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Gauge Model of a Crack

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Abstract—The paper proposes a model of a crack system as local symmetry breaking for a group of 3D rotations compensated by fictitious fields that make Lagrangian equations of elastic energy density covariant in the effective Riemannian space. Equilibrium equations are solved using the perturbation theory with the number of notches being a small parameter, and exact expressions are derived for stress and gauge fields in a 2D problem by applying Hilbert transform to orthogonal Chebyshev polynomials. The model is generalized to nonlinear elasticity (deformation theory of plasticity) and statistical mesomechanics models. Also presented is a solution for stress concentration in a system of arbitrarily oriented cracks which takes into account their mutual influence at any order of the perturbation theory and reduces to a system of linear equations with explicit exact solutions.

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1. INTRODUCTION

The strength of a material is determined by its structure. In particular for a metal, the structure means the arrangement, form, anisotropy, and other characteristics of equilibrium or nonequilibrium second phase particles, being all eventually responsible for its performance: homogeneity of strain, stress and fracture strain, *J* integral, critical stress intensity, etc. [1, 2].

For a fracture surface to arise, a main crack must develop through several stages: its near-isotropic nucleation at structural defects, e.g., second phase particles (often nonmetallic inclusions), passage through incoherent particles, constrained growth, and opening due to stress concentration, delamination, etc.

Fracture, as a rule, begins from a series of “bad” sites, like badly located inclusions, large segregations (so-called conchoidal fractures), etc. The concentration of such bad sites is small, falling on statistical distribution tails, which makes their statistical analysis difficult [3–5].

However, we can use the advantages of field models and statistical description, offering us universality, developed mathematical apparatus, feasibility of broad generalization and appropriate analogies. It is therefore expedient to include cracks in field descriptions. One of the variants of the theory of defects is to consider dislocations and disclinations as gauge fields recovering local symmetry the structural group of which is the well-known product of translations and rotations $T(3) \times SO(3)$.

The aim of the study is to generalize the field theory of defects to the theory of fracture and to derive exact (explicit) solutions for gauge and stress fields in a system of arbitrarily oriented notches.

2. GAUGE MODEL OF A CRACK

Let us consider a solid Ω with a boundary $\delta\Omega$ loaded by a force **P**, taking r_i as the initial point of a coordinate i at no load, $R(r_i)$ as its current coordinate, and $U_i = R(r_i) - r_i$ as the displacement vector. The initial space is Euclidian with a metric $\tilde{g}_{ij}$. The metric for the Cartesian space is [+++], and the four-valent curvature tensor for the Euclidian space is identically equal to zero: $R_{ijkl} = 0$. Let the co- and contravariant derivatives of the vector **R** be denoted as

$$\frac{\partial R_i}{\partial x^k} = R_{i,k}, \ \frac{\partial R^i}{\partial x^k} = R^i_{,k}.$$

The deformation of the solid can be described using the Almansi strain tensor $\varepsilon_{km} = 1/2\, R_{k,m} R_{k,m} - \delta^k_n \tilde{g}_{km}$, where δ^k_n is the Kronecker delta.

In linear order of displacement field derivatives, ε_{km} is expressed as

$$\varepsilon_{km} = \frac{1}{2}\left(\frac{\partial R_k}{\partial x^m} + \frac{\partial R_m}{\partial x^k}\right).$$

If the solid is free of cracks, the Lagrangian density of elastic energy with neglect of inertial terms is $L_0 = \lambda(\varepsilon_{mm}^2 + 2\mu/\lambda\, \varepsilon^{mn}\varepsilon_{mn})$, where λ, μ are Lame constants. Varying this expression in displacement fields **U** gives an ordinary equilibrium equation.

If the solid contains cracks, their presence can be described through a fictitious symmetric tensor field $\varphi^{\mu\gamma}$ (generally speaking, with spin state 2 or 0) that creates an effective Riemannian space with a metric $g^{\mu\gamma}$ introduced as

$$\sqrt{g}g^{\mu\gamma} = \sqrt{\tilde{g}}(\tilde{g}^{\mu\gamma} + \varphi^{\mu\gamma}),$$

where $\tilde{g} = \det(\tilde{g}^{\mu\gamma})$, $g = \det(g^{\mu\gamma})$.

Next follows expansion in terms of a small parameter $\gamma \ll 1$ characterizing the energy contribution of the field $\varphi^{\mu\gamma}$ to the total Lagrangian:

$$\varphi^{\mu\gamma} = \varphi_0^{\mu\gamma} + \varphi_1^{\mu\gamma} + \ldots.$$

In the zero approximation, i.e., with no cracks, we have $\varphi_0^{\mu\gamma} = 0$.

