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# Rock Mass as a Nonlinear Dynamic System. Mathematical Modeling of Stress-Strain State Evolution in the Rock Mass around a Mine Opening

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**Abstract**—The paper briefly reviews the fundamental (general) evolution properties of nonlinear dynamic systems. The stress-strain state evolution in a rock mass with mine openings has been numerically modeled, including the catastrophic stage of roof failure. The results of modeling the catastrophic failure of rock mass elements are analyzed in the framework of the theory of nonlinear dynamic systems. Solutions of solid mechanics equations are shown to exhibit all characteristic features of nonlinear dynamic system evolution, such as dynamic chaos, self-organized criticality, and catastrophic superfast stress-strain state evolution at the final stage of failure. The calculated seismic events comply with the Gutenberg–Richter law. The cut-off effect has been obtained in numerical computation (downward bending of the recurrence curve in the region of large-scale failure events). Prior to catastrophic failure, change of the probability density functions of stress fluctuations, related to the average trend, occurs, the slope of the recurrence curve of calculated seismic events becomes more gentle, seismic quiescence regions form in the central zones of the roof, and more active deformation begins at the periphery of the opening. These factors point to the increasing probability of a catastrophic event and can be considered as catastrophic failure precursors.

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## 1. ROCK MASS WITH MINE OPENINGS IS A TYPICAL NONLINEAR DYNAMIC SYSTEM. IS IT POSSIBLE TO PREDICT CATASTROPHIC FAILURE?

### *1.1. Basic Properties and Simulation Models of Nonlinear Dynamic Systems*

At present, the problems of risk prediction and assessment are relevant for most human activities. Among them is the problem of predicting dangerous rock pressure events, including rock bursts. In a broader context, this is the problem of predicting catastrophic failure of elements in the Earth's crust (geomedia), including earthquakes, as well as fracture of any solids and structural elements. From the theoretical view point, these problems can be solved with a deep understanding of the general laws of evolution of nonlinear dynamic systems, among which are geomedia and solids. The main task of studying the stress-strain state evolution in the loaded solid as a dynamic system is the prediction of extreme events or critical states. Currently, both the location and time of possible major earthquakes and rock bursts are empirically predicted, for example, by time-series analysis [1, 2] or from monitoring data and precursors.

A basic model of the geomedium evolution is usually the sandpile model that would be touched on below. Dangerous dynamic events of rock pressure, including rock bursts, are predicted using the data derived during mining and geophysical monitoring. Prognostic theories based on the fundamental laws of evolution of nonlinear dynamic systems are intensively developed [3–6]. The idea of scale invariance and self-similarity of fracture on different scales makes it possible to use extensive data on earthquakes as well as laboratory findings on fracture of small samples for the analysis of catastrophic failure of a rock mass with openings. There appear numerous works on the development of the fundamental theory of the seismic process regardless of the scale of events. Most works use the well-studied basic equations of nonlinear dynamics for this purpose. However, they lack details on the generation of the catastrophic failure source as well as on the seismic process. No commonly accepted physical model of fracture is available. This gave rise to simulation models that demonstrate some of the properties of real dynamic systems, for example, blow-up modes, power law distributions, and other important evolution features of dynamic systems [2]. The most popular are models of thermal structures based on the nonlinear heat transfer equation [7–9] as well as various modifications of the sandpile model proposed by Bak et al. [10].

The use of the model of thermal structures as a basic model of the theory of nonlinear dynamic systems stems from the following fundamental properties of solutions of the nonlinear heat equation: (i) the possibility of self-organization, i.e. the formation of thermal structure; (ii) the presence of blow-up modes, i.e. catastrophes. Blow-up modes were first investigated when studying the properties of nonlinear heat equation solutions [9].

From our point of view, the use of the nonlinear heat equation for simulating and explaining autocatalytic catastrophic failure is poorly argued. The following form of the equation provides the analysis of the general properties of the solutions:

$$T_t = (k(T)T_x)_x + Q(T), \quad (1)$$
$$-\infty < x < \infty, \; T(x,0) = T_0(x),$$

where $k(T) = k_0 T^{\sigma}$ is the nonlinear heat source and $Q(T) = q_0 T^{\beta}$ is the nonlinear function of heat conductivity, whereby $\beta = \sigma + 1$ and $\beta > 1$.

The system of hyperbolic dynamic equations of solid mechanics cannot be reduced to the single simulation equation of a parabolic type, and consequently deformation processes, including failure, are not reduced to such an equation.

As Akhromeeva et al. state [7], the model of thermal structures lacks two important features inherent to many nonlinear self-organized systems: (i) no new extrema can arise in it, and therefore no new structures can form; (ii) stable are only the simplest structures, and complex ordered structures do not form (at the predetermined initial conditions only).

To simulate complex stable structures and their formation process, it is necessary to pass to systems of equations (for example, to the system of solid mechanics equations).

The equation of the type

$$\frac{\mathrm{d}U}{\mathrm{d}t} = Q(U) \text{ with the constraint } \int_0^{\infty} \frac{\mathrm{d}U}{Q(U)} = c < \infty \quad (2)$$

also leads to blow-up modes.

The inequality in (2) means the finiteness of the blow-up time and is a necessary and sufficient condition for the existence of the blow-up mode [7].

In Eq. (2), $Q(U)$ stands for the source, for example, of damages in a loaded deformable medium. It is this equation that is used in our calculations of the stress-strain state evolution and failure in geomedia to set the damage accumulation rate in the medium [11, 12].

Various modifications of the sandpile model provide a more improved prediction of major earthquakes. Their quantitative algorithms developed around different earthquake scenarios, for example, activation and anti-activation ones, are adapted to the prediction of major earthquakes as well as rock bursts. An example is the widely known M8 algorithm [1, 13]. Sandpile model variants exhibit self-organized criticality; their evolution results in power law distributions of the number of avalanches $N$ over their sizes (energies) $E$ [2, 13–15]:

$$N(E) \sim E^{-b}, \quad (3)$$

where the exponent $b$ is close to unity at the given parameters, which meets the fundamental Gutenberg–Richter recurrence law. Power law distributions (3) are universal; they are characteristic of the evolution of the multitude of dynamic systems with complex behavior and are valid practically without exceptions in describing various catastrophes and natural disasters [16–18]. As will be shown further, any multiscale fracture of rock mass elements with openings corresponds to power laws. Seismic events calculated by solving the system of nonlinear equations of the deformable solid mechanics also fall into a typical recurrence diagram of seismic events (the Gutenberg–Richter law) that reflects the power law distribution of seismic events of different scales. This fact, firstly, means that the stress-strain state evolution in the rock mass with openings corresponds to typical evolution scenarios of nonlinear dynamic systems and, secondly, convincingly proves that solutions of the solid mechanics equations reflect dynamic system scenarios fitting power law distributions (of fracture scales in this case). It is common knowledge that the physical content of power laws is in the mutual dependence of events occurring in the system, which is impossible without information exchange between the dynamic system elements.

It is the information exchange between different parts of the system that provides its self-consistent coherent response to actions.

