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On a Potential-Velocity Formulation of Navier–Stokes Equations

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Abstract—Computational methods in continuum mechanics, especially those encompassing fluid dynamics, have emerged as an essential investigative tool in nearly every field of technology. Despite being underpinned by a well-developed mathematical theory and the existence of readily available commercial software codes, computing solutions to the governing equations of fluid motion remains challenging: in essence due to the non-linearity involved. Additionally, in the case of free surface film flows the dynamic boundary condition at the free surface complicates the mathematical treatment notably. Recently, by introduction of an auxiliary potential field, a first integral of the two-dimensional Navier–Stokes equations has been constructed leading to a set of equations, the differential order of which is lower than that of the original Navier–Stokes equations. In this paper a physical interpretation is provided for the new potential, making use of the close relationship between plane Stokes flow and plane linear elasticity. Moreover, it is shown that by application of this alternative approach to free surface flows the dynamic boundary condition is reduced to a standard Dirichlet–Neumann form, which allows for an elegant numerical treatment. A least squares finite element method is applied to the problem of gravity driven film flow over corrugated substrates in order to demonstrate the capabilities of the new approach. Encapsulating non-Newtonian behaviour and extension to three-dimensional problems is discussed briefly.

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1. MOTIVATION

As is well known, Bernoulli's equation is obtained as a first integral of Euler's equations in the absence of vorticity and viscosity if the velocity vector **u** is perceived as the gradient of a scalar potential. The so-called Clebsch transformation [1, 2] allows for a further extension to flows with non-vanishing vorticity. A similar methodology has recently been reported by Scholle et al. [3] for the case of two-dimensional incompressible viscous flow by making use of a representation of the fields in terms of complex coordinates. Besides a reduction of differential order the formulation of integrated Navier–Stokes equations allows for a convenient embodiment of the dynamic boundary condition as a Dirichlet–Neumann condition on the potential field in the case of free surface flows.

Initially the integration procedure is motivated and performed from a formal mathematical point of view in which a scalar potential field is introduced as an auxiliary variable to make the field equations integrable. Meanwhile, the question of physical interpretation of this "naturally" occurring "potential" motivates a review of complex methods in the field of fluid dynamics in which the exploration of the close relationship between plane Stokes flow and plane linear elasticity proves to be illuminating. In the case of Stokes flow the new potential velocity formulation in complex form can be shown to reproduce the well-known Kolosov–Muskhelishvili formulae [4–6] of plane linear elasticity, suggesting the potential to be a function of integrated stresses.

A short review of the first integral of Navier–Stokes equations [3] is provided in Sect. 2.1, followed by the derivation of the potential representation of the dynamic boundary condition in Sect. 2.2. Section 3 is devoted to the analysis and interpretation of the potential variable mentioned above. In Sect. 4 a least-squares finite element method, used to solve the fully non-linear problem of gravity-driven thin film flow over corrugated topography, is presented; it shows the general and convenient numerical applicability of the new formulation. Section 5 provides a summary and forward look.

2. TWO-DIMENSIONAL POTENTIAL-VELOCITY FORMULATION FOR STEADY INCOMPRESSIBLE NAVIER–STOKES EQUATIONS

Scholle et al. [3] developed an integration procedure for the case of two-dimensional incompressible viscous flow by making use of a representation of the fields in terms of complex coordinates. For convenience the essentials of the methodology are reviewed below.

2.1. Field Equations

In two-dimensions, the Navier–Stokes equations and the continuity equation governing a steady and incompressible flow, assuming that the external force on the fluid is conservative with a given potential energy density $U(x, y)$, are:

$$\rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \eta \Delta \mathbf{u} - \nabla U, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \text{ or } \operatorname{div} \mathbf{u} = 0, \tag{2}$$

where $\mathbf{u}$ denotes the velocity field and p the pressure field. Scholle et al. [3] show that a complex variable transformation $\xi := x + iy$, $\bar{\xi} := x - iy$ of the above field equations, together with the introduction of a complex velocity field $u := u_x + iu_y$, yields the equivalent formulation[1]:

