are employed at the microscale to describe crack growth or the evolution of damage. Conditions of crack closure and friction between crack faces are also accounted for in these models. A combined fracture mechanics and micromechanics approach has been adopted by Pichler et al. [9] to obtain the softening response commonly observed in geomaterials. The damage evolution law is obtained by implementing concepts of linear elastic fracture mechanics. Propagation of the microcracks results in strain softening at the macroscopic scale. While capable of addressing the physical modes of damage in the material adequately, the models do not consider the presence of an inclusion phase, which is an integral part of the composition of plain concrete. The presence of the coarse aggregates influences the response of concrete substantially and thus, should not be neglected in formulating the constitutive relations of the material.

In this study, a micromechanics based damage model is developed in order to simulate the mechanical response of plain concrete in which the mortar matrix is weakened by randomly distributed microcracks. The homogenized stiffness tensor is computed based on the classical solution of a matrix–inclusion system as put forward by Eshelby. Based on thermodynamical considerations, local damage variables characterizing the state of microcracks are defined. The kinetics of damage growth is governed by the strain energy release rate. The evolution of damage at the mesoscale and the corresponding macroscopic response are obtained by an incremental analysis. A highlight of this work is to assess the effect of each of the mesoscopic parameters on the behavior of concrete under uniaxial tension and a parametric study is carried out to this end.

2. MICROMECHANICAL MODEL

In order to determine the mechanical properties of composites, it is necessary to evaluate the stress and strain fields within the inclusions or inhomogeneities present in the material. An inclusion, in this context, is defined as a subregion (volume V') which experiences an eigenstrain or stress-free strain within a homogeneous material (volume V). The elastic properties are uniform throughout the composite material. The eigenstrain may be generated as a result of thermal expansion, phase transformation, initial strain, misfit strain etc.

Making use of the principle of superposition and Green's function, Eshelby [10] elegantly derived the strain field within the matrix and an ellipsoidal inclusion due to the eigenstrain. The resultant constrained strain $\boldsymbol{\varepsilon}^{\mathrm{c}}$ in the inclusion and the matrix is related to the eigenstrain $\boldsymbol{\varepsilon}^{*}$ by the relation $\boldsymbol{\varepsilon}^{\mathrm{c}} = \mathbf{L}:\boldsymbol{\varepsilon}^{*}$. $\mathbf{L}$ is a fourth order tensor, known as the Eshelby's tensor. For the ellipsoidal inclusion considered by Eshelby, the tensor was shown to be a constant, thus resulting in an uniform strain field within the inclusion. Eshelby's tensor is a function of the geometry of the inclusion and the elastic properties of the matrix in which the inclusion is present.

For a large class of composites, the commonly encountered problem is that of an inhomogeneity lying in a matrix and being subjected to external stress or strain fields. An inhomogeneity refers to a second phase material with different elastic properties (elastic stiffness tensor $\mathbf{C}'$) than that of the surrounding matrix material (stiffness tensor $\mathbf{C}$). An additional disturbance strain incurs in the material owing to the mismatch of the elastic properties of the constituent phases. Based on the solution of the inclusion under eigenstrain, the stress and strain fields in a material containing an inhomogeneity under external loads can be systematically deduced by the "equivalent inclusion method".

In the equivalent inclusion method, the inhomogeneity is replaced by a homogeneous inclusion subjected to a fictitious eigenstrain. The eigenstrain is introduced to account for the difference in the elastic properties of the two different phases. When subjected to a far field strain $\boldsymbol{\varepsilon}^{\infty}$, the strain in the equivalent inclusion can be expressed as:

$$\boldsymbol{\varepsilon}^{\mathrm{i}} = \boldsymbol{\varepsilon}^{\infty} + \boldsymbol{\varepsilon}^{\mathrm{c}} = \boldsymbol{\varepsilon}^{\infty} + \mathbf{L}:\boldsymbol{\varepsilon}^{*}, \quad (1)$$

where $\boldsymbol{\varepsilon}^{\mathrm{c}}$ is the disturbance strain arising due to the difference of the elastic properties and $\boldsymbol{\varepsilon}^{*}$ is the equivalent eigenstrain. The stress in the equivalent homogeneous inclusion is $\boldsymbol{\sigma}^{\mathrm{i}} = \mathbf{C}:(\boldsymbol{\varepsilon}^{\infty} + \boldsymbol{\varepsilon}^{\mathrm{c}} - \boldsymbol{\varepsilon}^{*})$ and that in the inhomogeneity is $\boldsymbol{\sigma}^{\mathrm{i}} = \mathbf{C}':(\boldsymbol{\varepsilon}^{\infty} + \boldsymbol{\varepsilon}^{\mathrm{c}})$. Assuming the elastic state of the inhomogeneity and that of the equivalent inclusion to be identical, the resultant stress fields are equated as follows:

$$\mathbf{C}:(\boldsymbol{\varepsilon}^{\infty} + \boldsymbol{\varepsilon}^{\mathrm{c}} - \boldsymbol{\varepsilon}^{*}) = \mathbf{C}':(\boldsymbol{\varepsilon}^{\infty} + \boldsymbol{\varepsilon}^{\mathrm{c}}). \quad (2)$$

