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MECHANICS

The Force Acting on a Cylinder in a Ring Flow of a Viscous Fluid with a Small Eccentric Displacement

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Abstract—A system consisting of two circular cylinders one inside the other with parallel axes is considered. The outer cylinder of radius R_2 is fixed, and the inner cylinder of radius R_1 rotates with a sufficiently large angular velocity. The region between the cylinders is filled with an incompressible viscous fluid and, in the case of coaxial cylinders, Couette flow along circular trajectories arises. Upon an eccentric small displacement of the axis of the inner cylinder, the symmetry of the flow is disturbed and a force exerted on the inner cylinder by the fluid is created. Within the ideal fluid model, the force depends linearly on the transverse velocities and accelerations of the cylinder. In a viscous fluid, the force depends on the previous motion of the cylinder. It is expressed in terms of the velocity functional by analogy with the Basset force acting on a ball moving in a viscous fluid with a variable velocity.

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1. PROBLEM STATEMENT

Consider a system consisting of two circular cylinders: an external rigidly fixed cylinder of radius R_2 and an inner mobile cylinder of radius $R_1 < R_2$. The space between the cylinders is filled with a viscous liquid, in which a plane-parallel flow arises. We study the plane problem of the flow of a viscous fluid between two circles in the cross section of the cylinders. If the centers of the circles coincide, then Couette flow along concentric circular trajectories arises. Upon an eccentric displacement of the center of the inner circle with the coordinates $x = x_0(t)$, $y = y_0(t)$ (Fig. 1), the symmetry of the flow is disturbed. This gives rise to a force exerted on the inner circle by the liquid. It is required to determine the force acting on the inner circle upon its given motion $x_0(t), y_0(t)$. P.L. Kapitsa [1] supposed that, on each element, a frictional force proportional to the square of the mean fluid velocity between the boundaries of the circles must act. According to this hypothesis, as a result of the integration, Kapitsa obtained a force the components S_x and S_y of which are proportional to the displacement of the center of the circle $S_x = -Ny$, $S_y = Nx$.

In a more accurate formulation but within the ideal fluid model, the force acting on the inner circle was found in [2]. The flow is assumed to be vertical, and the displacements x_0, y_0 are considered to be small. Earlier, in Chisotti's work [3, p. 120], the force acting on a fixed circle in the case of its finite displacement was found exactly.

Now let us turn to the formulation of the problem within the model of a viscous incompressible fluid. The fluid velocity field is sought in the form of an expansion in a small displacement x_0 of the center of

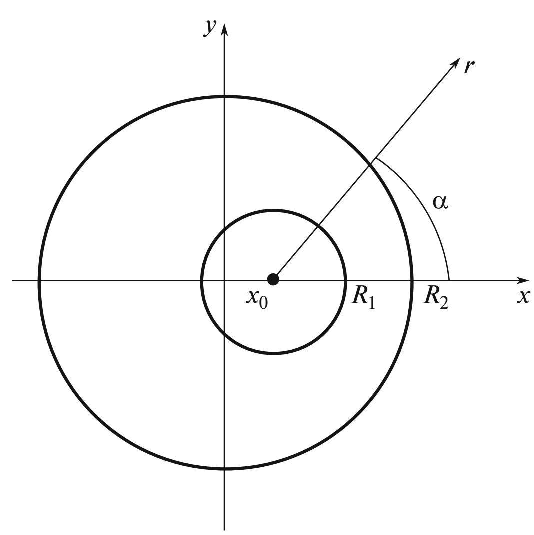

Fig. 1.

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the circle. We assume that the generating flow is caused by the rotation of the inner cylinder with an angular velocity Ω_0. It is the well-known Couette flow with the current function

$$\Psi_0 = \Omega_0 \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\left(-\ln\frac{r}{R_2} + \frac{r^2}{2R_2^2}\right),$$

which exactly satisfies both the Euler equations of an ideal fluid and the Navier–Stokes equations of a viscous fluid. In the polar coordinates (r, α), the radial (v_r) and transverse (v_α) velocity components are expressed in terms of the stream function Ψ_0:

$$v_r = \frac{\partial \Psi_0}{r\partial\alpha} = 0,$$

$$v_\alpha = -\frac{\partial \Psi_0}{\partial r} = \Omega_0 \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\left(\frac{1}{r} - \frac{r}{R_2^2}\right).$$

The velocity field satisfies the no-slip conditions at the boundaries of the circles $r = R_1$ and $r = R_2$: $v_\alpha(R_1) = \Omega_0 R_1$ and $v_\alpha(R_2) = 0$.