$$\frac{\partial}{\partial \bar{\xi}}\left[p + \rho \frac{\bar{u}u}{2} + U\right] + \rho \frac{\partial}{\partial \xi}\left(\frac{u^2}{2}\right) = 2\eta \frac{\partial^2 u}{\partial \bar{\xi} \partial \xi}, \tag{3}$$

$$\operatorname{Re}\left(\frac{\partial u}{\partial \xi}\right) = 0, \tag{4}$$

where Re denotes the real part of the subsequent complex expression. Now the introduction of a scalar real-valued potential Φ satisfying

$$p + \rho \frac{\bar{u}u}{2} + U = 4 \frac{\partial^2 \Phi}{\partial \bar{\xi} \partial \xi}, \tag{5}$$

allows for the integration of (3) with respect to ξ, giving:

$$\rho \frac{u^2}{4} - \eta \frac{\partial u}{\partial \bar{\xi}} + 2 \frac{\partial^2 \Phi}{\partial \bar{\xi}^2} = 0, \tag{6}$$

$$\operatorname{Re}\left(\frac{\partial u}{\partial \xi}\right) = 0. \tag{7}$$

[1] Note that in contrast to [3] the standard complex variable transformation is used here leading to slightly different equations.

Note, the resulting equations contain first order derivatives only of velocity in contrast to the original Navier–Stokes equation (1) and the corresponding two-dimensional streamfunction version in general contains second order derivatives only, whereas the classical streamfunction formulation results in a forth order equation.

A real representation of the complex system (6), (7) in tensor notation reads:

$$\eta\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_k}\delta_{ij}\right] - \rho\left[u_i u_j - u_k u_k \frac{\delta_{ij}}{2}\right] = 2\left[\frac{\partial^2 \Phi}{\partial x_i \partial x_j} - \frac{\partial^2 \Phi}{\partial x_k \partial x_k}\frac{\delta_{ij}}{2}\right], \tag{8}$$

together with

$$\frac{\partial u_k}{\partial x_k} = 0, \quad p + \frac{\rho}{2} u_k u_k + U = \frac{\partial^2 \Phi}{\partial x_k \partial x_k}, \tag{9}$$

in which the Einstein summation convention is used. The second equation in (9) is only relevant in applications where the recovery of pressure is of interest; the pressure can easily be computed subsequently.

2.2. Boundary Conditions

Common boundary conditions such as no-slip or periodic conditions can be used directly in connection with equations (8), (9). In these cases just the velocity components are constrained at the boundary, whereas the potential variable remains unconstrained. While free surface boundary conditions are more problematic they can be simplified substantially as shown in the following. Henceforth a simply connected domain with a free surface $\mathbf{x} = \mathbf{x}(s)$ is considered, in which the free surface is assumed to be parametrized with respect to arc length s. Furthermore normal and tangential unit vectors according to

$$n_i(s)\varepsilon_{ij} = t_j(s) = x_j'(s) \tag{10}$$

are defined along the free surface, where ε_{ij} denotes the Levi-Civita symbol. In two dimensions the surface shape is determined by one kinematic and two dynamic boundary conditions. The latter

$$\left[(p_0 - p)\delta_{ij} + \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]_{x_k = x_k(s)} \times n_j(s) = \sigma \frac{\mathrm{d}t_i}{\mathrm{d}s}, \tag{11}$$

with σ on the right-hand side denoting the surface tension, indirectly introduce the pressure as an undesired further variable into the solution process. Scholle et al. [3] found an elegant remedy for this by obtaining an integral formulation for the boundary condition revealing

(11) to be a pure condition on the potential gradient, namely:

$$\varepsilon_{ij}\left(\frac{\partial\Phi}{\partial x_j}\right)_{x_k=x_k(s)}=\frac{\sigma}{2}t_i(s)$$
$$-\frac{1}{2}\int_{s_0}^{s}\left[U(x_k(\tilde{s}))n_i(\tilde{s})+\sigma'(\tilde{s})t_i(\tilde{s})\right]\mathrm{d}\tilde{s}. \quad (12)$$