From Eq. (2), the eigenstrain is obtained as

$$\boldsymbol{\varepsilon}^{*} = [((\mathbf{C} - \mathbf{C}')\cdot\mathbf{L} - \mathbf{C})^{-1}\cdot(\mathbf{C}' - \mathbf{C})]:\boldsymbol{\varepsilon}^{\infty} = \mathbf{T}:\boldsymbol{\varepsilon}^{\infty}, \quad (3)$$

where the expression $((\mathbf{C}-\mathbf{C}')\cdot\mathbf{L}-\mathbf{C})^{-1}\cdot(\mathbf{C}'-\mathbf{C})$ is substituted by the tensor $\mathbf{T}$. Using the expression of the eigenstrain from Eq. (3), the strain field of the inhomogeneity is given by

$$\boldsymbol{\varepsilon}^{\mathrm{i}} = (\mathbf{I}+\mathbf{L}\cdot\mathbf{T}):\boldsymbol{\varepsilon}^{\infty}, \tag{4}$$

where $\mathbf{I}$ is the fourth order identity tensor. The resultant stress field in the inhomogeneity is

$$\begin{aligned}\boldsymbol{\sigma}^{\mathrm{i}} &= \mathbf{C}\cdot[\mathbf{I}+(\mathbf{L}-\mathbf{I})\cdot\mathbf{T}]:\boldsymbol{\varepsilon}^{\infty}\\ &= \mathbf{C}'\cdot(\mathbf{I}+\mathbf{L}\cdot\mathbf{T}):\boldsymbol{\varepsilon}^{\infty}.\end{aligned} \tag{5}$$

Thus, the strain and stress fields in an inhomogeneity are obtained from Eqs. (4) and (5), respectively. The results from the equivalent inclusion method have proved to be the stepping stone of an impressive amount of research in predicting the behavior of composites which have been applied to a diverse range of physical problems. Eshelby tensor can be used to obtain the stress and strain fields in heterogeneous materials containing reinforcements or defects such as voids, cracks, dislocations etc. Since the original work of Eshelby, a lot of research has been done to derive the analytical forms of the Eshelby's tensor for inclusions with different shapes and matrix with various elastic symmetries.

In the present work, the stress and strain fields of plain concrete containing matrix microcracks is computed by employing the solution of equivalent inclusion for the matrix–inclusion system. It should be mentioned here that the term "inclusion" has often been used to refer to the second phase particle or "inhomogeneity".

Concrete is considered as a two phase composite at the mesoscopic scale—the mortar phase and the aggregate phase designated by superscripts m and a, respectively. In addition to the coarse aggregates, microcracks are assumed to be distributed randomly in the mortar matrix. The orientation of a family of microcracks is identified by the normal to the crack face. The macroscopic constitutive relations are established through a two step homogenization procedure as illustrated in Fig. 1. In the first step, the homogenized stress–strain relations are obtained for the system containing coarse aggregates embedded in the mortar matrix. Employing the equivalent inclusion method and the Mori–Tanaka homogenization procedure, the macroscopic quantities are derived from the mesoscopic fields by consideration of nondilute distribution of coarse aggregates. The macroscopic behavior is significantly influenced by the microcracks present in the matrix which is addressed subsequently in the development of the model. The additional compliance due to the system of microcracks is added to the homogenized properties of the aggregate–matrix system derived in the first step to obtain the final macroscopic constitutive relations. The matrix cracks contribute to the inelastic strain $\boldsymbol{\varepsilon}^{\mathrm{int}}$, thereby imparting nonlinearity to the overall response of the composite. The governing equations of the numerical model are given in the next two sections.

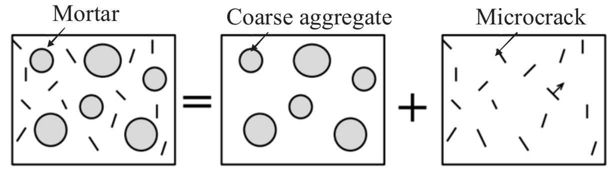

Fig. 1. Two-step homogenization of concrete with matrix microcracks.

2.1. Homogenized Stiffness due to Aggregate–Mortar System

Description of the macroscopic relations between the stress and strain of plain concrete containing mortar cracks involves a two-step homogenization as discussed previously. As a first step, a representative volume element consisting of circular coarse aggregates and the mortar matrix is considered. The assumption of an infinitesimal strain field allows for the additive split of the macroscopic strain $\bar{\boldsymbol{\varepsilon}}$ into an elastic part $\boldsymbol{\varepsilon}^{\mathrm{el}}$ and an inelastic part $\boldsymbol{\varepsilon}^{\mathrm{int}}$. The relation between the macroscopic stress and the elastic part of strain is described in this section. The overall stiffness of the composite comprising of the mortar matrix and coarse aggregates is obtained by the equivalent inclusion method.

The strain and stress fields of a single inclusion present in an infinite matrix subjected to external strain are given by Eqs. (4) and (5) respectively. For a composite material, containing inclusions whose volume fraction is given by f, the macroscopic stress $\boldsymbol{\Sigma}$ and macroscopic strain $\bar{\boldsymbol{\varepsilon}}$ are expressed in terms of the average stresses and strains of the phases (distinguished by superscripts m and a for matrix and aggregate) as:

$$\boldsymbol{\Sigma} = (1-f)\boldsymbol{\sigma}^{\mathrm{m}} + f\boldsymbol{\sigma}^{\mathrm{a}},\ \bar{\boldsymbol{\varepsilon}} = (1-f)\boldsymbol{\varepsilon}^{\mathrm{m}} + f\boldsymbol{\varepsilon}^{\mathrm{a}}. \tag{6}$$

The average strain of the aggregates is given in terms of the applied far field strain by Eq. (4). In the Mori–Tanaka method of homogenization, the interaction effects between the relatively high volume fraction of inclusions is incorporated in an approximate way by assuming that each of the inclusion is subjected to the average matrix strain instead of the far field strain. Replacing the far field strain $\boldsymbol{\varepsilon}^{\infty}$ in Eq. (4) by the average matrix strain, the resultant strain field in the aggregate phase is expressed as

$$\boldsymbol{\varepsilon}^{\mathrm{a}} = (\mathbf{I} + \mathbf{L} \cdot \mathbf{T}) : \boldsymbol{\varepsilon}^{\mathrm{m}}. \tag{7}$$

