General theory of micropolar elastic thin shells

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The paper formulates general hypotheses of micropolar elastic thin shells that are given asymptotic validation. Using these hypotheses and three-dimensional Cosserat (micropolar, asymmetric) theory of elasticity, general two-dimensional applied models of micropolar elastic thin shells with independent displacement and rotation fields, constrained rotation and low shear rigidity are constructed to suit dimensionless physical parameters of the shell material. The constructed micropolar shell models take into complete account transverse shear strain and related strain. Models of micropolar elastic thin plates and beams are particular cases of the constructed micropolar shell models. An axially symmetric stress-strain state problem of a hinged cylindrical micropolar shell is considered. Numerical analysis is used to demonstrate effective strength and rigidity characteristics of micropolar elastic shells.

Keywords: micropolar elastic shell, theory, strength and rigidity characteristics, efficiency

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1. Introduction

The progress in micro- and nanotechnologies sets up new problems assisting the research and development in physical mesomechanics and structural solid state mechanics [1–3]. The stress-strain state of heterogeneous solids is described to advantage by the micropolar theory of elasticity, which has been intensively developed in recent years [4–17]. Of urgency for modern applications is the construction of mathematical models of micropolar elastic thin beams, plates, and shells [18–29].

The main issue in the general theory of micropolar elastic thin rods, plates and shells is in approximate but adequate reduction of a three-dimensional micropolar elastic problem to a one- or two-dimensional boundary problem. In our opinion, it is appropriate to this end to use asymptotic integration of the three-dimensional micropolar elastic boundary problem of a thin shell or plate (rectangle) [30–33]. In terms of engineering practice, this idea can efficiently be implemented using the following approach. If we have qualitative results of asymptotic integration of a three-dimensional (two-dimensional) micropolar elastic boundary problem of a thin shell or plate (rectangle), we can then formulate rather general assumptions (hypothesis) that provide a possibility to go from a three-dimensional (two-dimensional) model to a two-dimensional (one-dimensional) model of micropolar shells and plates (rods).

In the work, using this approach and three-dimensional micropolar theory of elasticity with dimensionless physical parameters, general models of micropolar elastic thin shells with independent displacement and rotation fields, constrained rotation, and low shear rigidity are constructed taking complete account of transverse shear strain and related strain.

2. Problem statement

Let us consider an isotropic shell of constant thickness $2h$ as a three-dimensional micropolar elastic solid. We stem from the constitutive (tensor) equations of linear static micropolar elasticity with independent displacement and rotation fields [34–36]:

equilibrium equations:

$$\nabla_m \sigma^{mn} = 0, \ \nabla_m \mu^{mn} + e^{nmk}\sigma_{mk} = 0, \tag{1}$$

physical elasticity relations:

$$\begin{cases} \sigma_{mn} = (\mu+\alpha)\gamma_{mn} + (\mu-\alpha)\gamma_{nm} + \lambda\gamma_{kk}\delta_{nm}, \\ \mu_{mn} = (\gamma+\varepsilon)\kappa_{mn} + (\gamma-\varepsilon)\kappa_{nm} + \beta\kappa_{kk}\delta_{nm}, \end{cases} \tag{2}$$

geometric relations:

$$\gamma_{mn} = \nabla_m V_n - e_{kmn}\omega^k, \ \kappa_{mn} = \nabla_m \omega_n. \tag{3}$$

Here, $\hat{\sigma}, \hat{\mu}$ are the force and couple stress tensors; $\hat{\gamma}, \hat{\kappa}$ are the strain and bending-torsion tensors; $\mathbf{V}, \boldsymbol{\omega}$ are the displa-

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cement and independent rotation vectors; $\lambda, \mu, \alpha, \beta, \gamma, \varepsilon$ are elastic constants of the micropolar shell material. The subscripts m, n, k take on the values 1, 2, 3.

Constitutive equations (1)–(3) of the three-dimensional asymmetric theory of elasticity are added with appropriate boundary conditions.

The shell front faces are assigned boundary conditions of the first micropolar elastic boundary problem with free rotation, and the shell end faces Σ are assigned boundary conditions either in force and couple stresses, or displacements and rotations, or in a mixed form, depending on the type of external loading or fixing of shell points.

It should be noted that the main physical constant satisfying equations (1)–(3) of the micropolar theory of elasticity is the elastic modulus α (at $\alpha = 0$, the given system yields equations of the classical theory of elasticity).

In what follows, we use curvilinear orthogonal coordinates α_k (Lamé coefficients: $H_i = A_i(1+\alpha_3/R_i)$ $(i=1,2)$, $H_3 = 1$) accepted in the theory of shells [37]. In so doing, the physical tensor and vector components are left with the previous notation. The boundary conditions at the shell front faces $\alpha_3 = \pm h$ can now be written as

$$\sigma_{3i} = \pm q_i^{\pm}, \ \sigma_{33} = \pm q_3^{\pm}, \qquad \mu_{3i} = \pm m_i^{\pm}, \ \mu_{33} = \pm m_3^{\pm} \ (i=1,2). \tag{4}$$

Notice that from Hooke's equations (2) for the strains γ_{12}, γ_{21} and γ_{3i}, γ_{i3} $(i=1,2)$, it is sometime convenient to consider their sums and differences which, in view of geometric relations (3), in the accepted curvilinear coordinate system can be written as follows:

$$\frac{1}{H_1}\frac{\partial V_2}{\partial \alpha_1} - \frac{1}{H_1 H_2}\frac{\partial H_1}{\partial \alpha_2}V_1 + \frac{1}{H_2}\frac{\partial V_1}{\partial \alpha_2} - \frac{1}{H_1 H_2}\frac{\partial H_2}{\partial \alpha_1}V_2 = \frac{1}{2\mu}(\sigma_{12}+\sigma_{21}),$$
$$\frac{1}{H_i}\frac{\partial V_3}{\partial \alpha_i} - \frac{1}{H_i}\frac{\partial H_i}{\partial \alpha_3}V_i + \frac{\partial V_i}{\partial \alpha_3} = \frac{1}{2\mu}(\sigma_{i3}+\sigma_{3i}), \tag{5}$$

$$\omega_3 = -\frac{1}{2}\left[\left(\frac{1}{H_2}\frac{\partial V_1}{\partial \alpha_2} - \frac{1}{H_1 H_2}\frac{\partial H_2}{\partial \alpha_1}V_2\right) - \left(\frac{1}{H_1}\frac{\partial V_2}{\partial \alpha_1} - \frac{1}{H_1 H_2}\frac{\partial H_1}{\partial \alpha_2}V_1\right)\right] + \frac{1}{4\alpha}(\sigma_{21}-\sigma_{12}),$$
$$\omega_i = (-1)^j \frac{1}{2}\left[\left(\frac{1}{H_j}\frac{\partial V_3}{\partial \alpha_j} - \frac{1}{H_j}\frac{\partial H_j}{\partial \alpha_3}V_j\right) - \frac{\partial V_j}{\partial \alpha_3}\right] - (-1)^j \frac{1}{4\alpha}(\sigma_{j3}-\sigma_{3j}). \tag{6}$$

Hereinafter, $i, j = 1, 2$ and $i = j$.

It is assumed that the shell thickness is small compared to the characteristic curvature radius of the shell median surface. We stem from the following basic concept: in the static case, the general stress-strain state of the three-dimensional thin body forming the shell comprises the internal stress-strain state of the entire shell and the stress-strain state of boundary layers near the shell end face Σ. The construction of a general two-dimensional applied model of micropolar elastic thin shells is closely linked with the construction of an internal problem.

Considering that the method of hypotheses is extraordinary clear and is fast and rather easy in providing final results, we use the method to develop a theory of micropolar shells. The hypotheses as such are formulated reasoning from the result of asymptotic analysis of the three-dimensional micropolar elastic boundary problem in a thin shell region [32, 33].