Self-organized criticality of nonlinear dynamic systems is considered to be a universal mechanism of catastrophes [7, 10, 16]. Catastrophes are an inevitable and intrinsic feature of an evolving dynamic system. From a statistical viewpoint, this feature is indicative of scale-invariant processes occurring in the system [17].

Thus, the leading scale cannot be determined in the evolving dynamic system characterized by self-organized criticality. In such a system, processes, including fracture, develop throughout the scale hierarchy in a self-similar manner. Though the system as a whole is stable, its elements never reach equilibrium, evolving from one to another metastable state.

Self-similarity and scale invariance are fundamental evolution features of dynamic systems exhibiting self-organized criticality. These features are represented in some way by the basic simulation sandpile model as well as its various modifications. However, there is an opinion that self-organized critical systems are unpredictable [7, 16, 19]. Being of fundamental importance, this problem was first raised in the Nature journal [20] concerning the Bak–Tong–Wiesenfeld sandpile model [10] and the related theory of self-organized criticality. In fact, the debate was held (and periodically resumes) about the possibility of major event prediction using the Bak–Tong–Wiesenfeld model and its modifications. This debate was extended to the question of whether it is possible to predict catastrophic events, first of all earthquakes, in real natural and other systems characterized by power law distributions and therefore exhibiting scale invariance and self-organized criticality.

In some authors' opinion, the currently available sandpile model variants are much more predictable as their modified procedures are based on precursors adapted to the developed model dynamics. In these models, both activation and anti-activation scenarios are used [2, 19, 21].

As for the prediction of earthquakes and dangerous dynamic events of rock pressure in real media, including rock bursts, its horizon is unknown. For this reason, the problems of long-term and medium-term prediction of catastrophic failure are very debatable.

Another acute problem is a short-term prediction, including from multiple precursors of different nature. We will dwell on possible mechanical precursors formed during deformation and small-scale fracture that generate seismic noise prior to a large-scale catastrophe; the noise will be numerically reproduced.

### *1.2. The Cut-Off Problem*

When analyzing empirical data and studying various simulation models of self-organized systems, researchers meet with the so-called cut-off problem or energy cutoff, which is expressed in the downward bending of the recurrence curve in the region of major seismic events [2, 7, 22]. This effect means a decrease in the probability of large-scale catastrophic events. Behavior of the tail of the seismic event distribution in the region of the rare largest earthquakes is of paramount importance for their prediction [22]. However, observations are unhelpful in solving this problem because the empirical data are extremely limited. This lays special emphasis on numerical experiments performed with different models.

In the theoretical analysis of various simulation models, the cut-off effect may be related to the fact that the system reaches the scale limit in the specified computational domain. Power law distributions for many real nonlinear processes also demonstrate the cut-off effect. Figure 1 shows the statistics of catastrophes and disasters [23] with the same effect. Similar dependences are characteristic, for example, of the statistics in computer virus infection, epidemic spread, etc. [7].

From the aforesaid, we can conclude that the cut-off effect is an intrinsic evolution feature of a variety of nonlinear dynamic systems exhibiting self-organized criticality. This behavior is traced to special features and natural restrictions of processes that develop in real systems and are described by parameters of the corresponding models. For geomedia, the cut-off effect may be related to their rheology and fracture as well as to the restrictions imposed by the strength characteristics and structural organization on the maximum scale of a possible catastrophe. Evidently, a decisive role belongs to the long-range action of stress concentrators. This phenomenon also depends on the linear size $L$ of the system: cut-off $\sim L^{-2.2}$, but this dependence is of a model nature and probably cannot be extended to systems with very large $L$ [24, 25]. In the present numerical calculations of fracture based on solid mechanics equations, this effect is also seen in the recurrence diagram of the calculated seismic events.

### *1.3. Flicker Noise*

The fundamental property of all nonlinear dynamic systems (real and artificial) is the generation of flicker noise $1/f$ in them. The power spectrum of flicker noise on low frequencies obeys the power law

$$S(f) \sim f^{-\beta}, \tag{4}$$

where $\beta \sim 1$.

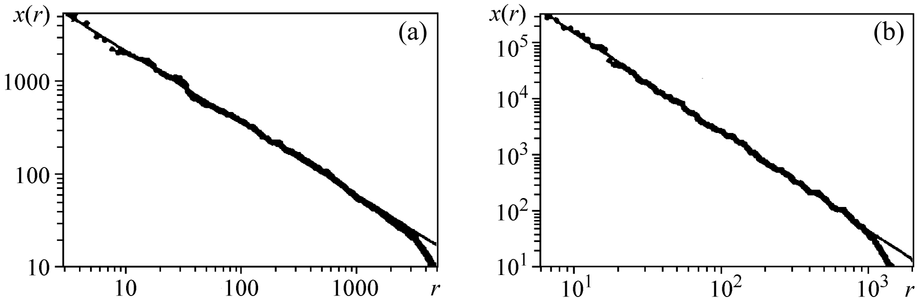


**Fig. 1.** Exponential statistics of catastrophes and disasters according to the database [23]; man-induced disasters ranked by the number of deaths (2047 major events) (a); natural disasters ranked by the number of the wounded (1084 events) (b). $x(r)$ is the dimension, $r$ is the rank in the $x$ descending order.

The physical nature of $1/f$ noise is thought to be unclear despite extensive studies of this phenomenon and a huge number of publications on this subject. For example, in electrical circuits, $1/f$ noise is associated with the presence of impurities. Intensive broadband noise is observed in regions of the critical nonequilibrium phase transition. The power law behavior of the power spectrum means the absence of characteristic frequencies in the system, and hence of characteristic times. It also shows that most of the energy results in slow processes corresponding to fluctuations of very high intensity, i.e. large-scale catastrophes. Additionally, the requirements for recurrence and periodicity of events are not met in systems for which the power spectrum obeys the power law. The system lacks a characteristic time that could be responsible for the most important (catastrophic) events. For this reason, long-term prediction is totally impossible in such systems [17]. Whatever extensive the accumulated information on the system evolution might be, nothing can be said about future important processes, with the preparation time comparable with the time required to study the system. That is obviously why the well-known project of the late 20th century on the San Andreas Fault was unsuccessful. Major earthquakes could not be predicted.

Thus, methods of medium- and short-term prediction as well as the study of numerous phenomena accompanying the formation of the source of catastrophic failure (precursors) acquire special importance.

In our model, we will analyze noise that accompanies failure, i.e. statistics of stress fluctuations around the average trend during the medium fracture or stress tremor (the analogue of seismicity and acoustic emission in geodynamic monitoring of the rock mass). On low frequencies, it exhibits properties of flicker noise ($\beta$ is close to unity).

Presently, the prediction of evolution of self-organized critical systems is associated with a new trend in nonlinear dynamics—the theory of channels and jokers [16]. We will not dwell on this poorly investigated subdiscipline of the theoretical nonlinear dynamics, especially with regard to the analysis of real dynamic systems. We mention only that the general idea of the prediction stems from the hypothesis of existence of regions of much smaller dimension in the phase space (channels) and a small number of variables (order parameters) responsible for the course of evolution on a certain time interval. These ideas are believed to be useful in the construction of simple models and local forecast. Areas where a trend is difficult to identify and the behavior looks random are called joker areas, and the corresponding rules for the system functioning are jokers. By managing jokers, you can change the game rules and investigate possible evolution scenarios of the system, which makes numerical simulation methods the most important tool for studying evolution processes.