Substituting $\boldsymbol{\varepsilon}^{\mathrm{a}}$ in Eq. (6), the average matrix strain $\boldsymbol{\varepsilon}^{\mathrm{m}}$ can be obtained in terms of the macroscopic strain $\bar{\boldsymbol{\varepsilon}}$ as

$$\boldsymbol{\varepsilon}^{\mathrm{m}} = \{f(\mathbf{I} + \mathbf{L} \cdot \mathbf{T}) + (1-f)\mathbf{I}\}^{-1} : \bar{\boldsymbol{\varepsilon}}. \tag{8}$$

Under the assumption of each of the phases being elastic and isotropic, the macroscopic stress is expressed in terms of the average mesoscopic strains as

$$\boldsymbol{\Sigma} = f\mathbf{C}^{\mathrm{a}} : \boldsymbol{\varepsilon}^{\mathrm{a}} + (1-f)\mathbf{C}^{\mathrm{m}} : \boldsymbol{\varepsilon}^{\mathrm{m}}, \tag{9}$$

where $\mathbf{C}^{\mathrm{a}}$ and $\mathbf{C}^{\mathrm{m}}$ are the stiffness tensors of the coarse aggregates and mortar matrix respectively. Using the relations between the aggregate strain and matrix strain with the macroscopic strain field, the macroscopic stress–strain relationship is finally obtained as

$$\boldsymbol{\Sigma} = [\{f(\mathbf{C}^{\mathrm{a}} \cdot (\mathbf{I} + \mathbf{L} \cdot \mathbf{T})) + (1-f)\mathbf{C}^{\mathrm{m}}\} \times \{f(\mathbf{I} + \mathbf{L} \cdot \mathbf{T}) + (1-f)\mathbf{I}\}^{-1}] : \bar{\boldsymbol{\varepsilon}}. \tag{10}$$

The homogenized stiffness tensor $\mathbf{C}_1^{\mathrm{hom}}$ for the composite consisting of mortar matrix and coarse aggregates is thus given by:

$$\mathbf{C}_1^{\mathrm{hom}} = \{f(\mathbf{C}^{\mathrm{a}} \cdot (\mathbf{I} + \mathbf{L} \cdot \mathbf{T})) + (1-f)\mathbf{C}^{m}\} \times \{f(\mathbf{I} + \mathbf{L} \cdot \mathbf{T}) + (1-f)\mathbf{I}\}^{-1}. \tag{11}$$

The inverse of the homogenized stiffness yields the homogenized compliance tensor $\mathbf{S}_1^{\mathrm{hom}} = \mathbf{C}_1^{\mathrm{hom}^{-1}}$. It can be seen from Eq. (11) that the overall behavior of the composite, which is governed by the tensor $\mathbf{C}_1^{\mathrm{hom}}$, is dependent on the elastic properties of the constituent phases, the volume fraction of the phases and the shape of the inclusion considered for the analysis. The effect of the shape of the inclusion enters into the formulation through the Eshelby's tensor $\mathbf{L}$. The components of $\mathbf{L}$ for a circular inclusion under plane stress conditions are

$$L_{ijkl} = \frac{3\nu^{\mathrm{m}} - 1}{8}\delta_{ij}\delta_{kl} + \frac{3 - \nu^{\mathrm{m}}}{8}(\delta_{il}\delta_{jk} + \delta_{jl}\delta_{ik}), \tag{12}$$

where ν^{m} is the Poisson's ratio for the matrix material and δ_{ij} is the second order identity tensor or the Kronecker delta.

$\mathbf{C}_1^{\mathrm{hom}}$ is the undamaged stiffness tensor which does not consider any inelastic effects. The randomly distributed mortar microcracks are the source of damage considered in the present analysis which result in inelastic strains and alter the macroscopic stiffness of cementitious materials. In the next section, the contribution of the matrix microcracks to the overall stiffness of the composite is estimated.

2.2. Additional Strains due to Microcracks in Mortar

The mortar matrix in cement based composites is characterized by the presence of a number of arbitrarily aligned microcracks. The cracks weaken the matrix and are responsible for the nonlinear behavior observed at the macroscale. The propagation and coalescence of the microcracks eventually lead to softening and final failure of the composite material. The inelastic strains and the additional compliance arising due to the presence of microcracks are deduced in this section.

The cracks are assumed to be line cracks or slit cracks in two dimensions with the length of the cracks being $2c$. For simplification of the present analysis, all the cracks are assumed to be of the same size. Further, since the response of the composite is considered under uniaxial tension, the cracks are assumed to be open at all stages of loading. The crack faces are assumed to be frictionless. Figure 2 shows a single microcrack inclined at an angle of θ with the x_1 axis. The local coordinate system of the crack is represented as $x_1' - x_2'$. The local coordinate system is chosen in a way such that the centre of the crack is at the origin of the coordinate system and the normal to the crack face coincides with the x_2' direction.

The effective elastic moduli of a composite material consisting of planar cracks in two dimensions are estimated by the procedure given by Nemat–Nasser and Hori [11]. The applied tensile stress σ_0 along x_1 direction can be resolved into normal component σ_{11}' and σ_{22}' in the local coordinate system. The crack thus experiences both tensile and shear stresses. The crack opening displacements in the normal and tangential directions in terms of the local stresses under plane stress conditions are

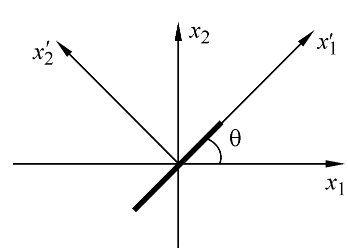

Fig. 2. An arbitrarily oriented mortar microcrack.

$$\|u_i\| = \frac{4}{E^{\mathrm{m}}}\sqrt{c^2 - x^2}\sigma'_{ii}, \tag{13}$$

where $i = 1, 2$ and $-c < x < c$, E^{m} is the Young's modulus of the matrix.

