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MECHANICS

Plane Motions of Rigid Body Controlled by Means of Movable Mass

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Abstract—The plane motions of a rigid body controlled by means of an auxiliary movable point mass are considered. Dry friction forces act between the body and the horizontal plane. It is shown that, under certain conditions, the body can be transferred to an arbitrary state in the plane so that the system is completely controllable.

Keywords: rigid-body dynamics, dry friction, control, mobile robots

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The problems of the dynamics of a rigid body that carries movable masses are relevant in connection with the creation of mobile robots, which have no external movable elements [1–3]. These robots, called capsule robots, can be sealed and able to move in aggressive and vulnerable environments, in pipes, inside living organisms, and to perform the operations of inspection, monitoring, diagnostics, etc. A number of studies have been devoted to the one-dimensional translational motion of these systems in the presence of external resistance forces, and the optimal driving modes are constructed [4, 5].

The two-dimensional plane motions, which are important for constructing rotations of mobile robots, were studied in [6, 7] in the presence of dry friction forces between the body and the horizontal plane. In these studies, it is assumed that the internal moving masses have two degrees of freedom with respect to the carrying body and consist of a rotor and a material point.

In this study, as in [8], we consider the case of one movable point, which controls the plane motion of a body in the presence of dry friction forces between it and the plane. It is shown that the system is completely controllable under fairly general assumptions. It can be transferred to an arbitrary state on the plane even if the point has only one degree of freedom and moves relative to the body along a certain curve.

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MECHANICAL SYSTEM

The system consists of a rigid body P of mass M and a material point Q of mass m (Fig. 1). Let us denote by C the center of mass of the body P and assume that one of the principal central axes of inertia of the body is vertical. The body slides along the fixed horizontal plane OXY in the gravity field leaning on three points A_i. In the case of three supporting points, the rigid body is a statically definable system; therefore, the normal reactions N_i at the points A_i are unambiguously determined. The dry friction forces $\mathbf{F}_i$ acting on the body P at the points A_i obey the Coulomb law with the friction coefficient f. If $\mathbf{v}_i$ is the velocity of the supporting point A_i, the friction forces are determined by the relations

$$\begin{gathered}\mathbf{F}_i = -\frac{f N_i \mathbf{v}_i}{v_i}, \quad \text{if} \quad v_i = |\mathbf{v}_i| > 0, \\ |\mathbf{F}_i| \le f N_i, \quad \text{if} \quad v_i = 0, \quad i = 1,2,3.\end{gathered} \tag{1}$$

The material point Q is equipped with an actuator and moves relative to the body P along a horizontal

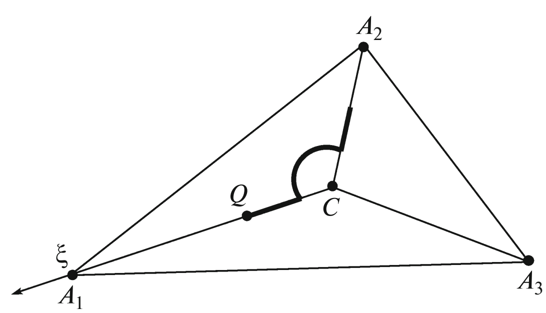

Fig. 1. Mechanical system.

plane parallel to the OXY plane. The point Q interacts with the body P and does not interact with the fixed plane OXY. Thus, the only external forces acting on the system $P+Q$ are the forces N_i and $\mathbf{F}_i$ of gravity and reaction in the supports A_i, $i = 1, 2, 3$.

For simplicity, we assume that the distances from the center of mass C and from the horizontal plane along which the point Q moves to the fixed plane OXY are small as compared to the horizontal linear dimensions of the body P. We also assume that the projections of the points C and Q onto the plane OXY lie inside the triangle $A_1A_2A_3$. These assumptions provide the positiveness of the normal reactions N_i and exclude the "overturning" of the body P. Therefore, we assume that the center of mass C and the point Q move in the OXY plane.