This flow consists of the potential flow of a point vortex with the angular velocity component

$$v_\alpha = \frac{\Gamma}{2\pi r} = \Omega_0 \frac{R_1^2 R_2^2}{(R_2^2 - R_1^2) r} \tag{1}$$

and the flow with a constant vortex:

$$2\omega_0 = -\Delta\Psi_0 = -\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\Psi_0 = -\Omega_0 \frac{2R_1^2}{R_2^2 - R_1^2}. \tag{2}$$

It is required to determine the force acting on the inner circle upon a small displacement x_0.

2. THE FORCE ACTING ON A CIRCLE IN AN IDEAL FLUID

If the center of the inner circle is displaced by a small amount x_0 along the axis x, then the stream function changes by a small quantity:

$$\Psi_{id} = \Psi_0(r) + \Psi_1(r,\alpha),$$

$$\Psi_1(r,\alpha) = \frac{R_1^2(R_2^2 - r^2)}{(R_2^2 - R_1^2) r}(\dot{x}_0 \sin\alpha - \dot{y}_0 \cos\alpha). \tag{3}$$

Here, the origin of the polar coordinates coincides with the center of the inner circle (Fig. 1).

The flow velocity field with the stream function $\Psi_1(r,\alpha)$ is potential and does not change the circulation Γ and the vortex ω of the main flow. Let us show that, in the linear approximation in x_0, $\dot{x}_0$, $\dot{y}_0$, it satisfies the conditions for the normal velocity at the boundaries of circles. Indeed, at the boundary of the inner circle, we have

$$v_r = \left.\frac{\partial \Psi_1}{r\partial\alpha}\right|_{r=R_1} = v_n = \dot{x}_0\cos\alpha + \dot{y}_0 \sin\alpha$$

and, at the boundary of the outer circle, $r = R_2 - x_0\cos\alpha + O(x_0^2)$, the normal velocity $v_r = \left.\frac{\partial\Psi_1}{r\partial\alpha}\right|_{r=R_2 - x_0\cos\alpha}$ has a quadratic smallness.

The tangential velocity at the boundary of the inner circle of the perturbed motion does not satisfy the no-slip condition:

$$v_{\alpha 1} = -\left.\frac{\partial\Psi_1}{\partial r}\right|_{r=R_1} = \frac{R_1^2 + R_2^2}{R_2^2 - R_1^2}(-\dot{x}_0\sin\alpha + \dot{y}_0\cos\alpha). \tag{4}$$

Upon a small displacement of the inner cylinder in the resulting flow with stream function (3), the pressure acting on the inner circle with force $\mathbf{F}(F_x, F_y)$ is created. It can be shown that the components of the force can be expressed by the Lagrangian

$$\begin{aligned}\frac{\Lambda}{\rho} &= \frac{\pi R_1^2(R_1^2 + R_2^2)(\dot{x}_0^2 + \dot{y}_0^2)}{2(R_2^2 - R_1^2)} \\ &+ \frac{(\Gamma(R_1^2 + R_2^2) + 4\pi a^2 R_2^2\omega)(\dot{x}_0 y_0 - \dot{y}_0 x_0)}{2(R_2^2 - R_1^2)} \\ &+ \frac{(\Gamma + 2\pi\omega R_1^2)(\Gamma + 2\pi\omega R_2^2)(x_0^2 + y_0^2)}{4\pi(R_2^2 - R_1^2)}\end{aligned} \tag{5}$$

by the Lagrange formulas

$$F_x = -\frac{d}{dt}\frac{\partial\Lambda}{\partial\dot{x}_0} + \frac{\partial\Lambda}{\partial x_0}, \quad F_y = -\frac{d}{dt}\frac{\partial\Lambda}{\partial\dot{y}_0} + \frac{\partial\Lambda}{\partial y_0}. \tag{6}$$