The resultant inelastic strain $\boldsymbol{\varepsilon}^{\mathrm{c}}$ for a single microcrack is given by

$$\varepsilon_{ij}^{\mathrm{c}} = \frac{1}{c^2}\int_{-c}^{c}(n_i \|u_j\| + \|u_i\| n_j)\,\mathrm{d}x'_1, \tag{14}$$

where n_i and n_j are the components of the unit normal on the crack face. Substituting Eq. (13) into Eq. (14), the inelastic strain can be expressed as a function of the locally applied stress fields as

$$\varepsilon_{ij}^{\mathrm{c}} = H'_{ijkl}\sigma'_{kl}. \tag{15}$$

H'_{ijkl} is the additional compliance resulting due to the presence of microcracks in the matrix. The nonzero components of the tensor $\mathbf{H}'$ are

$$H'_{2222} = \frac{2\pi}{E^{\mathrm{m}}},\qquad H'_{1212} = H'_{2121} = H'_{1221} = H'_{2112} = \frac{\pi}{2E^{\mathrm{m}}}. \tag{16}$$

The components of $\mathbf{H}'$ are transformed from the local coordinate system $x'_1 - x'_2$ of the cracks to the $x_1 - x_2$ coordinates through an orthonormal tensor $\mathbf{Q}$.

When a number of microcracks are present in the matrix, each of the cracks contributes to the inelastic strain as given by the previous equations. In order to obtain the total inelastic strain engendered by multiple microcracks, the crack density parameter d as introduced by Budiansky and O'Connell [12] is used. For two dimensional formulations, the crack density parameter is $d = Nc^2$, where N denotes the number of cracks of length $2c$ present per unit area of the representative volume element, d is the internal variable used in this micromechanical formulation to characterize the damage caused by microcracking.

In the present analysis, microcracks are assumed to be isotropically distributed, i.e., the same number of microcracks are present in all directions considered. The additional compliance due to the presence of microcracks in all possible directions is obtained from the following equation:

$$\mathbf{H} = \frac{d}{2\pi}\int_0^{2\pi}\mathbf{Q}^T\mathbf{H}'\mathbf{Q}\,\mathrm{d}\theta. \tag{17}$$

The equation yields the components of the additional compliance tensor $\mathbf{H}$ as

$$\mathbf{H} = \frac{\pi d}{E^{\mathrm{m}}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}. \tag{18}$$

The overall compliance tensor $\mathbf{S}^{\mathrm{hom}}$ of the composite includes the contributions of the coarse aggregate–mortar system and that of the randomly oriented microcracks and is given by

$$\mathbf{S}^{\mathrm{hom}} = \mathbf{S}_1^{\mathrm{hom}} + \mathbf{H}. \tag{19}$$

The existence and propagation of microcracks result in a reduction of the stiffness of the material. The homogenized stiffness tensor of concrete comprising of coarse aggregates, mortar matrix and matrix microcracks is

$$\mathbf{C}^{\mathrm{hom}} = \mathbf{C}_1^{\mathrm{hom}} + \mathbf{C}_1^{\mathrm{hom}} : \mathbf{J},\qquad \mathbf{J} = \mathbf{H} : \mathbf{C}^{\mathrm{m}}. \tag{20}$$

In the initial stages of loading, when the applied load is small, the cracks remain stationary and the macroscopic constitutive relation can be obtained by $\boldsymbol{\Sigma} = \mathbf{C}^{\mathrm{hom}} : \bar{\boldsymbol{\varepsilon}}$. However, as the intensity of the applied load increases, the cracks begin to propagate and the kinetics of crack propagation should enter into the formulation. A thermodynamic based framework is adopted in the present model in order to incorporate the growth of microcracks in the constitutive relations. The criteria determining the progress of damage at the mesoscale are discussed in the following section.

3. EVOLUTION OF DAMAGE

The propagation of microcracks at the mesoscale and the corresponding macroresponse of concrete is treated within a damage mechanics framework. The selection of the damage variable and its evolution are guided by micromechanical arguments, thus preserving the mechanistic basis of the model. The propagation of microcracks is an irreversible process which is accompanied by an increase of the inelastic strain in the material. The crack density parameter d has been used in the previous section to consider the effect of microcracks in the overall behavior of plain concrete. With increase in the applied load, the microcracks begin to propagate causing further damage in the material. The evolution of the damage variable d is obtained in this section. Energetic considerations based on damage mechanics, similar to some previous works [1, 5], are adopted in the present analysis.