In our model calculations, we will follow methods of seismic and acoustic monitoring of real systems. From the general set of dynamic variables describing the geomedium deformation, we set apart the stress-time variation, assuming that, owing to information exchange in the studied mass, stress fluctuations quite fully reflect the evolution pattern of the stress-strain state in the medium, including the formation of different-scale failure foci.

### *1.4. Integrity of Dynamic Systems*

Nonlinear dynamics says that self-organized critical states and catastrophes can arise only if the system features integrity or achieves it during the evolution. This means a self-consistent cooperative response to loading of various subregions of the dynamic system. From the statistical point of view, the system integrity is ensured by power-law spatial and temporal correlations between the system parts, which provide long-range interaction. In conventional systems with characteristic time and length scales, the correlations rapidly decrease (short-range interaction); the information about previous events and neighboring areas is rapidly forgotten.

In classical models of a sand pile, blocks and others [10, 16, 17], it is believed that the system achieves its integrity through the self-organization to a critical state. The mechanism providing the interaction in the whole system is the presence of two opposite processes, namely, natural evolution of the system (for example, an increase in local slopes of the sandpile due to added grains) and a selection process (avalanches). It is necessary that the rate of selection be much greater than the rate of natural evolution. There is no actual direct information exchange in such models. In our opinion, this is a significant disadvantage of the most simulation basic models of dynamic catastrophic events.

Solid mechanics equations that underlie the developed evolutionary approach to the description of deformation and catastrophic failure of a geomedium are without this drawback. In a real geomedium, deformation is always a slow process as compared to the information exchange rate in the medium.

The information is exchanged via stress waves with sound velocity. Every local disturbance generates stress waves in a loaded solid. The larger the scale and amplitude of a local disturbance, the larger the radius of its long-term action that can affect the stress-strain state evolution in the surrounding medium. The information exchange provides migration of the deformation activity when in one regions the strain rate slows down considerably due to relaxation while in other regions it increases. That is how seismic quiescence zones form when the deformation process moves to the periphery of this zone where the catastrophic failure source starts to actively develop.

In the presented numerical model, the time compression parameter $K$ is introduced, which makes it easy to convert from the conventional calculation time to the real deformation time:

$$t_{\mathrm{real}} = K t_{\mathrm{calc}}. \tag{5}$$

For the stability of numerical calculations, time steps are very small; the total time of the calculated deformation process does not exceed a few seconds or minutes on modern computers at the total number of time steps $\sim n\times 10^6$ (of course, we do not mean the computational time that can make up days in 3D modeling). Parameter $K$ is chosen for reasons of the system integrity. At each loading step (for example, face advance), the generated stress wave must make 1–3 passes through the calculation region, thus ensuring the information exchange in the medium and appropriate adjustment of the stress-strain state in the entire computational domain, as with real life.

In the evolutionary model of a deformable medium as a dynamic system, integrity is also provided by negative and positive feedbacks. Negative feedback stabilizes the deformation process, which is due to stress relaxation in local regions during inelastic deformation and/or fracture. Positive feedback acts as the process destabilizer, it accelerates the fracture process in the autocatalytic ultrafast mode (selection). This mechanism works in the model as follows. The deformation nonlinearity leads to localization of inelastic deformation, localization causes degradation of mechanical properties in these regions, which enhances localization processes and subsequent degradation. All this refers to local processes in small elements of the geomedium. A self-consistent cooperative response as a self-organization process in the self-organized critical system leads to major catastrophes.

### *1.5. Geophysical Monitoring of the Rock Mass with Openings*

The necessary safety of mining in modern conditions of high productivity and the acceptable risk level were ensured by the development and arrangement of technological and organizational measures based mainly on years of experience in mining engineering. Fundamental scientific studies on such new geomechanical problems as evolutionary scenarios of the failure foci formation have hardly been performed.

The geodynamic situation in a rock mass with openings is monitored using geodynamic methods, the most important of which are monitoring methods for microseismic events and the geoacoustic method [26].

As there are no standardized account for the time of geomechanical processes and estimates of the nonstationary damage growth in a rock mass (particularly, in the roof) near the opening, methods of real-time forecast of dangerous rock pressure events are gaining an importance. For example, seismic estimation methods based on monitoring systems are actively used in mines of Russia, Germany, Japan, Australia, the USA, South Africa, and China [18, 27, 28]. Many complex problems of control of geomechanical processes, which emerge under the modern conditions of high loads to mine faces, are also solved using methods of physical modeling of the geomechanical situation [26, 29], including rock bursts [30], which is extremely important but lacks a common methodology, in particular, quantitative evaluation of the proposed criteria. Mining engineering focuses on how to allow for the corrections to loading nonstationarity of the rock mass in order to take account of the dynamics of geomechanical processes, to estimate risks of negative dynamic phenomena, and to develop measures and technological solutions for their prevention. The main efforts are concentrated on experimental and theoretical methods, methods of physical modeling, and the development of methods of modern geodynamic monitoring.

Though most seismic monitoring systems enable continuous seismological observations of macroscale and microseismic processes at the mine as well as monitoring of rock bursts, roof caving, and the related dynamic fractures [31–34], there are no reliable methods for estimating the closeness of the rock mass to the critical state by the obtained data. Thus, the GITS seismic monitoring system developed by VNIMI and implemented at some Kuzbass mines covers the mine field ($5 \times 5$ km) with the event energy range from 100 J and the rate of occurrence up to 100 events per day. The output information of the GITS seismic monitoring system is seismic activity maps indicating rockburst hazard zones. However, criteria for rockburst hazard must be determined specifically for each mine field, but reliable methods for their quantitative evaluation are unavailable.

Rasskazov [26] reports modern methods of geoacoustic monitoring of a rock mass liable to bursts and discloses up to 20 predictive signs of the critical state of the mass in terms of the parameters and character of acoustic emission. The author states that the effectiveness of geoacoustic monitoring largely depends on the objective interpretation of measurement results and the validity of the used criteria of the rock mass state.

Conditionally it can be said that two tendencies are outlined to solve the actually common problems of prediction of catastrophic failure in the loaded geomedium during large earthquakes and in the rock mass with openings. Causes and mechanisms of fracture during earthquakes are often studied with theoretical approaches and nonlinear dynamics methods [36], while the requirements for the operative solution of mining safety problems force mining engineers and researchers to address the accumulated colossal empirical experience.

The study of the stress-strain state evolution of a rock mass with openings and of the formation mechanisms of failure foci using the modern nonlinear dynamics theory forms the fundamental theoretical basis for assessment of the geomechanical situation. It will provide an interpretation of a wealth of accumulated empirical data and result in effective analysis methods of geomonitoring data and reliable prognostic criteria.