EQUATIONS OF MOTION

We denote the velocities of the points C and Q relative to the plane OXY by $\mathbf{v}_c$ and $\mathbf{v}_Q$ and the force applied to the point Q from the side of the actuator by $\mathbf{F}$. Then the force $(-\mathbf{F})$ is applied to the body P at the point Q, and the equation of motion of the center of mass C of the body P has the form

$$M\dot{\mathbf{v}}_c = \sum_{i=1}^{3}\mathbf{F}_i - \mathbf{F}, \tag{2}$$

where the forces $\mathbf{F}_i$ are defined by formulas (1). The dots designate the derivatives with respect to the time t. We write the equation of motion of the point Q

$$m\dot{\mathbf{v}}_Q = \mathbf{F}$$

in expanded form representing its absolute acceleration as the sum of its absolute, Coriolis, and relative accelerations:

$$m\left[\dot{\mathbf{v}}_c + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + 2\boldsymbol{\omega}\times\mathbf{v} + \mathbf{w}\right] = \mathbf{F}. \tag{3}$$

Here, $\mathbf{r} = \overline{CQ}$ is the radius vector of the point Q relative to the center of mass C of the body P, $\mathbf{v}$ and $\mathbf{w}$ are the relative velocity and acceleration of the point Q relative to the body P, $\boldsymbol{\omega} = \omega\mathbf{k}$ is the angular velocity of the body P, and $\mathbf{k}$ is the unit vector directed vertically upward.

We also compose the equation of moments for the body P around the vertical axis passing through its center C. Denoting by J the moment of inertia of the body P about this axis, we obtain

$$J\dot{\omega} = \left(\sum_{i=1}^{3}\overline{CA_i}\times\mathbf{F}_i - \mathbf{r}\times\mathbf{F}\right)\mathbf{k}. \tag{4}$$

On the basis of Eqs. (2)–(4) of motion, we consider certain types of possible motions of the $P+Q$ system. We assume that the actuator is capable of generating a sufficiently large relative acceleration $\mathbf{w}$, arbitrary in direction, of the point Q, which plays the role of the control action.

SLOW AND RESIDUAL MOTIONS

If the point Q moves with respect to the motionless body P with a sufficiently small relative acceleration, then the body P remains at rest due to the forces of dry friction, which keep it in this state. These slow motions can be used for moving the point Q from the initial state of rest to an arbitrary terminal state of rest relative to the motionless body P.

If the point Q is motionless relative to the body P, then the system $P + Q$ is the rigid body of the mass $M + m$, which stops in a finite time under the action of the dry friction forces. We call such motions residual.

RECTILINEAR MOTIONS

Let the whole system $P + Q$ be at rest at the initial moment of time $t = 0$, and the point Q is on one of the straight lines CA_i, $i = 1, 2, 3$, for example, on the straight line CA_1 (Fig. 1). We show that, if the point Q moves rectilinearly along the straight line CA_1, then the translational motion of the body P occurs along the same straight line. Therefore, it suffices to make sure that all Eqs. (2)–(4) are satisfied for such a motion.

With the translational motion of the body P, the velocities $\mathbf{v}_i$ of all its support points are equal to $\mathbf{v}_c$ and in parallel to the straight line CA_1. According to Eqs. (1), all frictional forces $\mathbf{F}_i$ are also parallel to a straight line CA_1 and directed against the velocity $\mathbf{v}_c$. Therefore, the vector Eq. (2) is reduced to the scalar equation

$$M\dot{v}_c = -f\sum_{i=1}^{3}N_i q(v_c) - F, \tag{5}$$

where we introduced the following designation reflecting Coulomb law (1):

$$\begin{aligned} q(v_c) &= \operatorname{sgn} v_c \quad \text{for} \quad v_c \neq 0, \\ |q(v_c)| &\le 1 \quad \text{for} \quad v_c = 0. \end{aligned} \tag{6}$$

The sum of all normal reactions is equal to the weight of the system $P + Q$, i.e.,

$$N_1 + N_2 + N_3 = (M + m)g, \tag{7}$$

where g is the acceleration of gravity. Equation (5) subject to Eq. (7) takes the form

$$M\dot{v}_c = -f(M + m)gq(v_c) - F. \tag{8}$$

Equation (3) for the translational motion of the body P, i.e., at $\omega = 0$, is reduced to the scalar equation

$$m(\dot{v}_c + w) = F. \tag{9}$$

Ruling out F from Eqs. (8) and (9) and introducing the designation

$$\mu = \frac{m}{M + m}, \tag{10}$$

we obtain

$$\dot{v}_c = -\mu w - fgq(v_c). \tag{11}$$

Let us turn to Eq. (4) at $\omega = 0$. The moments of the forces $\mathbf{F}_1$ and $\mathbf{F}$ directed along a straight line CA_1 with respect to the point C are zero, and the moments of forces $\mathbf{F}_2$ and $\mathbf{F}_3$ balance each other, since the shoulders of these forces like those of the forces N_2 and N_3 are inversely proportional to the magnitudes of the forces.