The thermodynamic potential or the free energy of the composite material is expressed in terms of the effective stiffness as [1, 5]

$$\mathbf{W} = \frac{1}{2}\bar{\boldsymbol{\varepsilon}} : \mathbf{C}^{\mathrm{hom}} : \bar{\boldsymbol{\varepsilon}}, \tag{21}$$

where $\mathbf{W}$ is the free energy potential or the Helmholtz energy, $\overline{\boldsymbol{\varepsilon}}$ is the macroscopic strain and $\mathbf{C}^{\text{hom}}$ is the effective stiffness tensor of the composite. $\mathbf{C}^{\text{hom}}$ includes the effect of distributed microcracking. The derivative of $\mathbf{W}$ with respect to the macroscopic strain yields the constitutive relation as

$$\boldsymbol{\Sigma} = \frac{\partial \mathbf{W}}{\partial \overline{\boldsymbol{\varepsilon}}} = \mathbf{C}^{\text{hom}} : \overline{\boldsymbol{\varepsilon}}.$$

The degree of damage is dependent on the alignment of a crack with respect to the direction of the applied load. For example, the family of cracks lying perpendicular to the direction of the applied external loads is the most critical and begins to propagate first. Therefore, it is pertinent to define certain representative directions by considering isotropic distribution of microcracks. The damage state in each of the directions is given by a damage variable d_i. For a given direction i, the thermodynamic force associated with the damage variable d_i is derived from the free energy function W_i as a conjugate of the damage variable given by

$$F_i^{\text{d}} = -\frac{\partial W_i}{\partial d_i}. \tag{22}$$

The damage driving force F_i^{d} is the strain energy release rate of the ith crack which is used in the damage criterion to describe the growth of microcracks. The damage criterion is given by

$$f(F_i^{\text{d}}, d_i) = F_i^{\text{d}} - R(d_i) \leq 0. \tag{23}$$

Here $R(d_i)$ is the resistance offered by the matrix material to the growth of microcracks at the mesoscopic scale. The $R(d_i)$ curve is dependent on the state of damage and is capable of accommodating the local resistance due to the probable presence of some heterogeneities in the mortar matrix which cannot be distinguished at the mesoscale. The crack resistance function should ideally be obtained for different matrix materials by conducting experiments on mortar specimens. However, owing to the scarcity of experimental results, the exact form of the function is lacking at present. Similar to several studies available in the literature [1, 5, 6], in the present analysis, the crack resistance is assumed to depend linearly on the state of damage and is given by $R(d_i) = c_0 + c_1 d_i$. By using this relation, two variables c_0 and c_1 are introduced in the model, c_0 represents the initial damage threshold and can be interpreted as the fracture toughness of the matrix material. When the value of the damage variable is small, as in the case of initial stages of loading, damage begins to propagate when the strain energy release rate reaches the critical value given by c_0. With increase in the value of the damage variable, the resistance to the extension of the crack increases which is controlled by the variable c_1, depicting the kinematics of the damage evolution. An envelope of all the damage surfaces given by Eq. (23) gives the macroscopic damage criterion for the composite material considering various orientations of microcracks.

By imposing the normality rule, the damage evolution is finally obtained as

$$\dot{d}_i = \dot{\lambda}_i^{\text{d}} \frac{\partial f_i}{\partial F_i^{\text{d}}} = \dot{\lambda}_i^{\text{d}},$$
$$\dot{d}_i = \begin{cases} 0, \text{ if } f_i \leq 0 \text{ and } \dot{f}_i < 0, \\ \dot{\lambda}_i^{\text{d}}, \text{ if } f_i = 0 \text{ and } \dot{f}_i = 0, \end{cases} \tag{24}$$

$\dot{\lambda}_i^{\text{d}}$ is the damage multiplier of the ith family of microcracks. By making use of the consistency condition, $\dot{f}_i = 0$, the damage multipliers for K families of microcracks considered (i = 1, 2, ..., K) can be computed as

$$\dot{f}_i = \frac{\partial f_i}{\partial \overline{\varepsilon}} : \dot{\overline{\varepsilon}} + \sum_{s=1}^{K} \frac{\partial f_i}{\partial d_s} \dot{d}_s = \frac{\partial F_i^{\text{d}}}{\partial \overline{\varepsilon}} : \dot{\overline{\varepsilon}}$$
$$+ \sum_{s=1}^{K} \left(\frac{\partial F_i^{\text{d}}}{\partial d_s} - c_1 \frac{\partial d_i}{\partial d_s} \right) \dot{d}_s = 0. \tag{25}$$

Accounting for the response of all the microcracks oriented in various directions, the above equation results in a system of equations which can be represented by

$$[M]\{\dot{d}\} = \{F\} : \dot{\overline{\varepsilon}}. \tag{26}$$

The components of the matrix M are

$$M_{is} = -\left(\frac{\partial F_i^{\text{d}}}{\partial d_s} - c_1 \frac{\partial d_i}{\partial d_s} \right). \tag{27}$$

The vector $\{\dot{d}\}$ consists of all the damage variables in the rate form for the K different orientations of microcracks considered. The components of F are given by $\partial F_i^{\text{d}} / \partial \overline{\varepsilon}$. Thus, the solution of the set of equations provides the values of the damage variables associated with the respective microcrack family. The rate form of the constitutive relation can be written as $\dot{\boldsymbol{\Sigma}} = \mathbf{C}_t^{\text{hom}} : \dot{\overline{\boldsymbol{\varepsilon}}}$. $\mathbf{C}_t^{\text{hom}}$ is the tangential stiffness tensor which is obtained for each incremental step of loading.

Therefore, the macroscopic response can be predicted at each load step as damage progresses. The damaged state of the material is represented by a set of damage variables. Initially, damage in all the directions are assumed to be identical (assumption of isotropic distribution). However, as the applied load increases, the behavior of the microcracks vary depending on their orientations, which is encapsulated in the damage model described.

4. NUMERICAL IMPLEMENTATION

The damage model developed is suitable for materials in which the matrix contains a number of randomly aligned microcracks in addition to a second phase material or inhomogeneity. For application of the model to predict the response of composites, the crack density function d needs to be specified in all possible directions. The effect of microcracks is obtained by integration of d over all the directions. In the present formulation, certain discrete orientations are considered and numerical integration is performed over the chosen directions to obtain the effective behavior of the material. The Gauss quadrature is generally implemented to evaluate such integrals [1, 13, 14]. For the present problem, sixteen evenly spaced directions are taken and the appropriate weight functions are used for the evaluation of the integral. The weight function and the directions are dependent on the integration scheme adopted. A global damage variable D is derived by accounting for the damage d_i in all the directions as $D = \omega_i d_i$ where ω_i is the weight function related to the particular direction.