Basic equations of the nonlinear dynamics theory describe, at the most general qualitative level, possible evolution scenarios. In the majority of cases, they are not mathematical models of real physical processes but their simulation models. That is why classical nonlinear dynamics, though it has opened a new era in explaining the laws of the universe and become the first metascience after philosophy, runs into almost insurmountable difficulties when solving the problems of evolution and prediction of real dynamic systems. The exception is cases when adequate evolution equations, being quite strict mathematical models, can be written for real processes. Unfortunately, such cases are few in number; some examples can be given from economy and the physics of phase transitions.

The analysis procedure of the stress-strain state evolution of a rock mass with openings used in the present paper stems from the mathematical theory of evolution of loaded solids and media [6]. The essence of the theory is the system of solid mechanics equations. These equations model deformation processes, including fracture. With the negative and positive feedbacks as well as governing equations for inelastic strain and/or damage accumulation rates in the loaded medium, numerical solutions of the solid mechanics equations demonstrate all characteristic evolution features of nonlinear dynamic systems exhibiting integrity and self-organized criticality, as it was discussed elsewhere [6, 11, 12].

Thus, the developed approach is based on fundamental ideas of nonlinear dynamics and rigorous mathematical models of deformation and fracture of solids and media. The present paper shows that numerical solutions of the solid mechanics equations reveal the most important stages of evolution of loaded solids as typical nonlinear dynamic systems. Consequently, the proposed approach and methods for the analysis of the stress-strain state evolution can be used as the basis for the fundamental prognostic theory of catastrophic failure of solids, including the rock mass with openings.

## 2. MATHEMATICAL EVOLUTIONARY MODEL OF A LOADED MEDIUM

If the problem of mechanical response of a solid to loading is formulated as an evolutionary one, solutions of solid mechanics equations show all characteristic stages of evolution of nonlinear dynamic systems, including the self-organized critical state and fracture on different scales in the blow-up mode [6, 11, 12]. In fact, they describe the stress-strain state evolution of the medium, whose state at each loading stage can be used to understand how close the medium is to the macroscopic catastrophic failure.

The mathematical model describing deformation and fracture as evolutionary processes is presented elsewhere [6, 11, 12]. In the general case, it includes equations of the deformable solid mechanics that express the laws of conservation of mass, momentum, energy, geometric relations (6), governing equations, and responses. Negative feedback stabilizes the deformation process due to stress relaxation and positive feedback transfers the fracture process to the ultrafast autocatalytic stage in local degradation regions of mechanical parameters of the medium (Eqs. (7)–(10)). In this paper the medium is considered in the barotropic approximation, which yields a closed system of equations free of the law of conservation of energy. In this case, nothing is known about its thermodynamic state. In fact, we study an isothermal process, which appears to be justified by the relative slowness of the process and the closeness of the medium to thermodynamic equilibrium (meaning the thermal state):

$$\rho V = \rho_0 V_0,\ \rho \dot{v}_i = \sigma_{ij,j} + \rho F_i,$$
$$2\dot{\varepsilon}_{ij}^{\mathrm{T}} = v_{i,j} + v_{j,i},\ 2\dot{\omega}_{ij} = v_{i,j} - v_{j,i}, \quad (6)$$

$$\dot{P} = -K(\dot{\theta}^{\mathrm{T}} - \dot{\theta}^{\mathrm{p}}),\ \dot{\theta}^{\mathrm{T}} = \dot{\varepsilon}_{ii}^{\mathrm{T}},$$
$$\dot{S}_{ij} + S_{ik}\dot{\omega}_{kj} - S_{kj}\dot{\omega}_{ik} = 2\mu\left(\dot{\varepsilon}_{ij}^{\mathrm{T}} - \frac{1}{3}\dot{\theta}^{\mathrm{T}}\delta_{ij} - \dot{\varepsilon}_{ij}^{\mathrm{p}}\right), \quad (7)$$

$$f(\sigma_{ij}) = -\alpha P + \sqrt{J_2} - Y,$$
$$g(\sigma_{ij}) = J_2 - \Lambda P(2Y + \alpha P) + \mathrm{const},\ Y = Y_0(1 - D). \quad (8)$$

Hereinafter $\rho_0$ and $\rho$ are the initial and current densities of the material, $V_0$ and $V$ are the initial and current volumes of the material, $v_i$ is the velocity vector components, $P$ is the pressure, $\sigma_{ij}$ is the stress tensor components, $S_{ij}$ is the stress tensor deviator components, $F_i$ is the mass force vector components, $\dot{\omega}_{ij}$ is the velocity vector rotor, $\dot{\varepsilon}_{ij}^{\mathrm{T}}$ is the strain rate tensor components, $\delta_{ij}$ is the Kronecker symbol, $\dot{\lambda}$ is the plastic multiplier, $J_1$ is the first invariant of the stress tensor, $J_2 = 1/2\, S_{ij}S_{ij}$ is the second invariant of the stress tensor deviator, $\dot{\varepsilon}_{ij}^{\mathrm{p}}$ is the inelastic strain rate tensor components, $\dot{\theta}^{\mathrm{T}}$ is the volumetric strain rate, $\dot{\theta}^{\mathrm{p}}$ is the volumetric inelastic strain rate, $K$ is the bulk modulus, $\mu$ is the shear modulus, $\alpha$ is the internal friction coefficient, $\Lambda$ is the dilatancy coefficient, $g(\sigma_{ij})$ is the plastic potential, $D$ is the damage measure, $\mathrm{H}(x)$ is the Heaviside function, $\sigma_{\mathrm{int}}$ is the stress tensor intensity, $\sigma_0^{\mathrm{c}}$ and $\sigma_0^{\mathrm{t}}$ are the initial stresses at the elastic stage after which the material accumulates damages in compression and tension regions, respectively, $\mu_\sigma$ is the Lode–Nadai coefficient, $\sigma_{0*}$ is the parameter of the damage accumulation model, $S_1, S_2, S_3$ are the main values of the stress tensor deviator, $\sigma_{\mathrm{C}}$ is the generalized Coulomb stress, and $Y$ is the cohesion.

Rotation of the medium element as a whole is accounted for through the corotational Jaumann time derivative (the third equation in (7)); plastic potential meets the nonassociated flow rule, which makes the dilatancy process independent of internal friction. Cohesion (shear strength of the medium under zero pressure) decreases from the initial value $Y_0$ with damage accumulation $D$. The plastic multiplier $\dot{\lambda}$ in (9) is determined providing that stresses satisfy yield condition (8) (the first equation). From the basic relation of the plasticity theory $\dot{\varepsilon}_{ij}^{\mathrm{p}} = \dot{\lambda}\,\partial g(\sigma_{ij})/\partial\sigma_{ij}$ follows the relation for the inelastic strain rate tensor components [22]:

$$\dot{\varepsilon}_{ij}^{\mathrm{p}} = \left(S_{ij} + \frac{2}{3}\Lambda\left(Y - \frac{\alpha}{3}J_1\right)\delta_{ij}\right)\dot{\lambda},\ \dot{\theta}^{\mathrm{p}} = \dot{\varepsilon}_{ii}^{\mathrm{p}}. \quad (9)$$

A relaxation form of governing Eqs. (7) provides stress relaxation during inelastic strain and damage accumulation in the medium and serves as a negative response.