In order of magnitude, γ is proportional to the elastic energy increment in the solid with cracks compared to its elastic energy with no cracks, i.e., it is proportional to a quantity dependent on the crack concentration n.

For example, in a 2D elastic problem, the stress concentration is proportional to r^{-a}, where r is the distance from the crack tip, $a = 1/2$ for a notch, and $a = 1$ for a hole. The elastic energy is $\sim r^{-2a+2}$, and because $r \sim n^{-1/2}$, the relative contribution is $\sim n^{1-a}$, i.e., for real systems with $n = 10^{-2}$–10^{-4}, we have $\gamma \sim 10^{-4} - 10^{-2} \ll 1$, which does not certainly exclude a singularity in the vicinity of the crack tip.

Now we can replace the ordinary derivative in the effective Riemannian space by the covariant derivative $D_m R_m = R_{mn} + \Gamma^i_{mn} R_i$, where Γ^i_{mn} is the Christoffel symbol with $\Gamma^i_{mn} \to \tilde{\Gamma}^i_{mn}$ at $\gamma \to 0$; for the Cartesian space, $\tilde{\Gamma}^i_{mn} = 0$.

The stress field σ_n^m can be expressed by differentiating the Lagrangian with respect to the field E_m^n:

$$\sigma_n^m = -\frac{\partial L_0}{\partial E_m^n}.$$

Field variation of R_i gives equilibrium equations of the form

$$\frac{\partial \sigma_n^k}{\partial x^k} + \Gamma^n_{ks}\sigma_n^s - \Gamma^i_{sn}\sigma_i^s = 0$$

or, in view of symmetry of the tensor σ_n^s and expression of Christoffel symbols in $g^{\mu\gamma}$, we have

$$\frac{\partial \sigma_i^k}{\partial x^k} - \frac{1}{2}\frac{\partial g_{kl}}{\partial x_i}\sigma^{kl} = 0.$$

According to the principle of least action, we can obtain a Lagrangian of the field $\varphi^{\mu\gamma}$, restricting here to the field degree. Generally speaking, for particularization, it is sufficient to specify an infinitesimal transform of $\varphi^{\mu\gamma}$ so that the Lagrangian would differ by total divergence of an arbitrary vector.

For example, we can choose the much used transformation

$$\Delta\sqrt{g}g^{\mu\gamma} = \sqrt{g}g^{\mu\alpha}\mathrm{d}_\alpha\omega^\gamma + \sqrt{g}g^{\gamma\alpha}\mathrm{d}_\alpha\omega^\mu - \mathrm{d}_\alpha(\omega^\alpha\sqrt{g}g^{\gamma\mu}),$$

where ω^μ is an infinitesimal increment corresponding to the Lie algebra and $\mathrm{d}_\alpha = \partial/\partial x^\alpha$.

For this case supplemented by $\mathrm{d}_\mu(\sqrt{g}g^{\mu\gamma}) = 0$, which is an analogue of Lorenz gauge in electrodynamics, the transform $\sqrt{g} \to \sqrt{g} + \mathrm{d}_\mu(\sqrt{g}\omega^\mu)$ gives $\sqrt{g}R \to \sqrt{g}R + \mathrm{d}_\mu(\sqrt{g}R\omega^\mu)$, where R is the scalar curvature of $\varphi^{\mu\gamma}$-based effective Riemannian space. What is significant is that the introduced field provides $\mathrm{d}x = \mathrm{d}x^1$, i.e., the field $\varphi^{\mu\gamma}$ is a gauge one.

If, reasoning from the above, we choose the simplest expression $L_1 = -\lambda_1\sqrt{g}R$, its variation in $\sqrt{g}g^{\mu\gamma}$ or $\varphi^{\mu\gamma}$ gives respective motion equations which are obvious and thus omitted; the more so they are not required for the orders of the perturbation theory considered below.

Significantly, the Lagrangian of the field $\varphi^{\mu\gamma}$ cannot have the form $L_1 = -\lambda_1\sqrt{g}R - \lambda_2\sqrt{g}$ as it makes the motion equations noncovariant.

The derived relations represent a system of nine (3 + 6) nonlinear differential equations in twelve variables: six stress tensor components and six $\varphi^{\mu\gamma}$ field components (five are independent and one equation is coupling).