Thus, all Eqs. (2)–(4) are satisfied with the rectilinear motion of the point Q along a straight line CA_1 and the translational motion of the body P along this straight line. These equations are reduced to one dynamic Eq. (11). There are also the kinematic relations

$$\dot{\xi} = v, \quad \dot{v} = w, \quad \dot{x} = v_c, \tag{12}$$

where ξ is the displacement of the point Q along a straight line CA_1 (Fig. 1) measured from the initial position of this point, v is its velocity relative to the body P, and x is the absolute displacement of the center of mass C of the body P along the direction CA_1. The initial conditions for Eqs. (11) and (12) have the form

$$\xi(0) = v(0) = x(0) = v_c(0) = 0. \tag{13}$$

We set the control in the form of the piece-constant relative acceleration

$$\begin{aligned} w(t) &= w_1 \quad \text{for} \quad t \in (0, t_1), \\ w(t) &= -w_2 \quad \text{for} \quad t \in (t_1, t_2), \quad 0 < t_1 < t_2, \end{aligned} \tag{14}$$

where the constants w_1 and w_2 satisfy the restrictions

$$w_1 > fg\mu^{-1}, \quad 0 < w_2 \le fg\mu^{-1}. \tag{15}$$

Integrating the second equation in set (12) with $w(t)$ defined by Eq. (14) with initial conditions (13), we obtain

$$\begin{aligned} v(t) &= w_1 t \quad \text{for} \quad t \in (0, t_1), \\ v(t) &= w_1 t_1 - w_2(t - t_1) \quad \text{for} \quad t \in (t_1, t_2). \end{aligned} \tag{16}$$

The relative velocity $v(t)$ must vanish at $t = t_2$. We obtain

$$t_2 = \frac{(w_1 + w_2)t_1}{w_2}. \tag{17}$$

Let us integrate the first equation in set (12) for $v(t)$ from set (16) and initial conditions (13). As a result, we determine the total relative displacement of the point Q also using formula (17):

$$\Delta\xi = \xi(t_2) = \frac{w_1(w_1 + w_2)t_1^2}{2w_2}. \tag{18}$$

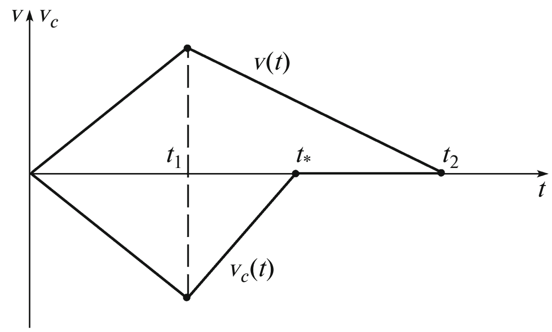

Fig. 2. Functions $v(t)$ and $v_c(t)$.

Referring to Eq. (11), we proceed from the fact that the inequality $v_c \le 0$ holds for all $t \in (0, t_2)$. Integrating Eq. (11) for $w(t)$ defined by Eqs. (14) under initial conditions (13), we obtain

$$\begin{aligned} v_c(t) &= -(\mu w_1 - fg)t \text{ for } t \in (0, t_1), \\ v_c(t) &= -(\mu w_1 - fg)t + (\mu w_2 + fg)(t - t_1) \text{ for } t > t_1. \end{aligned} \tag{19}$$