The damage accumulation process and the corresponding degradation of mechanical parameters are described by the damage measure $D = D(t, \mu_\sigma, \sigma_{\mathrm{y}})$ $(0 \le D \le 1)$ that depends on the stress state invariant $\sigma_{\mathrm{y}}$ and the stress state type defined via the Lode–Nadai coefficient $\mu_\sigma$:

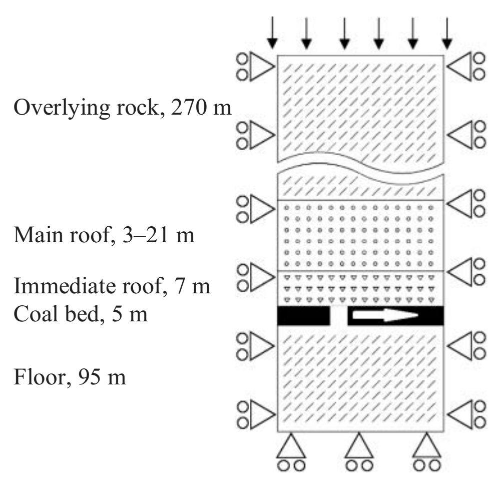


**Fig. 2.** Model stratigraphic column of a coal-bearing rock mass and boundary conditions for gravity and constraint deformation.

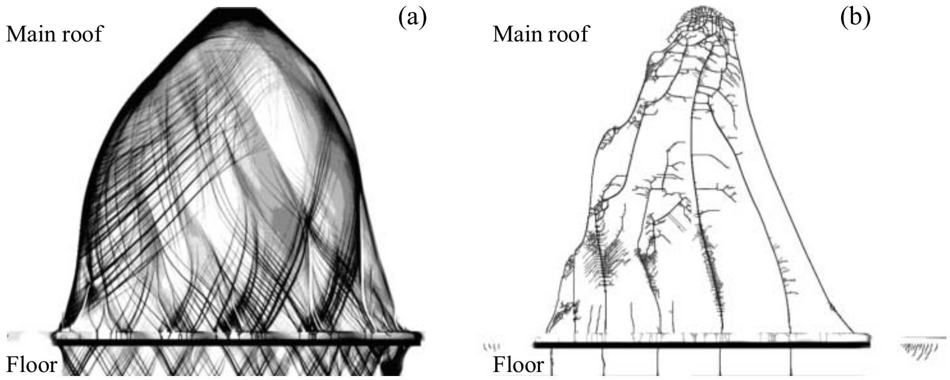


**Fig. 3.** Distribution of localized inelastic strains in the rock mass in the case of elastic-viscoplastic response (a) and fractal failure in the case of elastic-brittle response of the geomedium (b).

$$D=\int_{t_0}^{t}[\mathrm{H}(\mu_\sigma)(\sigma_{\mathrm{y}}-\sigma_0^{\mathrm{c}})^2+(1-\mathrm{H}(\mu_\sigma))$$
$$\times(\sigma_{\mathrm{y}}-\sigma_0^{\mathrm{t}})^2][\sigma_*^2(\mathrm{H}(\mu_\sigma)t_*+(1-\mathrm{H}(\mu_\sigma))t_*)]^{-1}\mathrm{d}t, \quad (10)$$
$$\sigma_*=\sigma_{0*}(1+\mu_\sigma)^2,\ \mu_\sigma=2\frac{S_2-S_3}{S_1-S_3}-1,$$
$$\sigma_{\mathrm{y}}=\sigma_{\mathrm{int}}-\alpha P,$$

$\sigma_0^{\mathrm{c}}$ and $\sigma_0^{\mathrm{t}}$ vary during deformation by the law similar to that in the last formula in (8) (degrade with damage accumulation). In so doing, $\sigma_0^{\mathrm{t}}$ is higher than $\sigma_0^{\mathrm{c}}$, thus damages in tension-shear regions ($\mu_\sigma < 0$) start to accumulate at lower stresses than at $\mu_\sigma > 0$ in compression-shear regions. Damage accumulation rates for local regions where $\mu_\sigma < 0$ are also much higher (by 1 to 2 orders of magnitude) than in compression-shear regions ($\mu_\sigma > 0$). This process is additionally controlled by the parameter $\sigma_*$ in (10), causing a much lower strength in shear-tension. The degradation function in the system provides a positive feedback, which leads to instability of the deformation process in damage localization regions and its strength degradation as well as accelerates the fracture process in the superfast catastrophic mode.

The system of Eqs. (6)–(10) is numerically solved by Wilkins' method [37]. All calculations of the stress-strain state evolution in a rock mass with openings are preceded by the solution of the problem of setting the distribution of all geomedium parameters responsible for a given depth. The presented system of equations is used to calculate, except for damage accumulation, all parameters for the selected region of the rock mass under constrained deformation (forbidden horizontal displacements at the lateral surfaces of the computational domain and vertical displacements at the lower surface, respectively (Fig. 2)) under gravity.

## 3. MODELING RESULTS OF THE STRESS-STRAIN STATE EVOLUTION OF THE ROCK MASS WITH OPENINGS

The main features of the stress-strain state evolution of the geomedium are demonstrated on the simplest structural organization of a rock mass with openings. Such a geological medium is schematized in Fig. 2. The mass includes a productive coal seam, immediate and main roof and floor, and overlying sedimentary rocks.

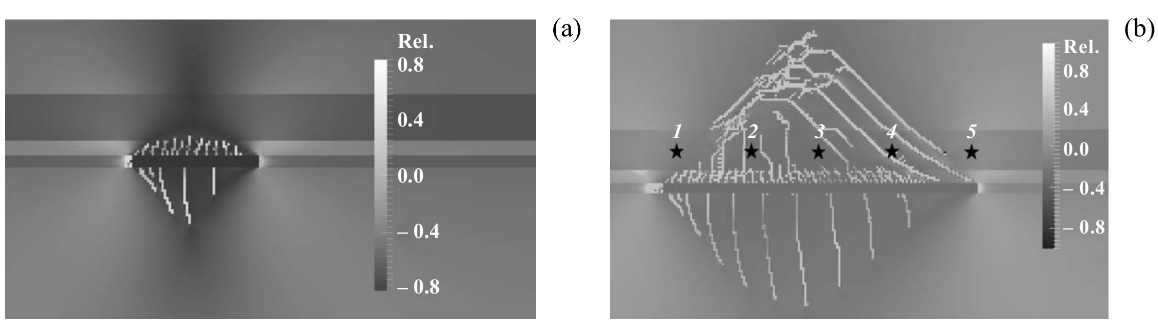


**Fig. 4.** Distribution of Coulomb stresses in the rock mass during the face advance. The main roof thickness is 19 m, and the face advance distance is 55 (a) and 150 m (b).