In determination of the internal (as well as boundary) stress-strain state of a shell [32, 33], an important role belongs to physical constants of the shell material. Therefore, we introduce the following dimensionless physical parameters:

$$\frac{\mu}{\alpha}, \frac{R^2\mu}{\beta}, \frac{R^2\mu}{\gamma}, \frac{R^2\mu}{\varepsilon}, \tag{7}$$

where R is the scale factor or the characteristic curvature radius of the shell median surface.

3. Model of micropolar elastic thin shells with independent displacement and rotations fields

Taking into account qualitative data of asymptotic solution of the system of equations (1)–(3) with the boundary conditions specified above and asymptotic integration of the boundary problem [32, 33] for the values of dimensionless physical parameters (7):

$$\frac{\mu}{\alpha} \sim 1, \frac{R^2\mu}{\beta} \sim 1, \ \frac{R^2\mu}{\gamma} \sim 1, \ \frac{R^2\mu}{\varepsilon} \sim 1, \tag{8}$$

we can put the following rather general assumptions (hypotheses) on which to base the theory of micropolar elastic thin shells.

1. Under deformation, initially straight fibers normal to the shell median surface rotate freely as a unity in space through a certain angle, keeping their length constant but changing the perpendicular orientation about the deformed median surface.

This hypothesis can be represented in the mathematical form: the tangential displacements and normal rotation are distributed within the shell thickness by the linear law:

$$V_i = u_i(\alpha_1,\alpha_2) + \alpha_3 \psi_i(\alpha_1,\alpha_2), \qquad \omega_3 = \Omega_3(\alpha_1,\alpha_2) + \alpha_3 \iota(\alpha_1,\alpha_2), \tag{9}$$

and the normal displacement and tangential rotations are independent of the traverse coordinate α_3, i.e.,

$$V_3 = w(\alpha_1,\alpha_2), \ \omega_i = \Omega_i(\alpha_1,\alpha_2). \tag{10}$$

Note that in terms of displacements, accepted hypothesis (9), (10) is in essence coincident with the Timoshenko kinematic hypothesis in the classical theory of elastic shells [38, 39]. Let hypothesis (9), (10) as a whole be named a generalized Timoshenko kinematic hypothesis of the theory of micropolar shells.

2. The force stress σ_{33} in generalized Hooke's law (2) can be neglected with respect to the force stress σ_{ii}.

3. For determination of strain, bending-torsion and force and couple stresses, the force stress σ_{3i} and couple stress μ_{33} are first taken as

$$\sigma_{3i} = \overset{0}{\sigma}_{3i}(\alpha_1, \alpha_2), \quad \mu_{33} = \overset{0}{\mu}_{33}(\alpha_1, \alpha_2). \tag{11}$$

Once the specified quantities are calculated, we can more accurately determine the values of σ_{3i} and μ_{33} by adding (11) with terms derived from integration of the first two or sixth equilibrium equation from (1) for which the averages over the shell thickness are required to be zero.

4. The quantities α_3/R_i compared to unity can be ignored.

Static hypothesis 3 differs from the corresponding Timoshenko hypothesis [38, 39]. Running ahead, we should say that the applied two-dimensional theory of micropolar shells based on hypotheses 1–4 will be an asymptotically exact theory. Formulated assumptions 1–4 provide a possibility to completely allow for shear strain and related strain in the theory of micropolar elastic thin shells.

According to generalized Timoshenko kinematic hypothesis (9), (10), equations (3) give the following expressions for the strain and bending-torsion tensor components:

$$\begin{aligned} &\gamma_{ii} = \Gamma_{ii}(\alpha_1, \alpha_2) + \alpha_3 K_{ii}(\alpha_1, \alpha_2), \\ &\gamma_{ij} = \Gamma_{ij}(\alpha_1, \alpha_2) + \alpha_3 K_{ij}(\alpha_1, \alpha_2), \\ &\gamma_{i3} = \Gamma_{i3}(\alpha_1, \alpha_2), \ \gamma_{3i} = \Gamma_{3i}(\alpha_1, \alpha_2), \end{aligned} \tag{12}$$

$$\begin{aligned} &\chi_{ii} = \kappa_{ii}(\alpha_1, \alpha_2), \ \chi_{ij} = \kappa_{ij}(\alpha_1, \alpha_2), \\ &\chi_{i3} = \kappa_{i3}(\alpha_1, \alpha_2) + \alpha_3 l_{i3}(\alpha_1, \alpha_2), \\ &\chi_{33} = \kappa_{33}(\alpha_1, \alpha_2), \ \gamma_{33} = 0, \ \chi_{3i} = 0, \end{aligned} \tag{13}$$

where

$$\begin{aligned} &\Gamma_{ii} = \frac{1}{A_i}\frac{\partial u_i}{\partial \alpha_i} + \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j} u_j + \frac{w}{R_i}, \\ &\Gamma_{ij} = \frac{1}{A_i}\frac{\partial u_j}{\partial \alpha_i} - \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j} u_i - (-1)^j \Omega_3, \\ &K_{ii} = \frac{1}{A_i}\frac{\partial \psi_i}{\partial \alpha_i} + \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\psi_j, \\ &K_{ij} = \frac{1}{A_i}\frac{\partial \psi_j}{\partial \alpha_i} - \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\psi_i - (-1)^j \iota, \\ &\Gamma_{i3} = -\vartheta_i + (-1)^j \Omega_j, \ \ \Gamma_{3i} = \psi_i - (-1)^j \Omega_j, \\ &\vartheta_i = -\frac{1}{A_i}\frac{\partial w}{\partial \alpha_i} + \frac{u_i}{R_i}, \end{aligned} \tag{14}$$

$$\begin{aligned} &\kappa_{ii} = \frac{1}{A_i}\frac{\partial \Omega_i}{\partial \alpha_i} + \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\Omega_j + \frac{\Omega_3}{R_i}, \\ &\kappa_{ij} = \frac{1}{A_i}\frac{\partial \Omega_j}{\partial \alpha_i} - \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\Omega_i, \\ &\kappa_{i3} = \frac{1}{A_i}\frac{\partial \Omega_3}{\partial \alpha_i} - \frac{\Omega_i}{R_i}, \ \ l_{i3} = \frac{1}{A_i}\frac{\partial \iota}{\partial \alpha_i}. \end{aligned} \tag{15}$$

Next, from generalized Hooke's law (2), equilibrium equations (1), and accepted hypotheses follows the constitutive formulae for the force and couple stress tensor components:

$$\begin{aligned} &\sigma_{ii} = \frac{E}{1-\nu^2}(\Gamma_{ii} + \nu\Gamma_{jj}) + \alpha_3 \frac{E}{1-\nu^2}(K_{ii} + \nu K_{jj}), \\ &\sigma_{ii} = [(\mu+\alpha)\Gamma_{ij} + (\mu-\alpha)\Gamma_{ji}] + \\ &\quad + \alpha_3[(\mu+\alpha)K_{ij} + (\mu-\alpha)K_{ji}], \\ &\sigma_{i3} = (\mu+\alpha)\Gamma_{i3} + (\mu-\alpha)\Gamma_{3i}, \\ &\sigma_{33} = \frac{q_3^+ - q_3^-}{2} + \frac{\alpha_3}{2h}(q_3^+ + q_3^-), \\ &\sigma_{3i} = \overset{0}{\sigma}_{3i}(\alpha_1,\alpha_2) + \\ &+ \alpha_3\left\{-\frac{1}{A_i A_j}\left[\frac{\partial\left(A_j \overset{0}{\sigma}_{ii}\right)}{\partial \alpha_i} + \frac{\partial\left(A_i \overset{0}{\sigma}_{ji}\right)}{\partial \alpha_j}\right] + \right. \\ &\left. + \frac{1}{A_i A_j}\frac{\partial A_j}{\partial \alpha_i}\overset{0}{\sigma}_{jj} - \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\overset{0}{\sigma}_{ij} - \frac{\sigma_{i3}}{R_i}\right\} + \\ &+ \left(\frac{\alpha_3^2}{2} - \frac{h^2}{6}\right)\left\{-\frac{1}{A_i A_j}\left[\frac{\partial\left(A_j \overset{1}{\sigma}_{ii}\right)}{\partial \alpha_i} + \frac{\partial\left(A_i \overset{1}{\sigma}_{ji}\right)}{\partial \alpha_j}\right] + \right. \\ &\left. + \frac{1}{A_i A_j}\frac{\partial A_j}{\partial \alpha_i}\overset{1}{\sigma}_{jj} - \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\overset{1}{\sigma}_{ij}\right\}, \end{aligned} \tag{16}$$