Here the integral limit s_0 can be chosen arbitrarily. In this context it is interesting to note that in the two-dimensional streamfunction formulation the potential boundary conditions on the free surface

$$\nabla\Phi=\mathbf{g}_1(s) \quad (13)$$

represent a natural counterpart to the streamfunction conditions on the solid wall

$$\nabla\Psi=\mathbf{g}_2(s). \quad (14)$$

Of course, knowledge of the potential gradient on the free surface further allows the derivation of Dirichlet and Neumann boundary conditions for the potential variable in a simply connected domain if Φ is specified at a single point. By taking the inner product of (12) with t_i, its tangential component takes the form:

$$n_i(s)\left(\frac{\partial\Phi}{\partial x_i}\right)_{x_k=x_k(s)}$$
$$=\frac{\sigma}{2}-\frac{t_i(s)}{2}\int_{s_0}^{s}\left[U(x_k(\tilde{s}))n_i(\tilde{s})+\sigma'(\tilde{s})t_i(\tilde{s})\right]\mathrm{d}\tilde{s}, \quad (15)$$

which is a Neumann boundary condition for the potential Φ. On the other hand, using (10) the inner product of (12) with n_i leads to the corresponding normal component:

$$t_i(s)\left(\frac{\partial\Phi}{\partial x_i}\right)_{x_k=x_k(s)}=\frac{\mathrm{d}}{\mathrm{d}s}\Phi(x_k(s))$$
$$=\frac{\varepsilon_{ji}t_j(s)}{2}\int_{s_0}^{s}f_i(\tilde{s})\mathrm{d}\tilde{s}, \quad (16)$$

$$f_i(s)=U(x_k(s))n_i(s)+\sigma'(s)t_i(s), \quad (17)$$

with a total derivative in (16). Thus, using (16), (10) and partial integration, an integrated form of a Dirichlet boundary condition for the potential Φ can be constructed:

$$\Phi(x_k(s))-\Phi(x_k(s_0))$$
$$=\frac{\varepsilon_{ji}}{2}\int_{s_0}^{s}x_j'(\tilde{s})\int_{s_0}^{\bar{s}}f_i(\bar{s})\mathrm{d}\bar{s}\mathrm{d}\tilde{s}, \quad (18)$$

$$\Phi(x_k(s))-\Phi(x_k(s_0))=\frac{\varepsilon_{ji}}{2}x_j(s)\int_{s_0}^{s}f_i(\tilde{s})\mathrm{d}\tilde{s}$$
$$-\frac{\varepsilon_{ji}}{2}\int_{s_0}^{s}x_j(\tilde{s})f_i(\tilde{s})\mathrm{d}\tilde{s}, \quad (19)$$

$$\Phi(x_k(s))-\Phi(x_k(s_0))$$
$$=\frac{1}{2}\int_{s_0}^{s}\left[U(x_k(\tilde{s}))\delta_{ij}+\sigma'(\tilde{s})\varepsilon_{ij}\right]$$
$$\times\left[x_i(s)-x_i(\tilde{s})\right]x_j'(\tilde{s})\mathrm{d}\tilde{s}. \quad (20)$$

The term $\Phi(x_k(s_0))$ can be set to zero without loss of generality and moreover, in the special case of $\sigma'(s)=0$, that is neglecting Marangoni effects, further simplification is possible.

The reduction of the dynamic boundary condition from its original form (11) to a standard Dirichlet–Neumann form (15), (20) is a key feature of the reformulation of the equations of motion in terms of the first integral of the Navier–Stokes equations (1) to allow for the construction of efficient solutions.

3. ON POTENTIALS AND STRESSES

So far, in Sect. 2, a potential-velocity formulation of the Navier–Stokes equations with corresponding boundary conditions has been found by a formal integration method. Note, this derivation is motivated and performed in a pure mathematical context, in which the scalar "potential" Φ appears rather as an auxiliary variable, introduced to make the field equations integrable, than as a physically meaningful quantity. It has to be distinguished from the classical meaning of a scalar field, the gradient of which represents a given vector field. As a physical interpretation tends to simplify analysis and numerical treatment later on, a closer look is taken at the character of this potential.