From the last equality of set (19), it follows that $v_c(t)$ vanishes at $t = t_*$, where

$$t_* = \mu(w_1 + w_2)t_1(\mu w_2 + fg)^{-1}. \tag{20}$$

Comparing Eqs. (17) and (20), we can make sure that $t_* \in (t_1, t_2)$. At $t \in (t_*, t_2)$ and inequalities (15), we have the inequality

$$|w(t)| \le fg\mu^{-1}. \tag{21}$$

Under condition (21), the second term on the right-hand side of Eq. (11) compensates for the first term because of property (6) so that we have $\dot{v}_c = 0$ and, therefore, $v_c \equiv 0$ for $t \in (t_*, t_2)$. The dependences $v(t)$ and $v_c(t)$ on the interval $(0, t_2)$ are shown in Fig. 2. We also determine the total displacement of the body P, which is achieved at $t = t_*$. Using Eq. (12) and initial conditions (13), we obtain

$$\Delta x = \frac{-\mu(w_1 + w_2)(\mu w_1 - fg)(\mu w_2 + fg)^{-1}t_1^2}{2}. \tag{22}$$

The control $w(t)$ specified by relations (14) and (15) provides the relative displacement of the point Q along a straight line CA_1 by distance (18) and, at the same time, the translational displacement of the body P by distance (22). At the beginning and end of the motion, the entire system is at rest. Both displacements are opposite in sign and proportional to t_1^2. Therefore, these motions can be arbitrarily small. After the end of the displacement maneuver, i.e., at $t > t_2$, the point Q can be transferred to the initial point $\xi = 0$ with the motionless body P by the slow motion with the acceleration satisfying condition (21). Repeating the

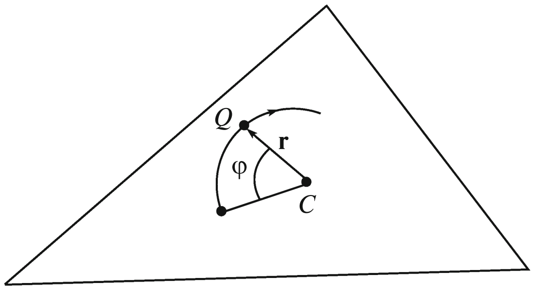

Fig. 3. Rotation.

maneuver described the desired number of times and selecting the parameters w_1, w_2, and t_1 each time, it is possible to displace the body P by an arbitrary distance along the direction CA_1 in a finite time. In this case, the point Q can move along an arbitrary finite segment along the straight line CA_1. The problems of optimization of motions as applied to Eq. (11) are considered in [4, 5, 8], where we constructed the motions with the highest average velocity.

ROTATION

Let us consider the motion of point Q along a circumference S of radius a around the point C (Fig. 3). We substitute the vector $\dot{\mathbf{v}}_c$ from Eq. (2) into Eq. (3), find the vector $\mathbf{F}$, substitute it into Eq. (4), and obtain after transformations

$$(J + \mu M r^2)\dot{\omega} = \left\{ \sum_{i=1}^{3} (\overline{CA_i} - \mu \mathbf{r}) \times \mathbf{F}_i - \mu M \left[2\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{v}) + \mathbf{r} \times \mathbf{w} \right] \right\} \mathbf{k}, \tag{23}$$

where μ is introduced by relation (10). We designate the angular coordinate and the angular velocity of the point Q as it moves along the circumference S as φ and $\Omega = \dot{\varphi}$. The relative velocity $\mathbf{v}$ of the point Q is directed perpendicular to $\mathbf{r}$ and is equal in magnitude to $v = a\Omega$, and its relative acceleration is the sum of the tangential acceleration $\dot{v} = a\dot{\Omega}$ directed perpendicular to the vector $\mathbf{r}$ and the normal acceleration equal to v^2/a and directed against $\mathbf{r}$. As a result of simplifications, Eq. (23) is reduced to the form

$$\dot{\omega} = -\mu_1 z + R, \tag{24}$$

where we introduced the designations

$$\mu_1 = \mu M a^2 (J + \mu M a^2)^{-1},$$
$$R = \left[\sum_{i=1}^{3} (\overline{CA_i} - \mu \mathbf{r}) \times \mathbf{F}_i \right] \mathbf{k} (J + \mu M a^2)^{-1}. \tag{25}$$

The angular acceleration $z = \dot{\Omega}$ plays the role of the control action. Let ψ denote the angle of rotation of the body P relative to the fixed plane OXY and write the kinematic equations

$$\dot{\varphi} = \Omega, \quad \dot{\Omega} = z, \quad \dot{\psi} = \omega, \tag{26}$$

and the initial conditions

$$\varphi(0) = \Omega(0) = \psi(0) = \omega(0) = 0, \tag{27}$$

similar to Eqs. (12) and conditions (13) for the rectilinear motions. Equation (24) is also similar to Eq. (11), but the expression for R from Eq. (25) is quite cumbersome, and, therefore, stronger assumptions are made here. On the basis of relations (1) and (7), we obtain the estimate

$$|R| \le R_o = (l + \mu a) f g (M + m)(J + \mu M a^2)^{-1} = T^{-2}, \quad l = \max CA_i, \quad i = 1, 2, 3, \tag{28}$$

where T is the characteristic value with the dimension of time.