$$\begin{aligned} &\mu_{ii} = \frac{4\gamma(\beta+\gamma)}{\beta+2\gamma}\kappa_{ii} + \frac{2\gamma\beta}{\beta+2\gamma}\kappa_{jj} + \frac{\beta}{\beta+2\gamma}\overset{0}{\mu}_{33}, \\ &\mu_{ij} = (\gamma+\varepsilon)\kappa_{ij} + (\gamma-\varepsilon)\kappa_{ji}, \\ &\iota = \frac{\beta+\gamma}{\gamma(3\beta+2\gamma)}\overset{0}{\mu}_{33} - \frac{\beta}{2\gamma(3\beta+2\gamma)}(\mu_{11}+\mu_{22}), \\ &\mu_{3i} = \frac{m_i^+ - m_i^-}{2} + \frac{\alpha_3}{2h}(m_i^+ + m_i^-), \\ &\mu_{33} = \overset{0}{\mu}_{33}(\alpha_1,\alpha_2) + \\ &+ \alpha_3\left\{-\frac{1}{A_1 A_2}\left(\frac{\partial(A_2 \overset{0}{\mu}_{13})}{\partial \alpha_1} + \frac{\partial(A_1 \overset{0}{\mu}_{23})}{\partial \alpha_2}\right) + \right. \\ &\left. + \left(\frac{\mu_{11}}{R_1} + \frac{\mu_{22}}{R_2}\right) - (\overset{0}{\sigma}_{12} - \overset{0}{\sigma}_{21})\right\} + \left(\frac{\alpha_3^2}{2} - \frac{h^2}{6}\right) \times \\ &\times \left\{-\frac{1}{A_1 A_2}\left(\frac{\partial(A_2 \overset{1}{\mu}_{13})}{\partial \alpha_1} + \frac{\partial(A_1 \overset{1}{\mu}_{23})}{\partial \alpha_2}\right) - (\overset{1}{\sigma}_{12} - \overset{1}{\sigma}_{21})\right\}, \end{aligned} \tag{17}$$

$$\mu_{i3}=\left[\frac{4\gamma\varepsilon}{\gamma+\varepsilon}\kappa_{i3}+\frac{\gamma-\varepsilon}{\gamma+\varepsilon}\frac{m_i^+-m_i^-}{2}\right]+$$
$$+\alpha_3\left[\frac{4\gamma\varepsilon}{\gamma+\varepsilon}l_{i3}+\frac{\gamma-\varepsilon}{\gamma+\varepsilon}\frac{m_i^++m_i^-}{2h}\right].$$

Here, $\overset{0}{\sigma}_{ii}, \overset{0}{\sigma}_{ij}, \overset{0}{\mu}_{i3}, \overset{1}{\sigma}_{ii}, \overset{1}{\sigma}_{ij}, \overset{1}{\mu}_{i3}$ are respectively the constant and linear-in-α_3 parts of the force stresses σ_{ii}, σ_{ij} and couple stresses μ_{i3}.

With the aim to reduce the three-dimensional micropolar elastic problem to the two-dimensional one, which is already done for displacements, strains, bending-torsions and force and couple stresses, we introduce, instead of the force and couple stress tensor components, their statically equivalent integral characteristics — forces $T_{ii}, S_{ij}, N_{i3}, N_{3i}$, moments $M_{ii}, H_{ij}, L_{ii}, L_{ij}, L_{i3}, L_{33}$ and hypermoments Λ_{i3} which, in view of assumption 4, are expressed as follows:

$$T_{ii}=\int_{-h}^{h}\sigma_{ii}\mathrm{d}\alpha_3,\quad S_{ij}=\int_{-h}^{h}\sigma_{ij}\mathrm{d}\alpha_3,$$
$$N_{i3}=\int_{-h}^{h}\sigma_{i3}\mathrm{d}\alpha_3,\quad N_{3i}=\int_{-h}^{h}\sigma_{3i}\mathrm{d}\alpha_3,$$
$$M_{ii}=\int_{-h}^{h}\alpha_3\sigma_{ii}\mathrm{d}\alpha_3,\quad H_{ij}=\int_{-h}^{h}\alpha_3\sigma_{ij}\mathrm{d}\alpha_3, \tag{18}$$
$$L_{ii}=\int_{-h}^{h}\mu_{ii}\mathrm{d}\alpha_3,\quad L_{ij}=\int_{-h}^{h}\mu_{ij}\mathrm{d}\alpha_3,\quad L_{33}=\int_{-h}^{h}\mu_{33}\mathrm{d}\alpha_3,$$
$$L_{i3}=\int_{-h}^{h}\mu_{i3}\mathrm{d}\alpha_3,\quad \Lambda_{i3}=\int_{-h}^{h}\alpha_3\mu_{i3}\mathrm{d}\alpha_3.$$

The main system of equations of micropolar elastic thin shells with independent displacement and rotation fields has the form:

equilibrium equations:

$$\frac{1}{A_i}\frac{\partial T_{ii}}{\partial\alpha_i}+\frac{1}{A_iA_j}\frac{\partial A_j}{\partial\alpha_i}(T_{ii}-T_{jj})+\frac{1}{A_j}\frac{\partial S_{ji}}{\partial\alpha_j}+$$
$$+\frac{1}{A_iA_j}\frac{\partial A_i}{\partial\alpha_j}(S_{ji}+S_{ij})+\frac{N_{i3}}{R_i}=-(q_i^++q_i^-),$$
$$\frac{1}{A_i}\frac{\partial M_{ii}}{\partial\alpha_i}+\frac{1}{A_iA_j}\frac{\partial A_j}{\partial\alpha_i}(M_{ii}-M_{jj})+\frac{1}{A_j}\frac{\partial H_{ji}}{\partial\alpha_j}+$$
$$+\frac{1}{A_iA_j}\frac{\partial A_i}{\partial\alpha_j}(H_{ji}+H_{ij})-N_{3i}=-h(q_i^+-q_i^-), \tag{19}$$
$$\frac{T_{11}}{R_1}+\frac{T_{22}}{R_2}-\frac{1}{A_1A_2}\left[\frac{\partial(A_2N_{13})}{\partial\alpha_1}+\frac{\partial(A_1N_{23})}{\partial\alpha_2}\right]=q_3^++q_3^-,$$
$$\frac{1}{A_i}\frac{\partial L_{ii}}{\partial\alpha_i}+\frac{1}{A_iA_j}\frac{\partial A_j}{\partial\alpha_i}(L_{ii}-L_{jj})+\frac{1}{A_j}\frac{\partial L_{ji}}{\partial\alpha_j}+$$
$$+\frac{1}{A_iA_j}\frac{\partial A_i}{\partial\alpha_j}(L_{ji}+L_{ij})+\frac{L_{i3}}{R_i}+$$
$$+(-1)^j(N_{j3}-N_{3j})=-(m_i^++m_i^-),$$