In terms of the potential Φ and a streamfunction Ψ, satisfying:

$$-2i\frac{\partial\Psi}{\partial\bar{\xi}}=u, \quad (21)$$

equations (6), (7) can be written as

$$\frac{\partial^2\Phi}{\partial\bar{\xi}^2}+i\eta\frac{\partial^2\Psi}{\partial\bar{\xi}^2}-\frac{\rho}{2}\left(\frac{\partial\Psi}{\partial\bar{\xi}}\right)^2=0, \quad (22)$$

in which the continuity equation is satisfied automatically. Considering the Stokes flow case and introducing a further complex potential $\chi=\Phi+i\eta\Psi$ allows equation (22) to be written as the simple bianalytic equation

$$\frac{\partial^2\chi}{\partial\bar{\xi}^2}=0, \quad (23)$$

the solution of which can be expressed in terms of two analytic functions

$$\chi=\bar{\xi}\omega_0(\xi)+\omega_1(\xi), \quad (24)$$

known as Goursat functions. On the one hand, this formula allows the problem of Stokes flow to be converted into one of complex analysis, namely that of finding ω_0 and ω_1 which satisfy appropriate conditions on the boundary. This approach leads to the well-known Sherman–Lauricella equations, see for example [7]. On the other hand, using $\Phi = \mathrm{Re}(\chi)$ the representation of the potential derivatives in terms of the Goursat functions reproduce the well-known Muskhelishvili–Kolosov formula [5, 6]:

$$\frac{\partial \Phi}{\partial x} + i\frac{\partial \Phi}{\partial y} = \omega_0(\xi) + \xi\overline{\omega_1'(\xi)} + \overline{\omega_2'(\xi)} \tag{25}$$

for the integrated components of stress in plane linear elasticity theory, revealing the potential to be closely connected with the Airy stress function.

At first glance, the analogy to linear elasticity theory provides an interpretation for the potential Φ in the linear Stokes case only, but rewriting the full Navier–Stokes equations (1) in terms of the streamfunction and Airy stress function actually reproduces the real version of equation (6). Accordingly, the constitutive law of a Newtonian fluid involving the convective momentum flux density[1] $R_{ij} = u_i u_j$ is adopted to account for inertial effects:

$$\sigma_{ij} = -p\delta_{ij} + \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \rho R_{ij}, \tag{26}$$

with σ_{ij} being a symmetric stress tensor, p the pressure and u_i the velocity components. Now introducing an Airy stress function such that

$$\sigma_{11} = \frac{\partial^2 \Phi}{\partial y^2},\ \sigma_{22} = \frac{\partial^2 \Phi}{\partial x^2},\ \sigma_{12} = -\frac{\partial^2 \Phi}{\partial x \partial y}, \tag{27}$$

leads to

$$-\rho u_x u_y + \eta\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \frac{\partial^2 \Phi}{\partial x \partial y} = 0, \tag{28}$$

$$\frac{\rho}{2}(u_x^2 - u_y^2) + \eta\left(\frac{\partial u_y}{\partial y} - \frac{\partial u_x}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 \Phi}{\partial x^2}\right) = 0, \tag{29}$$

which agrees with equation (8) except for a constant scaling of Φ. This result is not surprising, considering that the dynamic boundary condition (12) only constrains Φ on the one hand while essentially constituting a condition on the stresses on the other hand.

[1] By averaging Reynolds stress tensor is obtained from R_{ij}.

4. NUMERICAL FORMULATION AND SOLUTION

Complex variable methods have been known for a long time in hydrodynamics [8] and have contributed a great deal to the development of analytic and semi-analytic solutions of linear flow problems in simple two-dimensional geometries [8–10]. Also various forms of complex boundary integral methods have entered the arena of computational fluid dynamics which nowadays are ranked among the most efficient methods in their range of applicability [7, 9, 11, 12]. This development was partly due to realizing the close connection to the field of plane linear elasticity [4–6] which also allowed for a transfer of solution methods (see for example Greengard et al. [7]).