These formulas are consistent with the expressions for the forces obtained in [1]. With the help of expressions (1) and (2) for the circulation and vortex, formulas (5) and (6) are simplified. It is convenient to write the force in complex form:

$$\begin{aligned}\Lambda &= \rho\pi R_1^2\left[\frac{(R_1^2 + R_2^2)(\dot{x}_0^2 + \dot{y}_0^2)}{2(R_2^2 - R_1^2)} + \frac{\Omega_0 R_2^2(\dot{x}_0 y_0 - \dot{y}_0 x_0)}{R_2^2 - R_1^2}\right], \\ F_x + iF_y &= \rho\pi R_1^2\left[-\frac{(R_1^2 + R_2^2)\ddot{z}_0}{R_2^2 - R_1^2} + 2i\frac{\Omega_0 R_2^2 \dot{z}_0}{R_2^2 - R_1^2}\right].\end{aligned} \tag{7}$$

3. THE BOUNDARY VALUE PROBLEM FOR A VISCOUS FLUID

The vortex in a viscous liquid,

$$\omega = \omega_0 + \omega', \quad 2\omega' = -\Delta\Psi',$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\alpha^2}, \tag{8}$$

where ω_0 is a constant vortex (2) and ω' is the perturbation of the vortex, satisfies the vortex transport equation

$$\frac{\partial}{\partial t}\omega' + v_r\frac{\partial\omega'}{\partial r} + v_\alpha\frac{1}{r}\frac{\partial\omega'}{\partial\alpha} = \nu\Delta\omega', \tag{9}$$

where ν is the kinematic viscosity of the fluid.

With the Reynolds number $\mathrm{Re} = R_1^2\Omega_0/\nu \gg 1$, at the boundary of the inner circle, a boundary layer with a thickness on the order of $\delta = R_1/\sqrt{\mathrm{Re}} = \sqrt{\nu/\Omega_0}$ appears. In view of the estimates

$$\frac{\partial}{\partial r} \sim \frac{1}{\delta}, \quad \frac{\partial}{\partial r^2} \sim \frac{1}{\delta^2}, \quad \Delta\Psi = \frac{\partial^2\Psi}{\partial r^2}(1 + O(\delta)),$$
$$v_r = O_1, \quad \text{and} \quad v_\alpha = \Omega_0 + O_1,$$

where O_1 is a quantity of the first order of smallness x_0, $\dot{x}_0$, and $\dot{y}_0$, Eqs. (8) and (9) for the perturbed vortex $\omega'(t, r, \alpha)$, and the stream function in the boundary-layer approximation take the form

$$\frac{\partial\omega'}{\partial t} + \Omega_0\frac{\partial\omega'}{\partial\alpha} = \nu\frac{\partial^2\omega'}{\partial r'^2}, \qquad (10)$$
$$\omega' = -\frac{\partial^2\Psi'}{\partial r'^2}, \quad r' = r - R_1.$$

We assume that, at the initial instant of time, the perturbed vortex is zero:

$$\omega'(0, r', \alpha) = 0. \qquad (11)$$

Outside of the boundary layer, the perturbation of the vortex must tend to zero:

$$\omega'(t, \infty, \alpha) = 0. \qquad (12)$$

The perturbed velocity must satisfy the no-slip condition at the boundary of the inner circle. Hence, using (4), we obtain the no-slip boundary condition in the form

$$-\frac{\partial\Psi'}{\partial r'}\bigg|_{r'=0} = v'_\alpha\Big|_{r'=0} = -v_{\alpha 1}\Big|_{r'=0} \qquad (13)$$
$$= \frac{R_1^2 + R_2^2}{R_2^2 - R_1^2}(\dot{x}_0\sin\alpha - \dot{y}_0\cos\alpha).$$

Integrating the second equation in (10), $\frac{\partial\Psi'}{\partial r'} = \int_{r'}^{\infty}\omega' dr'$, and substituting it into boundary condition (13), we obtain

$$2\int_0^\infty \omega'(t, r', \alpha)dr' = \frac{R_1^2 + R_2^2}{R_2^2 - R_1^2}(-\dot{x}_0\sin\alpha + \dot{y}_0\cos\alpha) \qquad (14)$$
$$= \mathrm{Real}\left[-\frac{R_1^2 + R_2^2}{R_2^2 - R_1^2}ie^{-i\alpha}(\dot{x}_0 + i\dot{y}_0)\right].$$

4. SOLUTION OF THE BOUNDARY VALUE PROBLEM

We seek the solution to boundary value problem (10)–(12) and (14) in the form $2\omega' = \mathrm{Re}[e^{-i(\alpha+\Omega_0 t)}W(t, r')]$. Then, we obtain for the function $W(t, r')$ the following boundary value problem:

$$\frac{\partial W}{\partial t} = \nu\frac{\partial^2 W}{\partial r'^2}, \quad W(0, r') = 0, \qquad (15)$$
$$W(t, 0) = W_0(t), \quad W(t, \infty) = 0.$$

The function $W_0(t)$ is found from boundary condition (14).