$$\frac{L_{11}}{R_1}+\frac{L_{22}}{R_2}-\frac{1}{A_1A_2}\left[\frac{\partial(A_2L_{13})}{\partial\alpha_1}+\frac{\partial(A_1L_{23})}{\partial\alpha_2}\right]-$$
$$-(S_{12}-S_{21})=m_3^++m_3^-, \tag{20}$$
$$L_{33}-\frac{1}{A_1A_2}\left[\frac{\partial(A_2\Lambda_{13})}{\partial\alpha_1}+\frac{\partial(A_1\Lambda_{23})}{\partial\alpha_2}\right]-$$
$$-(H_{12}-H_{21})=h(m_3^+-m_3^-),$$

physical elasticity relations:

$$T_{ii}=\frac{2Eh}{1-\nu^2}(\Gamma_{ii}+\nu\Gamma_{jj}),$$
$$S_{ij}=2h[(\mu+\alpha)\Gamma_{ij}+(\mu-\alpha)\Gamma_{ji}],$$
$$M_{ii}=\frac{2Eh^3}{3(1-\nu^2)}(K_{ii}+\nu K_{jj}), \tag{21}$$
$$H_{ij}=\frac{2h^3}{3}[(\mu+\alpha)K_{ij}+(\mu-\alpha)K_{ji}],$$
$$N_{i3}=2h(\mu+\alpha)\Gamma_{i3}+2h(\mu-\alpha)\Gamma_{3i},$$
$$N_{3i}=2h(\mu+\alpha)\Gamma_{3i}+2h(\mu-\alpha)\Gamma_{i3},$$
$$L_{ii}=2h\left[\frac{4\gamma(\beta+\gamma)}{\beta+2\gamma}\kappa_{ii}+\frac{2\gamma\beta}{\beta+2\gamma}\kappa_{jj}\right]+\frac{\beta}{\beta+2\gamma}L_{33},$$
$$L_{ij}=2h[(\gamma+\varepsilon)\kappa_{ij}+(\gamma-\varepsilon)\kappa_{ji}],$$
$$L_{33}=2h[(\beta+2\gamma)\iota+\beta(\kappa_{11}+\kappa_{22})], \tag{22}$$
$$L_{i3}=2h\left[\frac{4\gamma\varepsilon}{\gamma+\varepsilon}\kappa_{i3}+\frac{\gamma-\varepsilon}{\gamma+\varepsilon}\frac{m_i^+-m_i^-}{2}\right],$$
$$\Lambda_{i3}=\frac{2h^3}{3}\left[\frac{4\gamma\varepsilon}{\gamma+\varepsilon}l_{i3}+\frac{\gamma-\varepsilon}{\gamma+\varepsilon}\frac{m_i^++m_i^-}{2h}\right].$$

To equilibrium equations (19), (20) and elasticity relations (21), (22) of micropolar shells we should add geometric relations (14), (15).

Let us represent "mitigated" boundary conditions on the boundary contour Γ of the shell median surface assuming that this contour coincides with the coordinate line $\alpha_1=\text{const}$:

$$T_{11}=T_{11}^*\ \text{or}\ u_1=u_1^*,\ S_{12}=S_{12}^*\ \text{or}\ u_2=u_2^*,$$
$$N_{13}=N_{13}^*\ \text{or}\ w=w^*, \tag{23}$$
$$M_{11}=M_{11}^*\ \text{or}\ K_{11}=K_{11}^*,\ H_{12}=H_{12}^*\ \text{or}\ K_{12}=K_{12}^*,$$
$$L_{11}=L_{11}^*\ \text{or}\ \kappa_{11}=\kappa_{11}^*,\ L_{12}=L_{12}^*\ \text{or}\ \kappa_{12}=\kappa_{12}^*, \tag{24}$$
$$L_{13}=L_{13}^*\ \text{or}\ \kappa_{13}=\kappa_{13}^*,\ \Lambda_{13}=\Lambda_{13}^*\ \text{or}\ l_{13}=l_{13}^*.$$

The system of equations (14), (15), (19)–(22) of micropolar elastic thin shells with independent displacement and rotation fields is a system of 18th-order differential equations with nine boundary conditions (23), (24) on each of the contours Γ of the shell median surface. The system consists of 52 equations in 52 unknown functions: $u_i, w, \psi_i, \Omega_i, \Omega_3, \iota, \vartheta_i, T_{ii}, S_{ij}, N_{i3}, N_{3i}, M_{ii}, H_{ij}, L_{ii}, L_{ij}, L_{33}, L_{i3}, \Lambda_{i3}, \Gamma_{ii}, \Gamma_{ij}, \Gamma_{i3}, \Gamma_{3i}, K_{ii}, K_{ij}, \kappa_{ii}, \kappa_{ij}, \kappa_{i3}, l_{i3}$.

Model (14), (15), (19)–(24) of micropolar elastic thin shells with independent displacement and rotation fields takes into complete account transverse shear strains and related strains.

If we formally take $\alpha = 0$ in model (14), (15), (19)–(24), we obtain the system of equations and boundary conditions of the classical theory of elastic shells with the Timoshenko hypothesis [38, 39] (surely, somewhat different due to static hypothesis 3).

If we neglect the transverse shear in model (14), (15), (19)–(24), i.e., if we put

$$\Gamma_{i3} + \Gamma_{3i} = 0 \text{ or } \psi_i = \vartheta_i, \tag{25}$$

we obtain a model of micropolar elastic thin shells with independent displacement and rotation fields in which the displacements are considered not from the Timoshenko hypothesis but from the Kirchhoff–Love classical hypothesis, namely:

$$V_i = u_i(\alpha_1, \alpha_2) + \alpha_3 \vartheta_i(\alpha_1, \alpha_2), \quad V_3 = w(\alpha_1, \alpha_2), \tag{26}$$

that should further be added with conditions for free rotations ω_i and ω_3 from formulae (9), (10):

$$\omega_i = \Omega_i(\alpha_1, \alpha_2), \quad \omega_3 = \Omega_3(\alpha_1, \alpha_2) + \alpha_3 \iota(\alpha_1, \alpha_2). \tag{27}$$

Assumptions (26), (27) as a whole will be termed a generalized Kirchhoff–Love kinematic hypothesis of the theory of micropolar shells.

The constitutive equations of the model of micropolar elastic shells with independent displacement and rotation fields for the generalized Kirchhoff–Love kinematic hypothesis (leaving assumptions 2–4 intact) are expressed as equilibrium equations (19), (20) to which we should add physical elasticity relations:

$$T_{ii} = \frac{2Eh}{1-\nu^2}[\Gamma_{ii} + \nu\Gamma_{jj}], \quad S_{ij} = 2h[(\mu+\alpha)\Gamma_{ij} + (\mu-\alpha)\Gamma_{ji}],$$
$$M_{ii} = \frac{2Eh^3}{3(1-\nu^2)}(K_{ii} + \nu K_{jj}), \tag{28}$$
$$H_{ij} = \frac{2h^3}{3}[(\mu+\alpha)K_{ij} + (\mu-\alpha)K_{ji}],$$

$$N_{i3} - N_{3i} = 4\alpha h(\Gamma_{i3} - \Gamma_{3i}),$$
$$L_{ii} = 2h\left[\frac{4\gamma(\beta+\gamma)}{\beta+2\gamma}\kappa_{ii} + \frac{2\gamma\beta}{\beta+2\gamma}\kappa_{jj}\right] + \frac{\beta}{\beta+2\gamma}L_{33},$$
$$L_{ij} = 2h[(\gamma+\varepsilon)\kappa_{ij} + (\gamma-\varepsilon)\kappa_{ji}],$$
$$L_{33} = 2h[(\beta+2\gamma)\iota + \beta(\kappa_{11} + \kappa_{22})], \tag{29}$$
$$L_{i3} = 2h\left[\frac{4\gamma\varepsilon}{\gamma+\varepsilon}\kappa_{i3} + \frac{\gamma-\varepsilon}{\gamma+\varepsilon}\frac{m_i^+ - m_i^-}{2}\right],$$
$$\Lambda_{i3} = \frac{2h^3}{3}\left[\frac{4\gamma\varepsilon}{\gamma+\varepsilon}l_{i3} + \frac{\gamma-\varepsilon}{\gamma+\varepsilon}\frac{m_i^+ + m_i^-}{2h}\right],$$