The nonlinear set of equations is solved by the following incremental iterative procedure.

Step 1: For load step $p + 1$, increase macroscopic strain $\boldsymbol{\varepsilon}$ by $\Delta\boldsymbol{\varepsilon}$.

Step 2: Initialize damage variables d_i for each of the K orientations of microcrack families considered, by taking the values of the previous converged load step p.

Step 3: Check damage criterion f_i^{d} for each direction and obtain the matrix [M].

Step 4: Solve $[M]\{\Delta d\}=\{f\}$ for Δd for K directions.

Step 5: If $\Delta d_i < 0$, update [M] as $f_i = 0$, $M_{is} = M_{si} = 0$ for $i \neq s$. Repeat previous step with updated values of $[M]$ until $\Delta d_i \geq 0$.

Step 6: Obtain the current values of damage variables $\Delta d_i^{p+1} = d_i + \Delta d_i$.

Step 7: Obtain the macroscopic stress in terms of macroscopic strain.

Step 8: Repeat steps 1 to 7 to obtain the complete stress-strain response.

5. RESULTS AND DISCUSSION

5.1. Model Validation: Response under Uniaxial Tension

The predictive capability of the proposed damage model is assessed by simulating the behavior of plain concrete under uniaxial tension. The results of the numerical simulations are compared with experimental data available in the literature. The data from the uniaxial tensile tests of Hordijk [15] and Gopalaratnam and Shah [16] are used for this purpose. The different material properties used for the analysis are listed in the table.

The damage variables d_i, $i = 1, 2, ..., K$, representing the density of microcracks present in each direction in the material, is chosen to have an initial value of 0.01 for both sets of experimental data. The material resistance to the growth of microcracks is characterized by a curve similar to the R-curve adopted in fracture mechanics as described previously. The resistance curve consists of two terms c_0 and c_1, c_0 depicts the initial resistance of the material while c_1 controls the evolution of damage. For the present numerical analysis, the resistance is assumed to be constant and c_1 is set to 0. The constant value of the resistance curve R represents the critical value of the strain energy release rate of the mortar matrix $G_{\mathrm{c}}^{\mathrm{m}}$ and is best suited to describe the behavior of a brittle matrix. The approximation is made to simplify the computations involved.

Figures 3 and 4 show the comparison of the results obtained from the numerical model with the experimental stress-strain response of plain concrete predicted by Hordijk [15] and Gopalaratnam and Shah [16] respectively. The strain represents the average strain considered in a material volume present in the damage zone or the fracture process zone wherein damage in the form of microcracking takes place. The results are able to reproduce the mechanical behavior of concrete at the macroscopic scale considerably well.

The corresponding evolution of the global damage index D with the macroscopic strain is presented in Fig. 5 for the experimental data of Hordijk [15]. It can be observed from the figure that initially the damage variable remains constant. This implies that all the microcracks, if any, that are present in the material,

Table. Material properties

Property	[15]	[16]
Young's modulus of coarse aggregate E^{a}, GPa	70	55
Young's modulus of mortar E^{m}, GPa	25	28
Poisson's ratio of coarse aggregate ν^{a}	0.20	0.20
Poisson's ratio of mortar ν^{m}	0.21	0.19
Aggregate diameter $2a$, mm	24	10
$c_0 \times 10^{-3}$, J/m^2	1.5	1.1

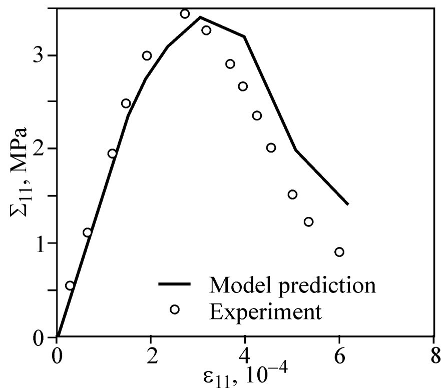

Fig. 3. Macroscopic stress–strain response under uniaxial tension compared with experimental results [15].

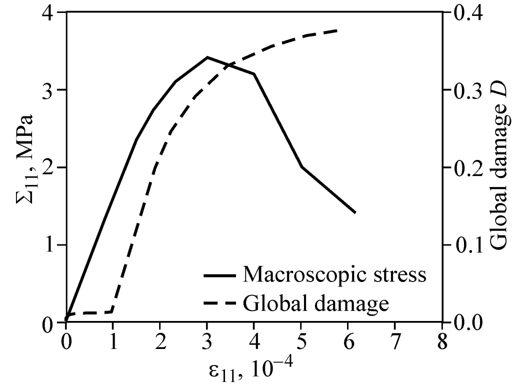

Fig. 5. Evolution of global damage with macroscopic strain.

are stationary. The stress–strain response is linear upto 48% of the peak stress recorded as seen in Fig. 3. Further increase of the strain causes microcracks in certain favorable directions to propagate as a result of which an increase in the global damage variable is observed. The gradual increase of the damage index with increasing strain is reflected in the nonlinearity in the macroscopic response before the peak stress followed by post-peak softening.

5.2. Parametric Study

The micromechanics based damage model developed to describe the behavior of plain concrete under uniaxial tensile loads involves different mesoscale parameters. It is essential to analyze the effect of each of the parameters on the macroscopic constitutive relationship of the composite material to gather sufficient knowledge about the influence of the microstructure. The different parameters involved in the proposed micromechanical damage model are the mix proportions or the volume fractions of the constituent phases, their elastic properties, the initial damage parameter and the material resistance provided by the mortar matrix to the growth of microcracks. The response observed at the macroscale is dominated by each of these parameters. This section presents the findings of a series of numerical experiments conducted to evaluate the influence of the mesoscale properties on the stress-strain behavior of concrete.