The solution on boundary value problem (15) has the form

$$W(t, r') = \frac{r'}{2\sqrt{\nu\pi}}\int_0^t \exp\left[-\frac{r'^2}{4\nu(t - t')}\right]\frac{W_0(t')dt'}{(t - t')^{3/2}}.$$

Using the equality

$$\int_0^\infty \exp\left(-\frac{r'^2}{4\nu(t - t')}\right)\frac{r' dr'}{2\nu(t - t')}$$
$$= \int_0^\infty \exp(-x^2)dx = \frac{\sqrt{\pi}}{2},$$

we find

$$\int_0^\infty W(t, r')dr'$$
$$= \int_0^\infty\int_0^t \exp\left(-\frac{r'^2}{4\nu(t - t')}\right)\frac{W_0(t')dt'}{(t - t')^{3/2}}\frac{r'}{2\sqrt{\nu\pi}}dr'$$
$$= \frac{\sqrt{\nu}}{2}\int_0^t \frac{W_0(t')dt'}{\sqrt{t - t'}}.$$

Substituting this expression into boundary condition (14), we obtain for the function $W_0(t)$ the Abel integral equation:

$$2\int_0^\infty \omega' dr' = e^{-i(\alpha+\Omega_0 t)}\int_0^\infty W(t, r')dr'$$
$$= e^{-i(\alpha+\Omega_0 t)}\frac{\sqrt{\nu}}{2}\int_0^t \frac{W_0(t')dt'}{\sqrt{t - t'}} = -\frac{R_1^2 + R_2^2}{R_2^2 - R_1^2}ie^{-i\alpha}\dot{z}_0$$
$$\Rightarrow \int_0^t \frac{W_0(t')dt'}{\sqrt{t - t'}} = -\frac{2}{\sqrt{\nu}}\frac{R_1^2 + R_2^2}{R_2^2 - R_1^2}ie^{i\Omega_0 t}\dot{z}_0,$$
$$\dot{z}_0 = \dot{x}_0 + i\dot{y}.$$

It has the solution

$$W_0(t) = K\frac{d}{dt}\int_0^t \frac{\dot{z}_0(t')e^{i\Omega_0 t'}dt'}{\sqrt{t - t'}}, \qquad (16)$$
$$K = -\frac{2}{\pi\sqrt{\nu}}i\frac{R_1^2 + R_2^2}{R_2^2 - R_1^2}.$$

5. TANGENTIAL STRESS AND FORCE

The complex tangential stress τ at the boundary of the inner circle on the complex plane $z = x + iy$ has the form

$$\tau = 2\mu i e^{i\alpha}\omega'(t,0) = \mu i e^{i\alpha}\mathrm{Re}[e^{-i(\alpha+\Omega_0 t)}W_0(t)]$$
$$= \mu i e^{i\alpha}(\cos(\alpha+\Omega_0 t)W_{0x}(t) + \sin(\alpha+\Omega_0 t)W_{0y}(t)),$$

where $\mu = \rho\nu$ is the dynamic viscosity of the fluid. Using the obvious formulas

$$\int_0^{2\pi} e^{i\alpha}W_{0x}\cos(\alpha+\Omega_0 t)d\alpha = \pi e^{-i\Omega_0 t}W_{0x},$$
$$\int_0^{2\pi} e^{i\alpha}W_{0y}\sin(\alpha+\Omega_0 t)d\alpha = i\pi e^{-i\Omega_0 t}W_{0y}$$

we calculate the resulting complex viscous friction force:

$$F = \int_0^{2\pi}\tau R_1 d\alpha = \mu i R_1 \pi e^{-i\Omega_0 t}W_0(t).$$

Substituting into this formula expression (16), we obtain the final expression for the viscous force:

$$F = 2\rho\nu^{1/2}R_1 e^{-i\Omega_0 t}\frac{R_1^2+R_2^2}{R_2^2-R_1^2}\frac{d}{dt}\int_0^t \frac{\dot{z}_0(t')e^{i\Omega_0 t'}dt'}{\sqrt{t-t'}}. \qquad (17)$$