geometric relations:

$$\Gamma_{ii} = \frac{1}{A_i}\frac{\partial u_i}{\partial \alpha_i} + \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}u_j + \frac{w}{R_i},$$
$$\Gamma_{ij} = \frac{1}{A_i}\frac{\partial u_j}{\partial \alpha_i} - \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}u_i - (-1)^j\Omega_3,$$
$$K_{ii} = \frac{1}{A_i}\frac{\partial \vartheta_i}{\partial \alpha_i} + \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\vartheta_j, \tag{30}$$
$$K_{ij} = \frac{1}{A_i}\frac{\partial \vartheta_j}{\partial \alpha_i} - \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\vartheta_i - (-1)^j\iota,$$
$$\vartheta_i = -\frac{1}{A_i}\frac{\partial w}{\partial \alpha_i} + \frac{u_i}{R_i},$$

$$\Gamma_{i3} - \Gamma_{3i} = 2[-\vartheta_i + (-1)^j\Omega_j],$$
$$\kappa_{ii} = \frac{1}{A_i}\frac{\partial \Omega_i}{\partial \alpha_i} + \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\Omega_j + \frac{\Omega_3}{R_i},$$
$$\kappa_{ij} = \frac{1}{A_i}\frac{\partial \Omega_j}{\partial \alpha_i} - \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\Omega_i, \tag{31}$$
$$\kappa_{i3} = \frac{1}{A_i}\frac{\partial \Omega_3}{\partial \alpha_i} - \frac{\Omega_i}{R_i}, \quad l_{i3} = \frac{1}{A_i}\frac{\partial \iota}{\partial \alpha_i},$$

boundary conditions (at $\alpha_1 = \alpha_{10}$):

$$T_{11} = T_{11}^* \text{ or } u_1 = u_1^*,$$
$$S_{12} + \frac{H_{12}}{R_2} = S_{12}^* \text{ or } u_2 = u_2^*,$$
$$N_{13} + \frac{1}{A_2}\frac{\partial H_{12}}{\partial \alpha_2} = N_{13}^* \text{ or } w = w^*, \tag{32}$$
$$M_{11} = M_{11}^* \text{ or } K_{11} = K_{11}^*,$$

$$L_{11} = L_{11}^* \text{ or } \kappa_{11} = \kappa_{11}^*,$$
$$L_{12} = L_{12}^* \text{ or } \kappa_{12} = \kappa_{12}^*,$$
$$L_{13} = L_{13}^* \text{ or } \kappa_{13} = \kappa_{13}^*, \tag{33}$$
$$\Lambda_{13} = \Lambda_{13}^* \text{ or } l_{13} = l_{13}^*.$$

If we formally put $\alpha = 0$ in the system of equations of micropolar shells (19), (20), (28)–(33), this system and boundary conditions gives constitutive equations and boundary conditions of the Kirchhoff–Love classical theory of elastic shells [37].

If equilibrium equations (19), (20) are incorporated, from the D'Alembert principle, with the inertia forces and moments:

$$2\rho h\,\partial^2 u_i/\partial t^2,\ 2\rho h\,\partial^2 w/\partial t^2,\ 2\rho h^3/3\,\partial^2\psi_i/\partial t^2,$$
$$2Jh\,\partial^2\Omega_i/\partial t^2,\ 2Jh\,\partial^2\Omega_3/\partial t^2,\ 2Jh^3/3\,\partial^2\iota/\partial t^2,$$

we obtain, using (14), (15), (19)–(24) and (19), (20), (28)–(33), general dynamic models of micropolar shells with independent displacement and rotation fields with and with no account of traverse shear.

The general theory of micropolar shells with independent displacement and rotation fields (14), (15), (19)–(24) (with regard for traverse shear) and (19), (20), (28)–(33) (without regard for transverse shear), assuming the curvilinear coordinates α_1, α_2 in the form [37]:

$\alpha_1 = R\xi,\ \alpha_2 = R\theta$

and hence putting for A_1, A_2

$A_1 = A_2 = R,$

gives corresponding models of micropolar circular cylindrical shells (R is the radius of the shell median surface).

Let us consider the axisymmetric stress-strain state problem of a hinged cylindrical micropolar elastic shell under a normally distributed force load of intensity $p = p_0 \times \times \sin(\pi x/l)$, where l is the shell length.

In this consideration we use the theory of micropolar shells with free rotation taking into account traverse shear (equations (14), (15), (19)–(22), boundary conditions (23), (24) for hinging) and neglecting traverse shear (equation (19), (20), (28)–(31), boundary conditions (32), (33) for hinging). For the above problems of a cylindrical micropolar elastic shell, exact solutions were derived and were used in numerical analysis. Table 1 presents the results of numerical calculation (note that micropolar materials, e.g., artificial bones [11, 12], are now available for which elastic constants were determined; however, the shell material chosen for calculations is here hypothetic).

It is seen from the data presented in Table 1 that the micropolar elastic shell material (both with and with no account of transverse shear in the micropolar theory with free rotation) displays very high strength and rigidity characteristics. The numerical results in Table 1 also show that account of transverse shear strain is very important for the theory of micropolar shells with free rotation. This means that one should be very careful in applying the theory of micropolar shells with free rotation based in the generalized Kirchhoff–Love kinematic hypotheses.

4. Model of micropolar elastic thin shells with constrained rotation

Let us consider the case where dimensionless physical parameters (7) take on the values:

$$\alpha \gg \mu,\ \frac{R^2\mu}{\beta} \sim 1,\ \frac{R^2\mu}{\gamma} \sim 1,\ \frac{R^2\mu}{\varepsilon} \sim 1. \quad (34)$$

The asymptotic analysis [32, 33] of boundary problem (1)–(6) for (34) shows that the asymptotic approximations of the rotation vector $\boldsymbol{\omega}$ are related to the approximations of the displacement vector $\mathbf{V}$ like in the classical theory of elasticity:

$$\boldsymbol{\omega} = \frac{1}{2}\mathrm{rot}\mathbf{V}. \quad (35)$$

This means that the constructed two-dimensional theory of micropolar shells lies in the domain of the micropolar theory with constrained rotation (or, in other words, Cosserat pseudocontinuum [40, 41]). It is readily seen from formula (6) of the general micropolar theory of elasticity that condition (35) is fulfilled when the physical material constant α is very high: $\alpha \to \infty$ [36, 40] (the first expression in (34) makes sure that the given condition, in this case, holds true).