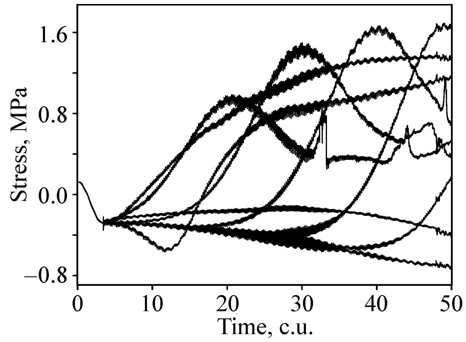


**Fig. 5.** Multisensor record of the stress tremor in the rock mass as the face advances.

From the mining chamber, the face moves to the right, increasing the worked out area. Deformation and fracture of the rock mass elements occurs under the action of gravity in accordance with the specified depth of mining. Tectonic stresses in these examples are given no account.

Figure 3 displays distributions of localized inelastic strains in the rock mass in the case of an elastic-viscoplastic response and of fractal fracture in the case of an elastic-brittle response of the geomedium. In the first case, different-scale blocks separated by localized deformation bands are formed in the geomedium. In these shear bands, the medium is damaged to different degrees ($0 < D < 1$) depending on the local situation. This example illustrates the possibility of modeling the formation of fractal structures in the numerical solution of the solid mechanics equations.

The Drucker–Prager model is convenient for the analysis of the stress-strain state evolution through the generalized Coulomb stresses $\sigma_C = \tau/(Y + \alpha P)$, where $\tau$ is