Complex variable methods are essentially based on rewriting the field equations in terms of the analytic Goursat functions given in equation (24) and solving the corresponding complex system. Though very efficient for plane Stokes flow, especially in the case of stress boundary conditions [7], their applicability remains limited. Note that this classical approach results as a special case of the more general first integral procedure of Sect. 2 and that the tensor version (8), (9) can naturally be extended to three dimensions as shown by Scholle et al. [3].

Since the overall goal is to construct a fairly general and flexible numerical method allowing for inertial effects and potentially being extendible to three dimensions, below the transformed real version of Navier–Stokes equations (8), (9) is discretized directly. Due to the close connection between the integrated real version of Navier–Stokes equations and a streamfunction / Airy stress function formulation as shown in Sect. 3, a review of the corresponding, though limited, literature [9, 11, 13–16] was undertaken.

For the purposes of the work reported here a least squares finite element method, inspired by the contribution to the field of Bolton and Thatcher [13], Cassidy [14], Thatcher [16], proved to be adequate. This way the highly efficient semi-analytic Ritz method developed by Scholle et al. [3] for the integrated Stokes equations is complemented by a more flexible method, allowing for the incorporation of inertial effects and more general applicability. For a comprehensive review of least squares methods, including a special treatment of Stokes and Navier–Stokes equations, the reader is referred to [17].

4.1. Least-Squares Finite Element Method

The least-squares finite element method has gained great popularity for the numerical solution of flow problems, allowing for the use of simple equal order elements as well as highly efficient multigrid solvers due to symmetry and positive definiteness of the resulting system matrices [17]. Arguments of practicality suggest rewriting the tensor equations (8), (9) in terms of velocity variables and first derivatives of the Airy stress function. Thus a system of four equations is obtained, including first order derivatives only, which is covered by the first order system least squares methodology.

By introducing the two variables $\Phi_x = \partial\Phi/\partial x$ and $\Phi_y = \partial\Phi/\partial y$ and a further equation

$$\frac{\partial\Phi_x}{\partial y} - \frac{\partial\Phi_y}{\partial x} = 0, \tag{30}$$

equations (8), (9) are transformed into the system:

$$-\rho u_x u_y + \eta\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) - \frac{\partial\Phi_x}{\partial y} + \frac{\partial\Phi_y}{\partial x} = 0, \tag{31}$$

$$\frac{\rho}{2}(u_x^2 - u_y^2) + \eta\left(\frac{\partial u_y}{\partial y} - \frac{\partial u_x}{\partial x}\right) + \frac{\partial\Phi_x}{\partial x} - \frac{\partial\Phi_y}{\partial y} = 0, \tag{32}$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0, \tag{33}$$

$$\frac{\partial\Phi_x}{\partial y} - \frac{\partial\Phi_y}{\partial x} = 0. \tag{34}$$

Applying a Newton linearisation, with $\tilde{u}_x$ and $\tilde{u}_y$ denoting the velocity components of the previous iteration step, leads to the following linearised system:

$$-\rho(u_x\tilde{u}_y + \tilde{u}_x u_y) + \eta\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) - \frac{\partial\Phi_x}{\partial y} + \frac{\partial\Phi_y}{\partial x} = -\rho\tilde{u}_x\tilde{u}_y := f_1, \tag{35}$$

$$\rho(u_x\tilde{u}_x - u_y\tilde{u}_y) + \eta\left(\frac{\partial u_y}{\partial y} - \frac{\partial u_x}{\partial x}\right) + \frac{\partial\Phi_x}{\partial x} - \frac{\partial\Phi_y}{\partial y} = \frac{\rho}{2}(\tilde{u}_x^2 - \tilde{u}_y^2) := f_2, \tag{36}$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 := f_3, \tag{37}$$

$$\frac{\partial\Phi_x}{\partial y} - \frac{\partial\Phi_y}{\partial x} = 0 := f_4, \tag{38}$$

which is written in condensed form as $L(W) = f$, with $W = (u_x, u_y, \Phi_x, \Phi_y)^{\mathrm{T}}$, here on in. Note, that here the linearisation is done before least squares minimization [18]. The corresponding least-squares functional to be minimized at each iteration step, can be written as:

$$J(W; f) = \sum_{i=1}^{4}\left\| L_i W - f_i \right\|_{L_2(\Omega)}^2. \tag{39}$$