The micropolar theory with constrained rotation has certain peculiarities [41]. If the displacement vector $\mathbf{V}$ is a vector differentiable a required number of times, constrained

Table 1

Strength and rigidity characteristics of a cylindrical micropolar elastic shell.
Model with independent displacement and rotation fields (with and with no account of traverse shear)

Shell dimensions			Generalized Timoshenko hypotheses		Generalized Kirchhoff–Love hypotheses	
R, mm	h, mm	l, mm	$\frac{\sigma_{11\max}^{\text{mic}}}{\sigma_{11\max}^{\text{cl}}}$	$\frac{w_{\max}^{\text{mic}}}{w_{\max}^{\text{cl}}}$	$\frac{\sigma_{11\max}^{\text{mic}}}{\sigma_{11\max}^{\text{cl}}}$	$\frac{w_{\max}^{\text{mic}}}{w_{\max}^{\text{cl}}}$
$\delta = h/R = 1/40$						
8	0.08	16	0.04525	0.374775	0.24424	0.24424
20	0.2	40	0.06515	0.387809	0.22892	0.22892
50	0.5	100	0.17292	0.458377	0.18672	0.18672
80	0.8	160	0.31875	0.553881	0.16531	0.16531
100	1	200	0.41412	0.616329	0.15805	0.15805
200	2	400	0.72957	0.822908	0.14612	0.14612
$\delta = h/R = 1/100$						
8	0.2	16	0.04533	0.375211	0.2446	0.2446
20	0.5	40	0.06526	0.388257	0.22925	0.22925
50	1.25	100	0.17317	0.458878	0.18698	0.18698
80	2	160	0.31914	0.554409	0.16554	0.16554
100	2.5	200	0.41455	0.616849	0.15827	0.15827
200	5	400	0.72992	0.823247	0.14632	0.14632

Note. The physical properties of the shell material: α = 1.6 MPa, μ = 2 MPa, λ = 3 MPa, $\gamma = \varepsilon$ = 3 kN, the load intensity p_0 = = 100 Pa.

rotation condition (35) gives the identity:

$$\operatorname{div}\boldsymbol{\omega} = 0.$$

This means that the first invariant of the bending-torsion tensor is equal to zero:

$$\chi_{11} + \chi_{22} + \chi_{33} = 0. \quad (36)$$

From formulae (2) for the first invariant of the couple stress tensor follows:

$$\mu_{11} + \mu_{22} + \mu_{33} = (3\beta + 2\gamma)(\chi_{11} + \chi_{22} + \chi_{33}). \quad (37)$$

In view of identity (36), we obtain:

$$\mu_{11} + \mu_{22} + \mu_{33} = 0; \quad (38)$$

whence it follows that

$$\mu_{33} = -(\mu_{11} + \mu_{22}). \quad (39)$$

Formula (39) means that the couple stress μ_{33} is not an independent function in the sense that for this quantity it is impossible to specify arbitrary boundary conditions on the shell front face $\alpha_3 = \pm h$ (i.e., of six boundary conditions on the shell front face $\alpha_3 = \pm h$ (see (4)), there are five left).

Moreover, as follows from (37), it makes no difference to the theory with constrained rotation what the value of the sum $3\beta + 2\gamma$ is (it can be stated that for this theory, the value of the physical constant β is immaterial). Thus, the micropolar theory with constrained rotation is determined by four elastic constants: λ, μ (or E, ν), γ, ε (or l, η [41]).

Now we turn to construction of a model of micropolar elastic thin shells with constrained rotation. On the grounds of asymptotic integration data for boundary problem (1)–(6) [32, 33] with the dimensionless physical properties of values (34), we take the following assumptions (hypothesis) for construction of the general applied two-dimensional theory of micropolar shells with constrained rotation:

1) assumptions 1–4 made in the previous section (assumption 3, in this case, should be applied only to the force stress σ_{3i}),

2) constrained rotation condition (35).

According to kinematic hypothesis (9), (10) and on the grounds of hypotheses accepted in this case, the formulae for the strain, bending-torsion and force and couple stress tensor components are the same as formulae (12)–(17), except that in formulae (17) the values for μ_{ii} and μ_{33} should be replaced by the simple expressions:

$$\mu_{ii} = 2\gamma\kappa_{ii}, \quad \mu_{33} = 2\gamma\iota. \quad (40)$$

The basic system of equations of the general applied two-dimensional theory of micropolar elastic thin shells with constrained rotation taking account of traverse shear strain and related strain has the form:

equilibrium equations:

$$\frac{1}{A_i}\frac{\partial T_{ii}}{\partial \alpha_i} + \frac{1}{A_i A_j}\frac{\partial A_j}{\partial \alpha_i}(T_{ii} - T_{jj}) + \frac{1}{A_j}\frac{\partial S_{ji}}{\partial \alpha_j} + \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}(S_{ji} + S_{ij}) + \frac{N_{i3}}{R_i} = -(q_i^+ + q_i^-),$$

$$\frac{1}{A_i}\frac{\partial M_{ii}}{\partial \alpha_i} + \frac{1}{A_i A_j}\frac{\partial A_j}{\partial \alpha_i}(M_{ii} - M_{jj}) + \frac{1}{A_j}\frac{\partial H_{ji}}{\partial \alpha_j} + \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}(H_{ji} + H_{ij}) - N_{3i} = -h(q_i^+ - q_i^-), \quad (41)$$

$$\frac{T_{11}}{R_1} + \frac{T_{22}}{R_2} - \frac{1}{A_1 A_2}\left[\frac{\partial (A_2 N_{13})}{\partial \alpha_1} + \frac{\partial (A_1 N_{23})}{\partial \alpha_2}\right] = q_3^+ + q_3^-,$$

$$\frac{1}{A_i}\frac{\partial L_{ii}}{\partial \alpha_i} + \frac{1}{A_i A_j}\frac{\partial A_j}{\partial \alpha_i}(L_{ii} - L_{jj}) + \frac{1}{A_j}\frac{\partial L_{ji}}{\partial \alpha_j} + \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}(L_{ji} + L_{ij}) + \frac{L_{i3}}{R_i} + (-1)^j (N_{j3} - N_{3j}) = -(m_i^+ + m_i^-),$$

$$\frac{L_{11}}{R_1} + \frac{L_{22}}{R_2} - \frac{1}{A_1 A_2}\left[\frac{\partial (A_2 L_{13})}{\partial \alpha_1} + \frac{\partial (A_1 L_{23})}{\partial \alpha_2}\right] - (S_{12} - S_{21}) = 0, \quad (42)$$

$$\frac{1}{A_1 A_2}\left[\frac{\partial (A_2 \Lambda_{13})}{\partial \alpha_1} + \frac{\partial (A_1 \Lambda_{23})}{\partial \alpha_2}\right] + H_{12} - H_{21} = 0,$$

physical relations:

$$T_{ii} = \frac{2Eh}{1-\nu^2}(\Gamma_{ii} + \nu\Gamma_{jj}),$$

$$M_{ii} = \frac{2Eh^3}{3(1-\nu^2)}(K_{ii} + \nu K_{jj}),$$

$$S_{12} + S_{21} = 4\mu h(\Gamma_{12} + \Gamma_{21}), \quad (43)$$

$$N_{i3} + N_{3i} = 4\mu h(\Gamma_{i3} + \Gamma_{3i}),$$

$$H_{12} + H_{21} = \frac{2h^3}{3} 2\mu (K_{12} + K_{21}),$$

$$L_{ii} = 4\gamma h \kappa_{ii},$$

$$L_{ij} = 2h[(\gamma + \varepsilon)\kappa_{ij} + (\gamma - \varepsilon)\kappa_{ji}],$$

$$L_{i3} = 2h\left[\frac{4\gamma\varepsilon}{\gamma + \varepsilon}\kappa_{i3} + \frac{\gamma - \varepsilon}{\gamma + \varepsilon}\frac{m_i^+ - m_i^-}{2}\right], \quad (44)$$

$$\Lambda_{i3} = \frac{2h^3}{3}\left[\frac{4\gamma\varepsilon}{\gamma + \varepsilon} l_{i3} + \frac{\gamma - \varepsilon}{\gamma + \varepsilon}\frac{m_i^+ + m_i^-}{2h}\right],$$

geometric relations:

$$\Gamma_{ii} = \frac{1}{A_i}\frac{\partial u_i}{\partial \alpha_i} + \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j} u_j + \frac{w}{R_i},$$

$$\Gamma_{ij} = \frac{1}{A_i}\frac{\partial u_j}{\partial \alpha_i} - \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j} u_i,$$

$$K_{ii} = \frac{1}{A_i}\frac{\partial \psi_i}{\partial \alpha_i} + \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\psi_j, \quad (45)$$

$$K_{ij} = \frac{1}{A_i}\frac{\partial \psi_j}{\partial \alpha_i} - \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\psi_i,$$