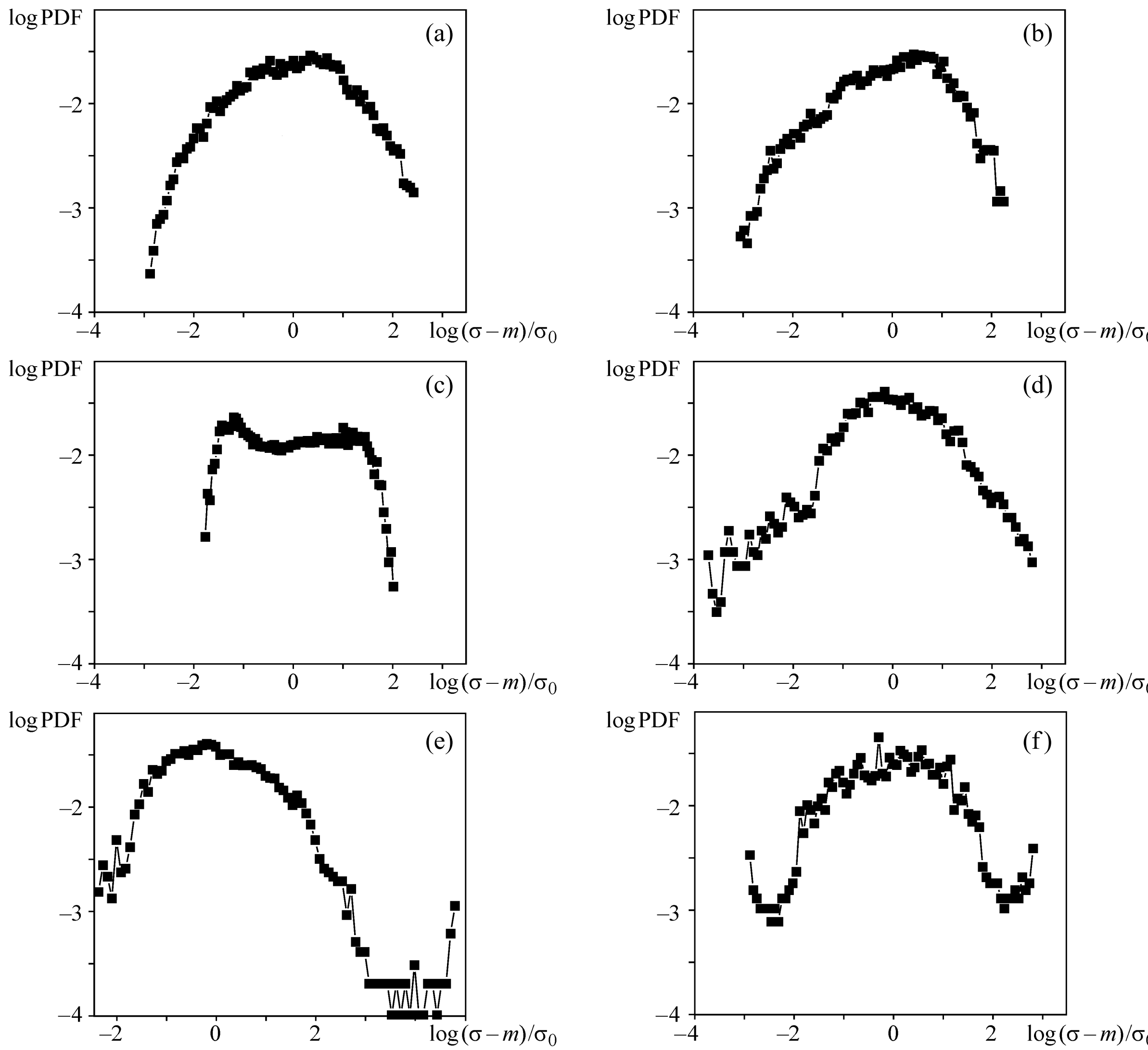


**Fig. 6.** Evolution of the PDF-dependence of the stress tremor as the face advances.

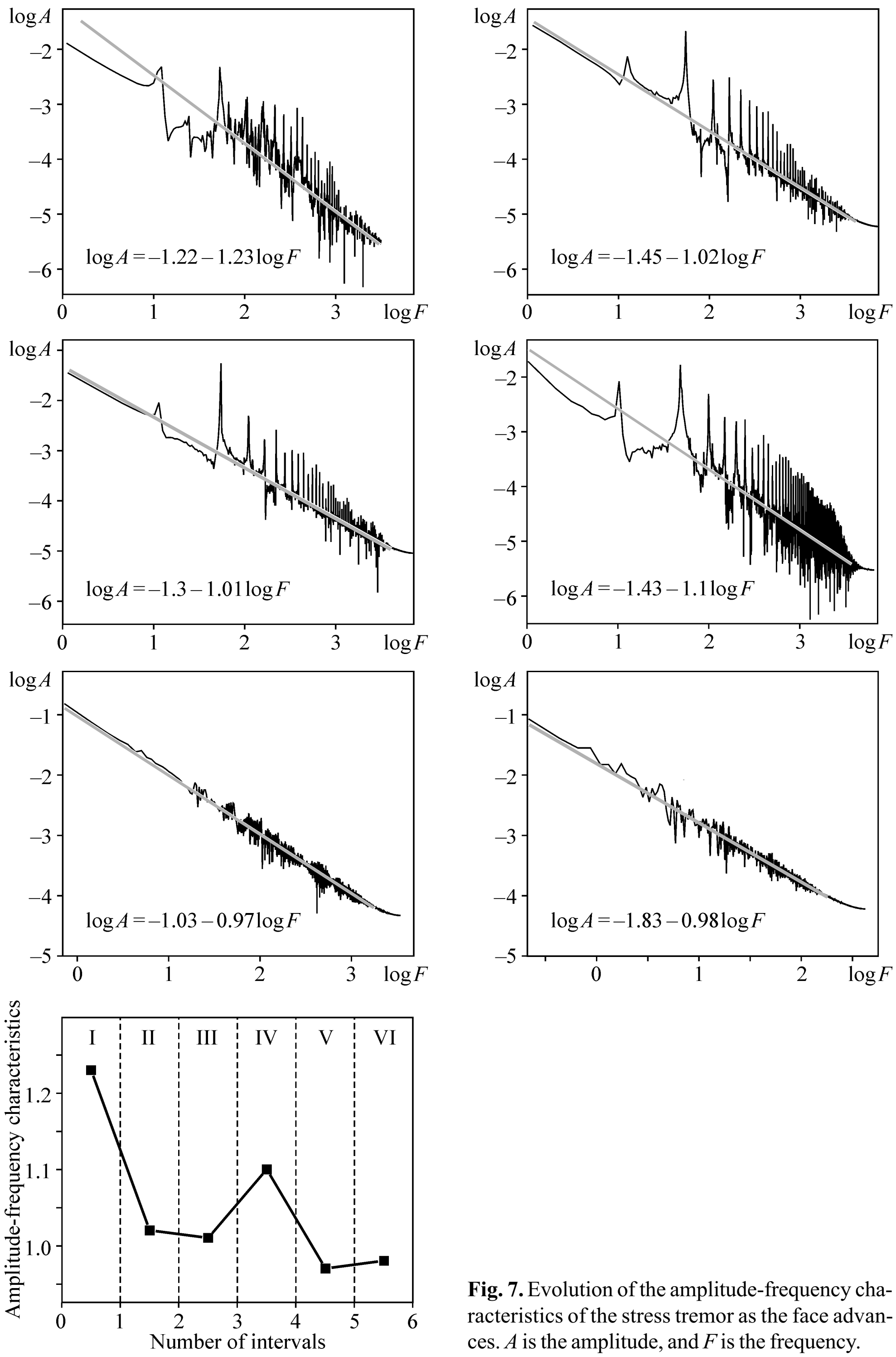


**Fig. 7.** Evolution of the amplitude-frequency characteristics of the stress tremor as the face advances. $A$ is the amplitude, and $F$ is the frequency.

the stress intensity, and $P$ is the hydrostatic pressure. As Coulomb stresses get closer to unity, the medium approaches the critical state.

Figure 4 gives an example of the Coulomb stress distribution in the roof and floor at the two time instants: at the face advance distance of approximately 55 m from the mining chamber (only the immediate roof and floor are damaged, Fig. 4a) and at the final stage before first caving ($L \approx 150$ m, Fig. 4b).

All small-scale fractures (growing cracks) generate stress waves, i.e. the analogue of microseismic events in the real rock mass. In the calculations, similarly to real conditions, the corresponding stress tremor is registered by sensors located in different parts of the modeled rock mass (by the sensor we mean continuous stress recording at the chosen geomedium points as a function of time; the sensor location is shown in Fig. 4b). Statistical processing of the calculated microseismic events allows a conclusion about the closeness of the medium to large-scale catastrophic failure. The stress tremor records from several sensors are shown in Fig. 5 (stress values are recorded from every time layer, thus the total number of fluctuating stress values can achieve up to million).

Figures 6 and 7 display the evolution of PDF dependencies (the stress tremor distribution density for events of different classes) and the corresponding amplitude-frequency characteristics, respectively, and correspond to the stress tremor evolution as the rock mass approaches the critical state. These figures are built using the results of statistical analysis of stress fluctuations recorded by the sensor closest to the failure zone at the roof caving. The entire time interval is divided into 6 conditional segments, with the construction of PDF dependences and amplitude-frequency characteristics of the stress tremor for each of them.

At the initial deformation stages, PDF dependencies are dome-shaped (Figs. 6a and 6b). This indicates both the power law distribution of the calculated seismic events and their weak dependence on each other (different-scale events are observed). However, as damaged areas gain in scale, the dependences drastically change their form: an inverted-U-shaped distribution (Fig. 6c) showing the presence of oscillation packages of the same frequency in the tremor turns to a pronounced heavy-tailed distribution with an explicit violation of spatiotemporal symmetry (Figs. 6d and 6e), which corresponds to the system in the critical state before first caving. After caving, the PDF dependence acquires a symmetrical form indicative of relaxation of a large amount of the stored energy and restoration of spatiotemporal symmetry (Fig. 6f).

Evolution of the amplitude-frequency characteristics is interesting in terms of the slope change. The calculations show that, as the rock mass approaches the critical state ending in first caving, the slope of the amplitude-frequency characteristics decreases. This means that in a sufficiently large range of failure scales, the events are flattened with regard to released energy. Intensification of the fracture process at all scale levels indicates an increase in the probability of large-scale failure (in this case, first caving).

A recurrence curve of the calculated seismic events is shown in Fig. 8. Seismic events in the rock mass during the face advance are calculated as follows. As the worked out area grows, rock mass elements locally lose their stability and pass to the inelastic state, as is clearly demonstrated by Fig. 4. There occurs dissipation of the accumulated elastic energy and emission of seismic waves. In the calculations, the energy released during inelastic deformation is registered to evaluate the event class by the well-known formula $k = \log E$, where $E$ is the event energy calculated by the ratio $E = 3G(d\varepsilon_{pl}^2)$, whereby $d\varepsilon_{pl}$ is the plastic strain rate increment.

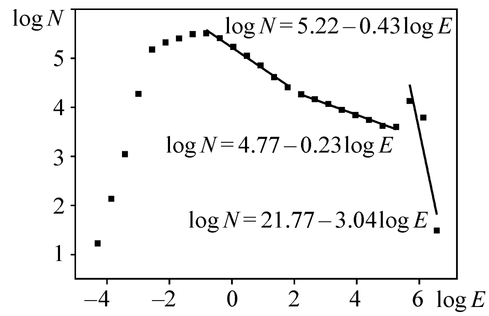


**Fig. 8.** Recurrence diagram of the calculated seismic events. $N$ is the number of events, and $E$ is the energy (J).

It is graphically seen that the distribution of events by class is multifractal. The final deformation stages at first caving are characterized by a small number of maximum-class events, as is also evidenced by Fig. 9. The cut-off effect is also pronounced (Fig. 8).

Most likely, such a strong downward bending of the recurrence curve in the region of large-scale catastrophes is caused by a highly limited size of the computational domain. This phenomenon requires additional research. It should be noted that such behavior of the recurrence curve (cut-off effect) is typical for almost all simulation models [2, 3, 25].

The spatial structure of the calculated seismic events related to the roof and floor failure during the face advance at the final stage ($L \approx 150$ m) is shown in Fig. 10.