For convenience a simply connected domain $\Omega \subset \mathbb{R}^2$ is assumed with prescription of either u_x and u_y or Φ_x and Φ_y on each part of the boundary $\partial\Omega$, where the potential boundary conditions are set according to (12) in case of problems involving a free surface. Note that in this formulation a free surface flow problem leads to pure Dirichlet boundary conditions on the velocity vector and on the potential gradient. Additionally in the case of pure velocity boundary conditions, $\nabla\Phi$ and $\Delta\Phi$ have to be specified at a single point in order to achieve a unique solution.

In the following $H^m(\Omega)$ denotes the standard Sobolev space of functions having square integrable derivatives of order up to m over Ω. Now, introducing test and solution spaces $\tilde{W}, \tilde{V} \subset [H^1(\Omega)]^4$, where the boundary conditions are assumed to be incorporated into the solution space, as well as corresponding finite element subspaces $\tilde{W}_h, \tilde{V}_h$, the problem of minimizing functional (39) can be rewritten as a variational formulation, namely, find $W_h \in \tilde{W}_h$ such that:

$$\int_\Omega L(W_h)L(V_h)\mathrm{d}\Omega = \int_\Omega L(V_h) f\,\mathrm{d}\Omega, \;\; \forall V_h \in \tilde{V}_h. \tag{40}$$

In producing the numerical results presented in the following subsection continuous finite element spaces of piecewise quadratic polynomials are employed for all test and solution spaces. Well-known problems associated with mass conservation in conjunction with least-squares methods [16] are addressed by an appropriate weighting of the continuity equation as suggested in [13] or by an augmented leastsquares method in severe cases [17].

4.2. Treatment of Free Surfaces

In two dimensions, three boundary conditions are required along the free surface, a kinematic condition (41) and two dynamic conditions (12). As two conditions are sufficient to determine a problem with fixed domain, the shape of the free surface can be found by iterating over one of the conditions while solving a sequence of flow problems with the other two conditions set on a fixed domain. The problem formulation provided suggests solving such problems with a given dynamic boundary condition (12) and to iterate over the kinematic condition

$$u_i(x_k(s))n_i(s) = 0. \tag{41}$$

A correction $H^{n+1}(x)$ of the previous free surface shape, given by a height function $H^n(x)$, is found by solving a differential equation of the form:

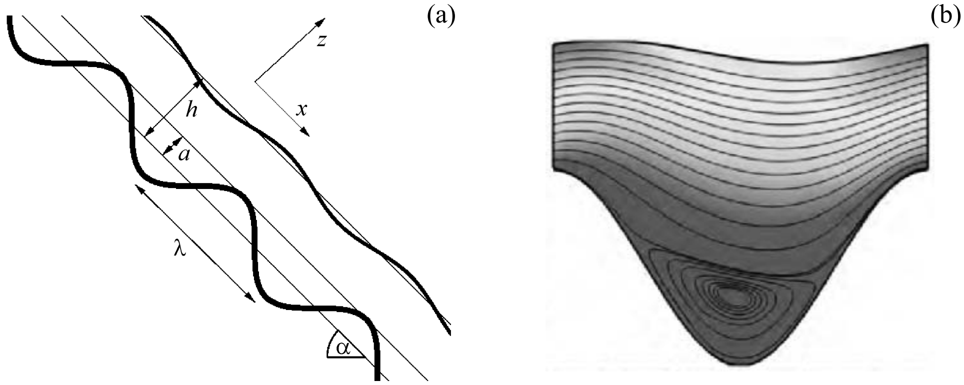

Fig. 1. Principle set-up for thin gravity-driven film flow over corrugated substrate with inclination angle α (a); streamlines with isotach lines as contour colours for the case of Re = 50 (b).

$$\frac{\mathrm{d}}{\mathrm{d}x}H^{n+1}(x)\Big|_{x=x(s)} = \frac{u_y^n(x,y)}{u_x^n(x,y)}\Bigg|_{(x,y)=(x(s),H^n(x(s)))}, \qquad (42)$$

via an explicit Runge–Kutta integration method. The update algorithm is supplemented by an appropriate damping in the case of strong variations.