$$\Gamma_{i3} = -\vartheta_i + (-1)^j \Omega_j, \quad \Gamma_{3i} = \psi_i - (-1)^j \Omega_j,$$

$$\vartheta_i = -\frac{1}{A_i}\frac{\partial w}{\partial \alpha_i} + \frac{u_i}{R_i},$$

$$\kappa_{ii} = \frac{1}{A_i}\frac{\partial \Omega_i}{\partial \alpha_i} + \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\Omega_j + \frac{\Omega_3}{R_i},$$
$$\kappa_{ij} = \frac{1}{A_i}\frac{\partial \Omega_j}{\partial \alpha_i} - \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\Omega_i,$$
$$\kappa_{i3} = \frac{1}{A_i}\frac{\partial \Omega_3}{\partial \alpha_i} - \frac{\Omega_i}{R_i}, \quad l_{i3} = \frac{1}{A_i}\frac{\partial \iota}{\partial \alpha_i}, \tag{46}$$
$$\Omega_i = -(-1)^j \frac{1}{2}(\psi_j + \vartheta_j), \quad \Omega_3 = \frac{1}{2}(\Gamma_{12} - \Gamma_{21}),$$
$$\iota = \frac{1}{2}(K_{12} - K_{21}).$$

To the system of equations (41)–(46) of the general applied two-dimensional theory of micropolar shells with constrained rotation we add "mitigated" boundary conditions (23), (24) on the boundary contour Γ of the shell median surface.

The system of equations (41)–(46) of the theory of micropolar shells with constrained rotation has the 18th order with nine boundary conditions (23), (24) on each edge of the shell median surface. This system contains 51 equations with 51 unknown functions: $T_{ii}, M_{ii}, S_{ij}, N_{i3}, N_{3i}, H_{ij}, L_{ii}, L_{ij}, L_{i3}, \Lambda_{i3}, \Gamma_{ii}, K_{ii}, \Gamma_{ij}, K_{ij}, \Gamma_{i3}, \Gamma_{3i}, \kappa_{ii}, \kappa_{ij}, \kappa_{i3}, l_{i3}, u_i, w, \psi_i, \vartheta_i, \Omega_i, \Omega_3, \iota$.

If we ignore transverse shear in the system of equations (41)–(46), i.e., we take formula (25), the model of micropolar elastic shells with constrained rotation appears with the generalized Kirchhoff–Love hypotheses rather than with the generalized Timoshenko kinematic hypotheses.

The constitutive equations of this model of micropolar shells with constrained rotation include:
equilibrium equations:

$$\frac{1}{A_1}\frac{\partial(M_{11}+L_{12})}{\partial\alpha_1} + \frac{1}{A_2}\frac{\partial(H_{21}+L_{22})}{\partial\alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}[(M_{11}+L_{12})-(M_{22}-L_{21})] + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}[(H_{21}+L_{22})+(H_{12}-L_{11})] + \frac{L_{23}}{R_2} - N_{13} = -h(q_1^+ - q_1^-) - (m_2^+ + m_2^-),$$
$$\frac{1}{A_1}\frac{\partial(H_{12}-L_{11})}{\partial\alpha_1} + \frac{1}{A_2}\frac{\partial(M_{22}-L_{21})}{\partial\alpha_2} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}[(M_{22}-L_{21})-(M_{11}+L_{12})] + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}[(H_{12}-L_{11})+(H_{21}+L_{22})] - \frac{L_{13}}{R_1} - N_{23} = -h(q_2^+ - q_2^-) + (m_1^+ + m_1^-), \tag{47}$$

physical relations:

$$T_{ii} = \frac{2Eh}{1-\nu^2}(\Gamma_{ii} + \nu\Gamma_{jj}),$$
$$M_{ii} = \frac{2Eh^3}{3(1-\nu^2)}(K_{ii} + \nu K_{jj}),$$
$$S_{12} + S_{21} = 4\mu h(\Gamma_{12} + \Gamma_{21}), \tag{48}$$
$$H_{12} + H_{21} = \frac{2h^3}{3}2\mu(K_{12} + K_{21}),$$

$$L_{ii} = 4\gamma h \kappa_{ii},$$
$$L_{ij} = 2h[(\gamma+\varepsilon)\kappa_{ij} + (\gamma-\varepsilon)\kappa_{ji}],$$
$$L_{i3} = 2h\left[\frac{4\gamma\varepsilon}{\gamma+\varepsilon}\kappa_{i3} + \frac{\gamma-\varepsilon}{\gamma+\varepsilon}\frac{m_i^+ - m_i^-}{2}\right], \tag{49}$$
$$\Lambda_{i3} = \frac{2h^3}{3}\left[\frac{4\gamma\varepsilon}{\gamma+\varepsilon}l_{i3} + \frac{\gamma-\varepsilon}{\gamma+\varepsilon}\frac{m_i^+ + m_i^-}{2h}\right],$$

geometric relations:

$$\Gamma_{ii} = \frac{1}{A_i}\frac{\partial u_i}{\partial\alpha_i} + \frac{1}{A_iA_j}\frac{\partial A_i}{\partial\alpha_j}u_j + \frac{w}{R_i},$$
$$K_{ii} = \frac{1}{A_i}\frac{\partial\vartheta_i}{\partial\alpha_i} + \frac{1}{A_iA_j}\frac{\partial A_i}{\partial\alpha_j}\vartheta_j,$$
$$\Gamma_{ij} = \frac{1}{A_i}\frac{\partial u_j}{\partial\alpha_i} - \frac{1}{A_iA_j}\frac{\partial A_i}{\partial\alpha_j}u_i, \tag{50}$$
$$K_{ij} = \frac{1}{A_i}\frac{\partial\vartheta_j}{\partial\alpha_i} - \frac{1}{A_iA_j}\frac{\partial A_i}{\partial\alpha_j}\vartheta_i,$$

$$\kappa_{ii} = \frac{1}{A_i}\frac{\partial \Omega_i}{\partial \alpha_i} + \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\Omega_j + \frac{\Omega_3}{R_i},$$
$$\kappa_{ij} = \frac{1}{A_i}\frac{\partial \Omega_j}{\partial \alpha_i} - \frac{1}{A_i A_j}\frac{\partial A_i}{\partial \alpha_j}\Omega_i, \tag{51}$$
$$\kappa_{i3} = \frac{1}{A_i}\frac{\partial \Omega_3}{\partial \alpha_i} - \frac{\Omega_i}{R_i}, \quad l_{i3} = \frac{1}{A_i}\frac{\partial \iota}{\partial \alpha_i}, \quad \iota = \frac{1}{2}(K_{12} - K_{21}),$$
$$\Omega_i = -(-1)^j \vartheta_j, \quad \Omega_3 = \frac{1}{2}(\Gamma_{12} - \Gamma_{21}),$$
$$\vartheta_i = -\frac{1}{A_i}\frac{\partial w}{\partial\alpha_i} + \frac{u_i}{R_i}.$$

To the system of equations of micropolar shells with constrained rotation based on generalized Kirchhoff–Love kinematic hypothesis (47)–(49) we should add the boundary conditions on the boundary contour Γ of the shell median surface $(\alpha_1 = \alpha_{10})$:

$$T_{11} = T_{11}^* \text{ or } u_1 = u_1^*, \quad S_{12} + \frac{H_{12} - L_{11}}{R_2} = S_{12}^* \text{ or } u_2 = u_2^*,$$
$$N_{13} + \frac{1}{A_2}\frac{\partial(H_{12} - L_{11})}{\partial\alpha_2} = N_{13}^* \text{ or } w = w^*, \tag{52}$$
$$M_{11} - L_{12} = M_{11}^* \text{ or } K_{11} = K_{11}^*,$$
$$L_{13} = L_{13}^* \text{ or } \kappa_{13} = \kappa_{13}^*, \quad \Lambda_{13} = \Lambda_{13}^* \text{ or } l_{13} = l_{13}^*.$$