Three additional equations follow from standard compatibility conditions or from strain expressed in terms of displacement vector. In fact, we have "geometrization" for a system of cracks.

The system of equations, given its boundary conditions, can be solved numerically (first or second problem of the elasticity theory). However, we are interested in analytical solutions for some intriguing and important cases considered further.

Besides, if we go to covariant description in the Riemannian space with actual inclusion of cracks in the system of field equations, rather than as boundary conditions (standard schemes), we have significant complications: the elastic problem becomes nonlinear and assumes a larger number of equations. However, the number of independent constants remains the same and equal to the number of isotropic elastic constants λ, μ.

3. PERTURBATION THEORY

The equations for the fields σ_n^m, φ_m^n are solved in terms of a formal perturbation series in small γ:

$$\sigma_m^n = \sigma_m^{n0} + \sigma_m^{n1} + \ldots, \ \varphi_{nm} = \varphi_{nm}^0 + \varphi_{nm}^1 + \ldots.$$

In the zeroth order approximation, $\varphi_m^n = 0$ is absent and the field $\sigma_n^{m0} = \sigma_n^{m0}(x)$ is know; $\partial\varphi_{kl}/\partial x^i = S_{ikl}$ is as further introduced.

In the first order, the equilibrium equation takes the form

$$\frac{\partial \sigma_i^{k1}}{\partial x^k} - \frac{1}{2} S_{ikl}^1 \sigma^{ki0} = 0.$$

Let us represent the solution σ_i^{m1} as

$$\sigma_{pl}^1 = \int \Omega_{pl}^1 (x - x_1) S_{ikl}^1 (x_1) \sigma^{kl0} (x_1) \mathrm{d}x_1,$$

where $\Omega_{pl}^i (x - x_1)$ is a known action function at a point x for concentrated force at a point x_1.

Calculating $\sigma_{pl}^1(x)$ allows a similar second-order calculation for S_{ikl}^1 and $\sigma_{il}^2(x)$, and so on.

In the most interesting case for a two-dimensional crack, the action function can be expressed in terms of complex potentials, and the field correction in terms of analytical extension.

4. TWO-DIMENSIONAL EXACT SOLUTIONS

Let us consider a plane notch of length $2d = 2$ along axis *1* in an inhomogeneous field of a large plane crack of length $2D \gg 2d$ (Fig. 1). The second crack is so large that the effect of the first on the second can be ignored. The entire system experiences tensile stress σ along axis *2*.

The additional condition of unloading at the crack surface allows us to express the fields in terms of two orthogonal Chebyshev polynomials. For the stress field along the notch (–1, 1), we have

$$\sigma_{22} + \int_{-1}^{1} \frac{1}{x_1 - y_1} \sigma_{22}(y_1) S_{122}^1 (y_1) \mathrm{d}y_1 = 0,$$

$$\sigma_{12} + \int_{-1}^{1} \frac{1}{x_1 - y_1} \sigma_{12}(y_1) S_{112}^1 (y_1) \mathrm{d}y_1 = 0,$$

because $\Omega_{pl}^i (x - x_1) \sim 1/(x - y)$.

Further, we apply the Hilbert integral relation[1] to $U_k(x_1)$-order Chebyshev polynomials of the first and

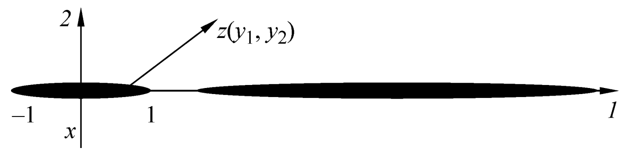

Fig. 1. Crack in an inhomogeneous external field.

second kind:

$$\frac{1}{\pi} \int_{-1}^{1} \frac{T_{k+1}(y_1)}{y_1 - x_1} \frac{\mathrm{d}y_1}{\sqrt{1 - y_1^2}} = U_k(x_1), \ |x_1| < 1.$$