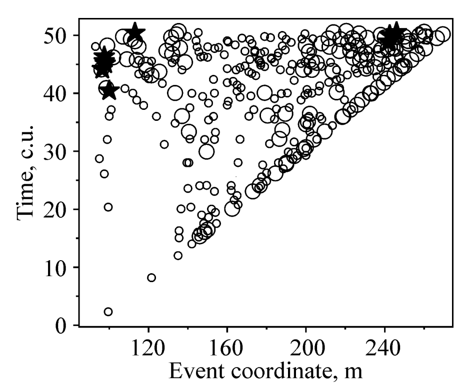


**Fig. 9.** Spatiotemporal distribution of the calculated seismic events in the rock mass as the face advances.

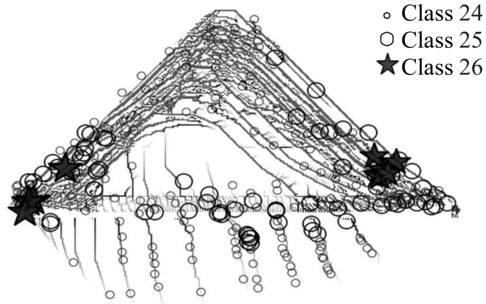


**Fig. 10.** Spatial distribution of the calculated seismic events in the rock mass for the whole time of mining.

The main events are moved to the periphery to a region between the roof and the unworked-out area (marked with asterisks in Fig. 10). A seismic quiescence region is pronounced in the roof center.

The wavelet analysis of the stress tremor recorded by one of the sensors reveals the following. When the original signal from the sensor in the immediate vicinity of the failure zone is divided into 9 sublevels using the Daubeshies-10 wavelet, which is the standard tool of the Matlab program, we observe a seismic quiescence region on all frequencies. This region appears prior to catastrophic failure of rock mass elements, which represents first roof caving. The corresponding decomposition levels demonstrate peaks of much higher amplitude as compared to the background activity (Fig. 11). First caving is followed by peaks of lower amplitude, which are associated with further failure (aftershocks).

These calculations convincingly demonstrate the effectiveness of the developed evolutionary approach to modeling of deformation and catastrophic failure. The performed statistical analysis of the stress fluctuation evolution elucidates how close the fracture process is to the catastrophic stage.

## 4. CONCLUSION

A real geomedium is a multiscale hierarchically organized nonlinear dynamic system. This is evidenced by long-term observations of the seismic process: the Gutenberg–Richter recurrence law, the Omori law for the aftershock process, and observation of high-frequency seismic noise.

High-frequency seismic noise differs from both purely deterministic and random (with an infinite number of modes) signals; the signal structure is fractal. This fact indicates that the signal carries information about a critical nonequilibrium process developing in the medium

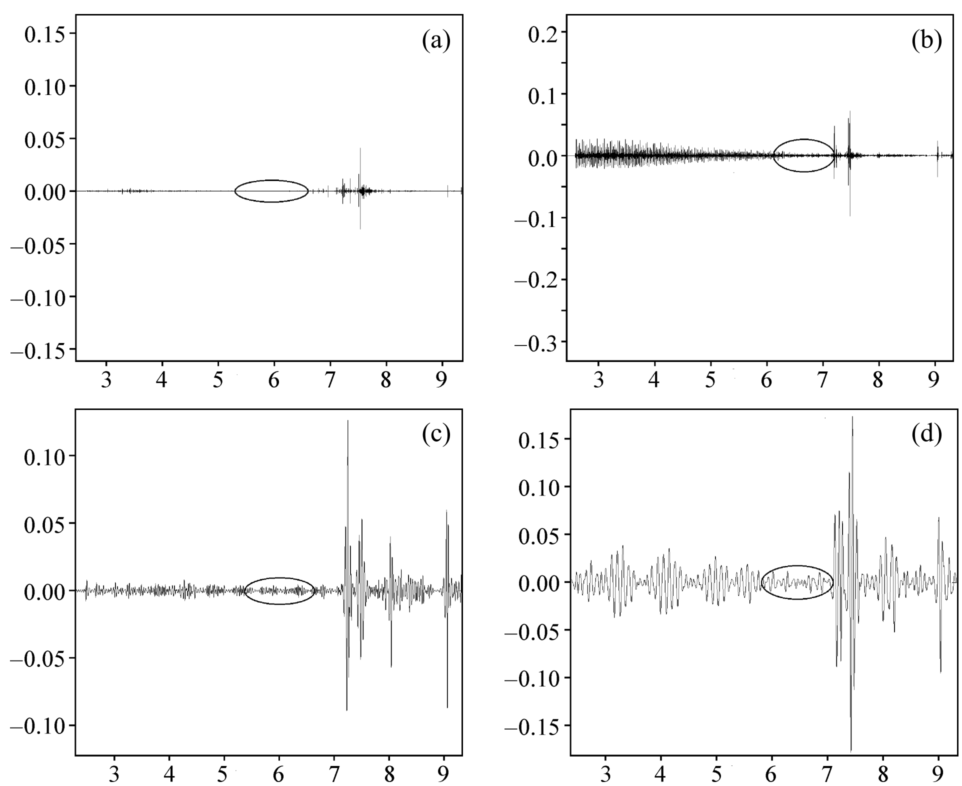


**Fig. 11.** Wavelet analysis of the stress tremor. Decomposition levels d1 (a), d5 (b), d8 (c), d9 (d) correspond to different frequency ranges of the original signal.

[36]. It also bears information about local rearrangements in the geomedium and the kinetics of preparation of large-scale failure [36].

Works that use the theory of nonlinear dynamic systems to study numerous observations of various seismic events and attendant phenomena are continuously increasing in number. However, the fundamental theory of this complex process has not yet been developed.

Numerous theoretical models of catastrophic failure in geomedia employ the basic equations of nonlinear dynamics, which are not mathematical models of deformation processes, including fracture. They are more or less successful simulation models, and for this reason they cannot be predictive.

Analysis of the solutions of these equations provides useful information about both possible evolution scenarios of critical events and the evolution of nonlinear dynamic systems as a whole. However, these results are of a qualitative nature.

Nonlinear dynamics is ready to solve complex problems of the evolution of real dynamic systems. For this purpose, rigorous mathematical models of processes in them should be developed, the model problem should be formulated as an evolutionary one, and dynamics of the corresponding solutions should represent the modeled process evolution as the real nonlinear dynamic system evolution.

The present paper attempted to develop such a theory. It is shown that solutions of solid mechanics equations have all characteristic features of the nonlinear dynamic system evolution if the problem is formulated as evolutionary.

We are far from the idea that in real situations critical events can be predicted from a mathematical model alone. Any structural and physical model is far from the real medium. From the theory of nonlinear dynamic systems it is known that small deviations at initial stages can lead to large deviations at later stages. This argues only for the qualitative agreement of the model and real process. Modeling solves other tasks. They are to study possible evolution scenarios of a loaded geomedium, to reveal catastrophic failure precursors, to develop data processing methods and new analysis methods of geodynamic monitoring data, based on the fundamental deformation theory of the geomedium as a nonlinear dynamic system, for the prediction of its closeness to catastrophic failure. Among them are well adopted Fourier and wavelet analysis methods employed here for the statistical analysis of fluctuations of rock mass elements with openings.

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