5.2.1. Aggregate volume fraction

The coarse aggregates form 40–80% of the total volume of plain concrete and, as demonstrated by experimental studies, tensile strength and fracture behavior of concrete are significantly affected by the volume fraction of coarse aggregates in the concrete mix [17, 18]. The inclusion of the aggregate phase in the present model is an improvement over the models available in the literature [1, 5, 6] which have dealt with

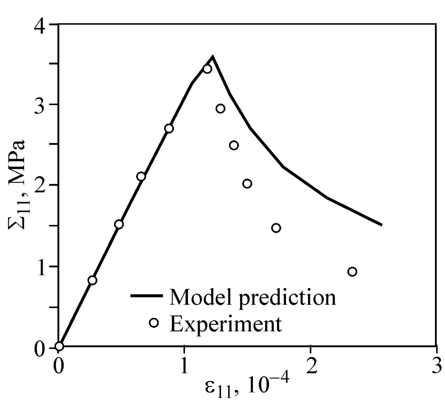

Fig. 4. Macroscopic stress–strain response under uniaxial tension compared with experimental results [16].

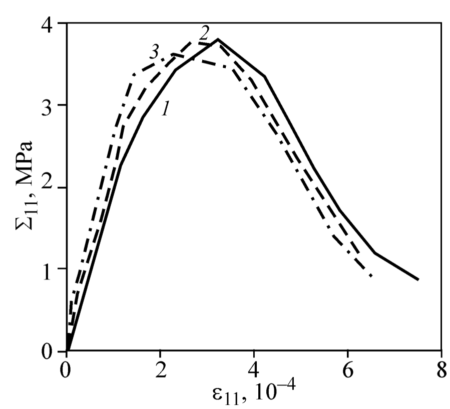

Fig. 6. Effect of aggregate volume fraction on the macroscopic response of plain concrete under uniaxial tension. Aggregate volume fraction $f = 0.45$ (*1*), 0.60 (*2*), 0.80 (*3*).

microcracks distributed in mortar without consideration of the presence of aggregates. Figure 6 shows the macroscopic stress–strain response of concrete for three different values of aggregate volume fraction f considered. The effective stiffness of the material is a function of the stiffnesses of the constituent phases. For plain concrete, the coarse aggregate particles are much stiffer than the mortar matrix. Thus, increasing the volume fraction of aggregate particles in concrete results in an increase of the overall stiffness of the material. The tensile strength is observed to decrease with the increase in the aggregate content. The stress distribution in the matrix is perturbed by the presence of the aggregates which is accounted for in the model. The variation in the elastic properties between the phases results in stress concentration, thereby increasing the average stress of the matrix. This leads to the lowering of the strain at which damage begins to propagate; thus a lower tensile strength is recorded for the composite as a whole.

5.2.2. Elastic modulus of mortar

The relative stiffness of each of the phases has a considerable effect on the macroscopic behavior of a composite. In the present analysis, concrete is modeled as a two-phase material. To understand the role played by the elastic properties of the phases, the Young's modulus of mortar E^{m} is altered and its effect on the overall stress–strain response is analyzed. The results are illustrated in Figure 7 for two different values of E^{m} considered. The derived homogenized stiffness of concrete is a function of the elastic moduli of both the phases. Hence, increasing the value of E^{m} results in an increased value of the overall stiffness of concrete. The crack opening displacement or the displacement jump across the crack faces $\| u_i \|$ is inversely proportional to E^{m}. As the mortar matrix is made stiffer, the crack opening displacement reduces thereby resulting in lowering the inelastic strain $\varepsilon^{\mathrm{int}}$. The contribution of $\varepsilon^{\mathrm{int}}$ to the added compliance is decreased and the stress at which the propagation of the microcracks start increases. As a result, a higher macroscopic tensile strength is achieved with an increase of the elastic modulus of the mortar.

5.2.3. Initial damage in mortar

The crack density parameter d is an important variable which characterizes the damage caused by the randomly distributed microcracks. The macroscopic behavior of concrete is governed by the evolution of the crack density parameter. The damage parameter is a function of the number of cracks N present per unit area of the representative volume element and the length of the crack $2c$. Through microscopic analysis of cored cylindrical specimens subjected to uniaxial tension, Dhir and Sangha [19] have given an estimation of the microcrack density in plain concrete. Figure 8 presents the effect of variation of the density of microcracks N on the overall response of the composite for three different values considered. Keeping the size of the microcrack constant, increasing the value of N implies an increased value of the initial damage in the material. The effective stiffness of the composite material is reduced with an increased value of N. This is evident as increasing the damage d of the material makes it more compliant. Increasing the initial damage also results in lowering the tensile strength of the material as the initial resistance to crack growth

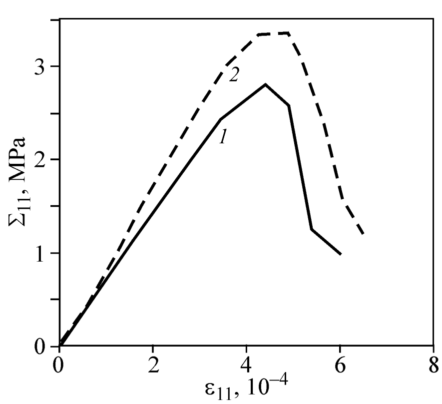

Fig. 7. Effect of elastic modulus of mortar on the macroscopic response of plain concrete under uniaxial tension. The Young's modu.lus of mortar $E^{\mathrm{m}} = 10$ (*1*), 25 GPa (*2*).