4.3. Numerical Results

Consider now the case of gravity-driven free surface film flow over a corrugated substrate inclined at angle of α to the horizontal, as shown in Fig. 1. Assume Ω to be a symmetric part of the domain and impose periodic velocity boundary conditions to the left and right, no-slip conditions on the lower fixed boundary and $\Phi_x = g_x$, $\Phi_y = g_y$ on a fixed approximation to the upper free surface with functions g_x, g_y constructed via (12). The potential energy density U in (12) is given by

$$U(x,y) = \rho g[y\cos\alpha - x\sin\alpha] \qquad (43)$$

where ρ denotes the density and g the gravitational constant.

Numerical results are found using a structured criss-cross grid in combination with a moving mesh method for grid adaptation. In the finite element space, piecewise continuous quadratic functions are employed. Figure 2 shows the impact of varying film thickness on the resulting flow structure. As the film thickness increases, the free surface becomes smoother and eddies form in the corrugations. The solutions obtained are in accord with comparable experimental and computational results [19–21].

In a further parameter study the Reynolds number is varied in order to demonstrate the capability of the new approach (Fig. 3). From the global perspective, the appropriate choice of h as reference length leads to the following definition of a global Reynolds number [21]:

$$\mathrm{Re} = \frac{\rho^2 g h^3 \sin\alpha}{2\eta^2}, \qquad (44)$$

which is important as far as the stability of the flow and

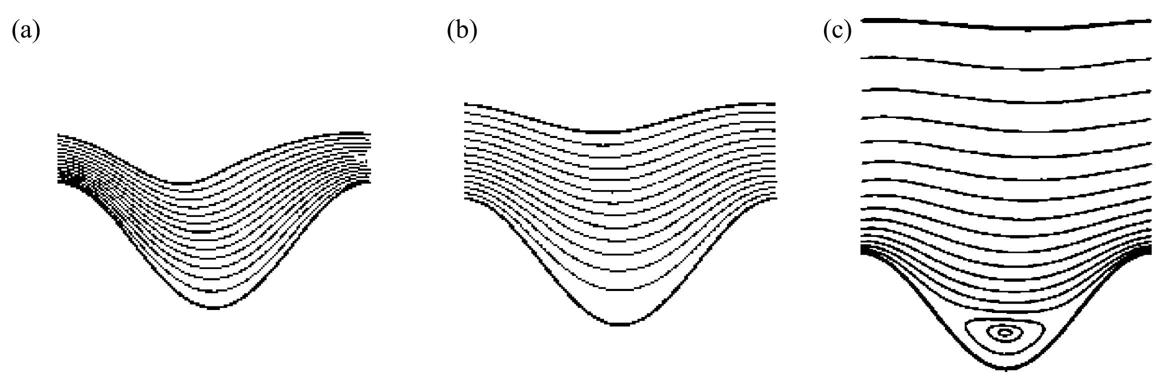

Fig. 2. Varying film thickness effect. Constant substrate geometry and fluid data λ = 1 cm, a = 0.2 cm, $\alpha = \pi/8$, η = 5.78 Pa s, σ = 0.074 N m^{-1}, g = 9.81 m s^{-2} and ρ = 972 kg m^{-3}; the film thickness is h = 0.15 (a), 0.3 (b) and 0.8 cm (c).