The Chebyshev polynomials have the form

$$U_0(x) = 1, \ U_1(x) = 2x, \ U_3(x) = 4x^2 - 1, \ldots,$$

$$T_0(x) = 1, \ T_1(x) = x, \ T_2(x) = 2x^2 - 1, \ldots.$$

Expanding the field along the notch (–1, 1) in terms of Chebyshev polynomials of the first kind as

$$\sigma_{12}(y_1) = \sum_{k=0}^{N-1} c_k^1 U_k(y_1), \ \sigma_{22}(y_1) = \sum_{k=0}^{N-1} c_k^2 U_k(y_1)$$

gives the expressions for the field in the range (–1, 1):

$$S_{122}^1(y_1) = \frac{\sum_{k=0}^{N-1} c_k^1 T_{k+1}(y_1)}{\sum_{k=0}^{N-1} c_k^1 U_k(y_1)} \frac{1}{\sqrt{1 - y_1^2}},$$

$$S_{112}^1(y_1) = \frac{\sum_{k=0}^{N-1} c_k^2 T_{k+1}(y_1)}{\sum_{k=0}^{N-1} c_k^2 U_k(y_1)} \frac{1}{\sqrt{1 - y_1^2}},$$

which automatically satisfy the variational form of equations.

For the schematic in Fig. 1, the only nonzero component is the component S_{112}^1. In the simplest case of the notch (–1, 1) at homogeneous σ

$$S_{122}^1(y_1) = \frac{1}{\sqrt{1 - y_1^2}}.$$

It is easy to solve a similar problem for the gauge field of the notch (–1, 1) in the region of a hole of diameter $2D \gg 1$ and other problems like this.

Figure 2 presents a gauge field for a crack in an inhomogeneous external field induced by a large crack of size $2D$ and hole of radius D. The external field is $\sigma = 1$; the order of Chebyshev polynomials is no greater than 4.

If the mutual position of the notch, crack or hole is changed, we are to calculate another component of the field $\varphi^{\mu\gamma}$, which is easy. In particular, we can obtain a “shadow effect” with the small crack above or below the large notch.

Now let us calculate the fields $\sigma_{il}^1(x)$. The analytical extension of Hilbert transform for Chebyshev polynomi-

[1] The Hilbert transform in the field problem of a crack system was first used by M.A. Shtremel [4].

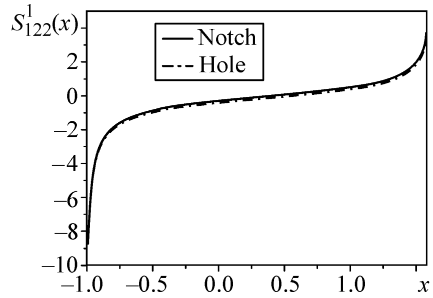

Fig. 2. Field $\varphi^{\mu\gamma}$ for a small crack in the field of a large notch and hole of diameter $2D = 16$; external stress $\sigma = 1$.

als has the form [6]:

$$\Psi(z,k)=\frac{1}{\pi}\int_{-1}^{1}\frac{T_{k+1}(z)}{(z-x)\sqrt{1-x^2}}\,\mathrm{d}x$$
$$=\frac{1}{\sqrt{z^2-1}}(z-\sqrt{z^2-1})^{k+1},\ |z|>1.$$

Next, we can put

$$\mathrm{Re}\frac{1}{x-z}=\frac{x-y_1}{(x-y_1)^2+y_2^2}$$

and introduce

$$Q(z)=-y_2\frac{\mathrm{d}\Psi(z,k)}{\mathrm{d}z},$$

and then, using direct substitution, it is easy to show that the action functions take the form:

$$\Omega^1_{22}(y_1,y_2)=cy_2+\mathrm{d}y_2^2,\ \Omega^1_{12}(y_1,y_2)=cy_2+\mathrm{d}y_1y_2,$$
$$\Omega^1_{11}(y_1,y_2)=-3cy_2-\mathrm{d}y_2^2,\ \Omega^2_{22}(y_1,y_2)=cy_1+\mathrm{d}y_1y_2,$$
$$\Omega^{21}_{12}(y_1,y_2)=cy_2+\mathrm{d}y_2^2,\ \Omega^2_{11}(y_1,y_2)=cy_1+\mathrm{d}y_1y_2,$$
$$\Omega^2_{11}=-\Omega^1_{11},\ \Omega^2_{12}=\Omega^1_{12},\ \Omega^2_{22}=\mathrm{Im}(-3\Psi+Q),$$
$$\Omega^1_{11}=-\mathrm{Re}(-\Psi+Q),\ \Omega^1_{12}=-\mathrm{Re}(-\Psi+Q),$$
$$\Omega^1_{22}=-\mathrm{Re}(-\Psi+Q).$$

Series expansion in the vicinity of $z\sim\pm1$ with the dominant terms retained gives us the ordinary expression for stress concentration in the vicinity of the crack tip:

$$\sigma_{mn}(r\gg1)\sim\frac{K_{1,2}}{\sqrt{r}}J(Q)\Omega^2_{11},\ z=re^{i\theta}.$$

The respective stress intensity factors $K_{1,2}$ for the right crack end (+1) and left crack end (–1) have the form $K^+_{1,2}=\sum c^{1,2}_k$, $K^-_{1,2}=\sum(-1)^k c^{1,2}_k$, i.e., are expressed in terms of expansion coefficients of the gauge field **S**.