We set the control $z(t)$ in the form

$$z(t) = Z \quad \text{for} \quad t \in (0, \tau), \qquad z(t) = -Z \quad \text{for} \quad t \in (\tau, 2\tau), \tag{29}$$

where the constants Z and τ are such that

$$\mu_1 Z / R_o \sim \varepsilon^{-2}, \quad \tau / T \sim \varepsilon, \quad \varepsilon \ll 1. \tag{30}$$

Neglecting the terms of the order of ε^2 in Eq. (24), we have at $\varepsilon \to 0$

$$\dot{\omega} + \mu_1 \dot{\Omega} = 0,$$

from which we obtain under initial conditions (27)

$$\omega(t) + \mu_1 \Omega(t) = 0, \qquad \psi(t) = -\mu_1 \varphi(t) \quad \text{for} \quad t \in (0, 2\tau). \tag{31}$$

The total angular displacements of the point Q and the body P for the time 2τ are determined by the equalities following from Eqs. (29), (26), (27), and (31):

$$\Delta\varphi = \varphi(2\tau) = Z\tau^2, \quad \Delta\psi = \psi(2\tau) = -\mu_1 Z \tau^2. \tag{32}$$

From relations (30) and (32), it follows that angular displacements (32) are finite at $\varepsilon \to 0$, proportional to each other, and opposite in sign. By choosing τ, they can be made arbitrarily small. Performing the maneuver described the desired number of times and choosing the parameter τ each time, it is possible to rotate the body by a set angle. Between these maneuvers, the point Q can move along the circumference S by means of slow motions. As a result, it is possible to rotate the body P at a set angle and displace the point Q from the initial position to an arbitrary position on the circumference S. The center of mass C of the body P can move during the described maneuver. After its completion, the rotation of the body P and the relative motion of the point Q cease, but the translational motion of the body P can continue. In this residual motion, the system $P + Q$ comes to rest in a finite time.

CONTROLABILITY

Let us show that with the help of the considered motions, the $P + Q$ system can be transferred from an arbitrary initial state of rest to a set terminal state of rest.

(1) First, using a slow motion, we move the point Q from the initial position to a certain point on the circumference S. In this case, the body P does not move.

(2) With the help of several rotational maneuvers, we rotate the body P so that its final orientation coincides with the set terminal orientation. There may be slow and residual motions between rotational maneuvers, as described above. At the end of this stage, the system is at rest and the orientation of the body P coincides with the set one.

(3) With the help of rectilinear motions in two of three arbitrary possible directions CA_i, $i = 1, 2, 3$, we move the body P to a set final position in the plane. Straight-line maneuvers can alternate with slow motions for displacing the point Q along the straight lines CA_i. In this case, the motion of the body P is translational; its orientation does not change. At the end of the motion, the point Q can be transferred to the set position with a slow motion.

Thus, it was established that the system under consideration can be transferred from an arbitrary initial position to a set final position if the movable point Q can move relative to the body P with a sufficiently large acceleration. Under this condition, the system is quite controllable.

For implementing the displacement specified, it suffices that the point Q can move relative to the body P along a certain (arbitrary) arc of the circumference S centered at the point C and along two arbitrary rectilinear segments CA_i, $i = 1, 2, 3$. It suffices to require, for example, that the point Q can move relative to the body P along a curve consisting of the straight-line segment CA_1, an arc of the circumference S, and the straight-line segment CA_2 (the bold curve in Fig. 1). Thus, the point Q can have only one degree of freedom with respect to the body P.

CONCLUSIONS

A rigid body moving along a horizontal plane in the presence of dry friction forces and controlled by a moving mass can be transferred from an arbitrary initial state of rest to an arbitrary terminal state of rest in a finite time. The controllability of the system takes place if the moving mass can move relative to the body along a certain curve with a sufficiently large relative acceleration.

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