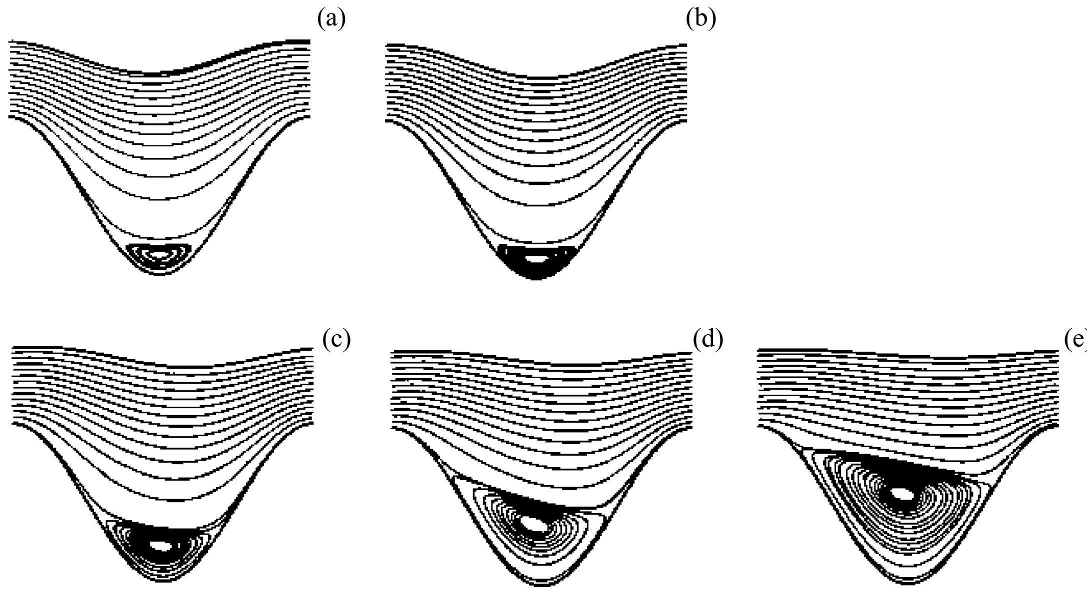

Fig. 3. Varying Reynolds number effect. Constant substrate geometry and fluid data $\lambda = 6$ cm, $a = \pi/2$ cm, $\alpha = \pi/4$, $\eta = 5.78$ Pa s, $\sigma = 0.074$ N m^{-1}, $g = 9.81$ m s^{-2} and $h = 12\pi/25$ cm; the Reynolds number Re = 0.3 (a), 10 (b), 30 (c), 50 (d), 100 (e).

dynamics of the free surface are concerned. Although thin gravity-driven film flow appears unstable at the surface at a sufficiently high Reynolds number, as indicated in Haas [8], the region below the free surface in the vicinity of the eddy is heuristically stable and resolved correctly. These results are in accordance with Haas [19], Pollak and Aksel [20], Scholle et al. [21]. Although problems of mass conservation in conjunction with least-squares methods tend to increase when other than pure velocity boundary conditions are imposed, the example flows considered shows the method to produce accurate results even in the case of periodic and free boundary conditions.

5. SUMMARY AND OUTLOOK

It is shown by use of complex variables that a first integral of the two-dimensional incompressible and steady Navier–Stokes equations can be established, the order of which is lower than that of the original Navier–Stokes equations. The procedure results in either a single complex valued equation of second order depending on a potential and the streamfunction or a system of two equations in the case when velocities are used. Alternatively in terms of Cartesian coordinates a tensor formulation can be given.

The potential field is formally introduced as an auxiliary variable to make the field equations integrable, while posing the question of physical interpretation. A look at the integrated equations in the Stokes flow case and the analogy between plane Stokes flow and plane linear elasticity for a complex formulation, on the one hand, shows that the new formulation reproduces the well-known complex Kolosov–Muskhelishvili formulas of linear elasticity, while on the other hand, revealing the potential to be equivalent to an Airy stress function except for constant scaling. This interpretation is shown to be cogent also in the non-linear case.

Motivated by the above integration procedure a representation of the dynamic boundary condition as a pure Dirichlet–Neumann condition on the potential is derived. This formulation gives rise to the development of new numerical methods for free surface flows. An efficient least-squares finite element method is developed and shown to accurately solve a fully non-linear problem; that of gravity-driven film flow over corrugated substrate for different film thickness and different Reynolds number.

As indicated in [3] the tensor representation (8), (9) of the first integral of Navier–Stokes equations gives rise to a natural generalization to three dimensions. Apart from this, a natural generalization to non-Newtonian materials is obvious by replacing the stress tensor for a Newtonian fluid by the stress tensor for arbitrary materials. The presentation of a generalized theory will be the subject of subsequent publications.

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