The total force acting on the circle is the sum of the force determined by the ideal fluid model with the help of (7) and viscous force (17). Viscous force (17) is an analog of the hereditary Basset force acting on a sphere of radius R_1, moving by the law $z_0(t)$ [4, p. 132]:

$$F_B = -6\rho\nu^{1/2}R_1^2\int_0^t \frac{\ddot{z}_0(t')dt'}{\sqrt{\pi(t-t')}}.$$

6. SPECIAL CASES

Let us consider a special case in which the center of the inner circle moves from the center of the outer circle by some law $z_0(t)$ and stops at point z_0 at time t_0: $z(0) = 0$, $z(t_0) = z_0$, $\dot{z}(t_0) = 0$. Then the integral in (17) at a sufficiently large time $t \gg t_0$ has the asymptotic expansion

$$\int_0^t \frac{\dot{z}_0(t')e^{i\Omega_0 t'}dt'}{\sqrt{t-t'}}$$
$$= \int_0^{t_0}\dot{z}_0(t')e^{i\Omega_0 t'}\frac{1}{\sqrt{t}}\left(1+\frac{t'}{t}+\ldots\right)dt' = \frac{A}{\sqrt{t}} + O(t^{-3/2}),$$
$$A = \int_0^{t_0}\dot{z}_0(t')e^{i\Omega_0 t'}dt',$$

and tends to zero as $t \to \infty$.

Thus, the part of the viscous force linear with respect to the displacement, acting on the inner fixed cylinder, is zero. Only the force corresponding to the ideal fluid approximation (7) remains. However, by virtue of the equalities $\dot{z}_0 = \ddot{z}_0 = 0$, it is also zero.

Let us consider the second case, in which the center of the inner cylinder moves along a circular trajectory of a small radius r with an angular velocity Ω_1. In complex form, the motion of the center can be written as $z_0 = re^{i\Omega_1 t}$. In this case, $\dot{z}_0 = i\Omega_1 z_0$, $\ddot{z}_0 = -\Omega_1^2 z_0$, and, using the ideal fluid model, we find from (7) that

$$F = \rho\pi R_1^2 z_0\frac{(R_1^2+R_2^2)\Omega_1^2 - 2R_2^2\Omega_0\Omega_1}{R_2^2-R_1^2}.$$

The viscous force component is found with the help of the integral

$$\int_0^t \frac{\dot{z}_0(t')e^{i\Omega_0 t'}}{\sqrt{t-t'}}dt' = \frac{ie^{it(\Omega_0+\Omega_1)}\sqrt{\pi}r\Omega_1 Erf[\sqrt{it(\Omega_0+\Omega_1)}]}{\sqrt{i(\Omega_0+\Omega_1)}}$$
$$= \frac{e^{it(\Omega_0+\Omega_1)}\sqrt{\pi}r\Omega_1\sqrt{i(\Omega_0+\Omega_1)}}{\Omega_0+\Omega_1} + O(t^{-1/2}).$$

Substituting this expression into formula (17) for the viscous force, we obtain

$$F' = i2\rho(i\pi\nu(\Omega_0+\Omega_1))^{1/2}\frac{R_1^2+R_2^2}{R_2^2-R_1^2}R_1\Omega_1 z_0.$$

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REFERENCES

1. P. L. Kapitsa, Zh. Tekh. Fiz. **9** (2), 124 (1939).
2. V. F. Kopiev, S. A. Chernyshev, and M. A. Yudin, J. Appl. Math. Mech. **81** (2), 148 (2017).
3. H. Lamb, *Hydrodynamics*, 6th ed. (University Press, Cambridge, 1932; OGIZ, Moscow–Lenigrad, 1947).
4. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1975; Nauka, Moscow, 1986).
5. A. G. Petrov, *Analytical Hydrodynamics* (Fizmatlit, Moscow, 2009) [in Russian].

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