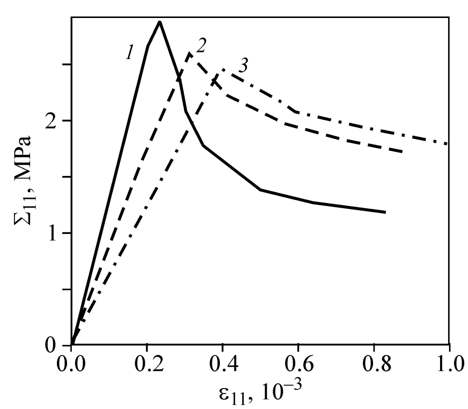

Fig. 8. Effect of microcrack density on the macroscopic response of plain concrete under uniaxial tension. Microcrack density $N = 10^3$ (*1*), 10^4 (*2*), $2\times10^4\ \mathrm{m}^{-2}$ (*3*).

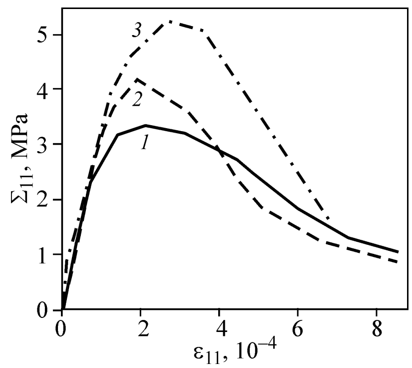

Fig. 9. Effect of parameter c_0 on the macroscopic response of plain concrete under uniaxial tension. $c_0 = 0.5\times10^{-3}$ (*1*), 1×10^{-3} (*2*), 2×10^{-3} N/m (*3*).

decreases. The slope of the post-peak curve is observed to be steeper for a lower value of *N*. This is because increasing the value of *N* results in much more distributed microcracking in the matrix, which dissipates higher energy, which is manifested in the post-peak response.

5.2.4. Fracture resistance of mortar

In this work, the damage mechanism under consideration is the growth of microcracks present in the mortar matrix. As the intensity of the applied strain increases, the cracks begin to propagate, resulting in a nonlinear macroscopic stress–strain response. The criterion for initiation and propagation of damage used in the present analysis is based on the strain energy release rate which is obtained from the free energy potential of the material. The matrix material offers resistance to the growth of the microcracks which is represented by the crack growth resistance curve $R(d_i)$.

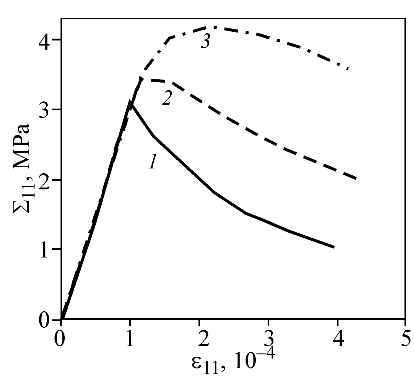

Fig. 10. Effect of parameter c_1 on the macroscopic response of plain concrete under uniaxial tension. $c_1 = 10^{-4}$ (*1*), 10^{-3} (*2*), 3×10^{-3} (*3*).

Due to lack of experimental studies which relate the resistance of the material at the mesoscopic scale with the corresponding state of damage d_i, the *R* curve is assumed to depend linearly on the damage index. The two constants appearing in the relation are c_0 which is the initial threshold of damage (comparable to the fracture toughness of the mortar matrix) and c_1 which influences the kinetics of the damage. The values of c_1 for plain concrete are variously reported to be in the range of 0 and 3×10^{-3} in the literature under tension [1, 5]. The macroscopic stress-strain behavior with the variation of c_0 and c_1 is depicted in Figs. 9 and 10 respectively. The initial stiffness is not altered as the parameters enter into the formulation once damage start to evolve. The tensile strength increases with an increase in the value of c_0 and c_1. This is because the resistance to the propagation of microcracks increases, as a result of which a higher stress is required for damage to initiate. The evolution of damage is determined by the value of c_1 which can be seen from the variation in the resultant post-peak softening curve.

6. CONCLUSIONS

A micromechanical analysis is carried out to understand the effect of damage caused by the propagation of microcracks randomly distributed in the matrix on the response of cementitious materials under tensile loads. The mesoscale model consists of a number of slit cracks in addition to the aggregate phase and the mortar phase. The crack density parameter is used to define the state of damage at the mesoscopic scale. The strain energy release rate at the mesoscale serves as the criterion for damage. Different orientations of the microcracks are considered and the cumulative effect on the macroscopic constitutive relation is established. The model is used to simulate the behavior of plain concrete under monotonic load and the results are in good agreement with experimental data from the literature. The model involves a quantity, known as the fracture resistance of mortar, similar to the *R*-curve commonly used in fracture mechanics, which is used to formulate the condition of damage evolution. The curve should be determined by conducting experiments at the pertinent material scale, i.e., the mesoscale in the context of the present problem, to describe the dependence of the fracture resistance on the state of damage. However, owing to the deficiency in experimental data, a precise form of the resistance curve is unavailable and it is assumed to be a linear function of damage. This introduces an additional parameter c_1 in the model. With the exception of c_1, all other parameters appearing in the model can be obtained from mechanical tests performed on concrete. The damage mechanism occurring at the mesoscale is explicitly modeled and its manifestation at the macroscale is studied in detail. An advantage of the model is its ability to enunciate the role of different microstructural properties such as the aggregate content, elastic properties of the constituent phases, microcrack density and the mesoscopic resistance to damage on the fracture and fatigue behavior of plain concrete, thus providing a scope to design the material optimally.

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