If we neglect the moments and hypermoments due to couple stress in the system of equations (41)–(46) and in boundary conditions (23), (24) of the theory of micropolar shells with constrained rotation (with regard for traverse shear) or in the system of equations (47)–(50) and boun-

Table 2

Strength and rigidity characteristics of a cylindrical micropolar elastic shell.
Model with constrained rotation (with and with no account of traverse shear)

Shell dimensions			Generalized Timoshenko hypotheses		Generalized Kirchhoff–Love hypotheses	
R, mm	h, mm	l, mm	$\frac{\sigma_{11\max}^{\text{mic}}}{\sigma_{11\max}^{\text{cl}}}$	$\frac{w_{\max}^{\text{mic}}}{w_{\max}^{\text{cl}}}$	$\frac{\sigma_{11\max}^{\text{mic}}}{\sigma_{11\max}^{\text{cl}}}$	$\frac{w_{\max}^{\text{mic}}}{w_{\max}^{\text{cl}}}$
$\delta = h/R = 1/40$						
60	1.5	120	0.06231	0.38632	0.2675	0.2675
80	2	160	0.20167	0.47753	0.393653	0.393653
90	2.25	180	0.26774	0.52077	0.451054	0.451054
100	2.5	200	0.32974	0.56135	0.503577	0.503577
150	3.75	300	0.56953	0.71828	0.695347	0.695347
200	5	400	0.71318	0.81229	0.802279	0.802279
$\delta = h/R = 1/100$						
60	0.6	120	0.0622	0.38588	0.267271	0.267271
80	0.8	160	0.20138	0.47702	0.393374	0.393374
90	0.9	180	0.26739	0.52025	0.450764	0.450764
100	1	200	0.32935	0.56082	0.503284	0.503284
150	1.5	300	0.56909	0.71782	0.695099	0.695099
200	2	400	0.71282	0.81194	0.802094	0.802094

Note. The physical properties of the shell material: μ = 2 MPa, λ = 3 MPa, $\gamma = \varepsilon$ = 3 kN, the load intensity p_0 = 100 Pa.

dary conditions (51) (without regard for transverse shear), we obtain appropriate classical theories of elastic shells (with and with no account of traverse shear).

From the general models of micropolar elastic thin shells with constrained rotation with and with no account of traverse shear we can pass to appropriate equations and boundary conditions for cylindrical shells.

Let us consider a stress-strain problem, similar to that posed in Sect. 3, for a hinged cylindrical shell based on the theory of micropolar shells with constrained rotation with and with no account of traverse shear. Numerical data for the problem are presented in Table 2. These data suggest that in the model of micropolar shells with constrained rotation (with and with no account of traverse shear), the micropolar material also displays very high strength and rigidity characteristics. It should also be noted that in the theory of micropolar shells with constrained rotation, account of traversal shear strains is of significance.

5. Model of micropolar elastic thin shells with "low shear rigidity"

Let us consider the case where

$$\alpha \sim \mu,\ \frac{R^2\alpha}{\beta} \ll 1,\ \frac{R^2\alpha}{\gamma} \ll 1,\ \frac{R^2\alpha}{\varepsilon} \ll 1. \tag{53}$$

On the grounds of asymptotic analysis [32, 33] of boundary problem (1)–(6) in a three-dimensional thin region of a shell for (52), we can take the following asymptotically substantiated assumptions (hypothesis):

1) assumptions 1–4 from Sect. 3,

2) in the moment equilibrium equations from (1) we can neglect the differences of the force stresses $\sigma_{ij} - \sigma_{ji}$, $\sigma_{i3} - \sigma_{3i}$, but for dimensionless physical parameters (52) (the physical constant α at a given R is small), these differences in formulae (6) are retained.

It is significant that in the model of micropolar elastic shells with the hypotheses taken (we term it a model with low shear rigidity in the sense that the physical constant α is a shear modulus of sort like the classical shear modulus μ), the "moment" part of the problem is an independent boundary problem.

The basic system of equations and boundary conditions of the model of micropolar elastic thin shells with low shear rigidity and complete account of traverse shear strains is written as follows: for the moment part of the problem, we have equilibrium equations (20) without regard for the differences $S_{12} - S_{21}$, $N_{i3} - N_{3i}$, $H_{12} - H_{21}$ to which we should add physical relations (22), geometric relations (15) and boundary conditions (24); for the force part of the problem, we have equilibrium equations (19), elasticity relations (21), geometric relations (14) and boundary conditions (23).

The basic system of equations and boundary conditions of the model of micropolar elastic shells with low shear rigidity without regard for traverse shear strain is as follows: for the moment part, we have the same boundary problem as in the model with regard for transverse shear strain; for the force part, we have equilibrium equations (19), elasticity relations (28), geometric relations (30) and boundary conditions (32).

Table 3

Strength and rigidity characteristics of a cylindrical micropolar elastic shell. Model with low shear rigidity (with and with no account of traverse shear)

Relative shell thickness $\delta = h/R$	Generalized Timoshenko hypotheses		Generalized Kirchhoff–Love hypotheses	
	$\frac{\sigma_{11\max}^{\text{mic}}}{\sigma_{11\max}^{\text{cl}}}$	$\frac{w_{\max}^{\text{mic}}}{w_{\max}^{\text{cl}}}$	$\frac{\sigma_{11\max}^{\text{mic}}}{\sigma_{11\max}^{\text{cl}}}$	$\frac{w_{\max}^{\text{mic}}}{w_{\max}^{\text{cl}}}$
1/100	0.041363	0.3722294	0.24776	0.247762

Note. The physical properties of the shell material: $\alpha = 1.6$ MPa, $\mu = 2$ MPa, $\lambda = 3$ MPa, $\gamma = \varepsilon = 3$ kN, the load intensity $p_0 = 100$ Pa, the shell length $l = 2R$.

These two models of micropolar shells with low shear rigidity (with and with no account of traverse shear) are characterized by the following: when the moment boundary problem has a zero solution $(\omega_i \equiv 0, \omega_3 \equiv 0, \iota \equiv 0)$, which is the case with homogeneity of the corresponding equations and boundary conditions, the force boundary problem will differ from the appropriate problem of the classical theory of elastic shells (with or without regard for traverse shear), because these equations will have terms with a physical constant α.

The foregoing general models of micropolar shells with low shear rigidity allow us to derive constitutive equations and boundary conditions for circular cylindrical micropolar shells (with and with no account of traverse shear strain).

Let us consider a problem, similar to those posed in Sect. 3 and 4, based on the constitutive equations and boundary conditions for cylindrical micropolar elastic shells with low shear rigidity (with and with no account of traverse shear). Numerical data for the problem are presented in Table 3. Analysis of these data allows the conclusion that in the model of micropolar shells with low shear rigidity (with and with no account of traverse shear), the micropolar material displays very high strength and rigidity characteristics, and account of transverse shear in this model is also of significance.

6. Conclusion

In the work, using asymptotic analysis of boundary problems of the three-dimensional micropolar theory of elasticity in a thin shell region and depending on values of the dimensionless physical properties, we formulated the assumptions (hypothesis) and constructed the general models of micropolar shells with free rotation, constrained rotation, and low shear rigidity with and with no account of traverse shear.

The constructed models of micropolar shells were applied to the specific stress-strain state problem (in the axisymmetric statement) of a hinged circular cylindrical shell. The calculations gave very high strength and rigidity properties of the micropolar material. Although the calculations refer to the chosen hypothetic material, similar effective properties are surely characteristic of micropolar material as such. This conclusion is of interest from the standpoint of materials science, physics and mechanics of new advanced materials. Another important conclusion from the calculations is that in the micropolar theory of shells, account of traverse shear strains is of significance.

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