The stress in the vicinity of the left and right crack ends in the notch and hole regions is presented in Fig. 3.

Although the gauge fields for the large notch and hole differ little (Fig. 2), the stress concentrations at the near and far ends of the small crack differ greatly: the stress concentration in the notch region is 1.7–2.0 times higher than that in the hole region (Fig. 3).

5. DISCUSSION

The two-dimensional exact solutions considered above can be extended to a system of arbitrarily orientated plane notches. In this case, completing the system of equations requires expansion of the field at the site of an arbitrary notch in terms of Chebyshev polynomials of the order $2n$, where n is the number of notches. Calculations of the field for each crack thus reduce to solving a system of $2n$ algebraic equations with subsequent substitution in (1).

The solution derived for stress concentration in a system of arbitrarily oriented cracks takes into account their mutual effect at any order of the perturbation theory. It reduces to a system of linear equations for which an explicit exact solution is always obtainable. At $n\gg1$, the problem reduces to inversion of the nongenerate matrix $2n\times2n$; the algorithm of this operation is standard for numerical methods.

The proposed model does not allow for compensating fields in the zeroth order of the perturbation theory, as opposed to gauge models of defects (dislocations and disclinations). In other words, these are fictitious fields

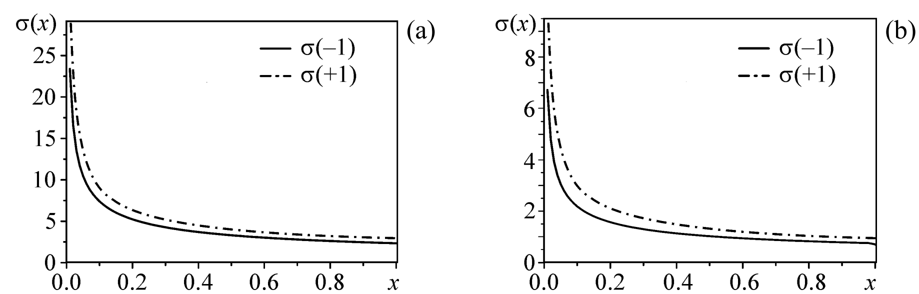

Fig. 3. Stress concentration for the left $\sigma(-1)$ and right $\sigma(+1)$ crack ends in inhomogeneous fields of a large notch (a) and hole (b); external field $\sigma = 1$.

introduced for notch surface unloading; they are required for solving a singular integral equation.

Because the covariant expressions were passed to without any assumptions on the elastic character of deformation, the obtained algorithm of description can be extended to inelastic models by putting, in particular, gauge-invariant deviators $J_2 = (E_i^n E_n^i)$, instead of invariants of strain deviators, in the deformation theory of plasticity and by expressing the plastic potential in the form

$$V(S_{nmk}, R_i) = \lambda_0 J_2 + \lambda_1 J_2^2 + \lambda_2 J_2^4 + \ldots.$$

Thus, we can extend the field description of cracks to the range of plastic strain.

Another aspect of application of the approach is in the possibility to pass from the classical description of deformation to so-called nonlinear pseudocontinuum models [3] that focus not on stress and strain fields but on their correlation functions of different orders the evolution and features of which determine the critical behavior of a deformed solid.

Thus, if we pass to covariant relations with subsequent continual integration for fields **s** and calculation of full correlation functions of strain field fluctuations, we arrive at so-called renormalized models, which is indirect evidence for correctness of the approach.

The model has analogues in the quantum field theory: confinement of quarks, in our case, is realized as attraction—motion of two cracks toward each other in an external field, which is observed in experiments on thin foils [4]. There is also a considerable difference: quarks are point objects, and we deal with a $(d-1)$-dimensional manifold (crack) in a d-dimensional space.

6. CONCLUSION

Thus, the covariant Lagrangian of elastic energy was obtained for a medium with plane notches, and the respective variational equations were solved in the framework of the perturbation theory.

The exact expression for gauge fields in a two-dimensional problem was derived through Hilbert transform of orthogonal Chebyshev polynomials.

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