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MECHANICS

Singular Solutions of Contact Problems and Block Elements

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Abstract—In this work, we consider mixed problems of elasticity theory, in particular, contact problems for cases that are nontraditional. They include mixed problems with discontinuous boundary conditions in which the singularities in the behavior of contact stresses are not studied or the energy of the singularities is unbounded. An example of such mixed problems is contact problems for two rigid stamps approaching each other by rectilinear boundaries up to contact but not merging into one stamp. It has been shown that such problems, which appear in seismology, failure theory, and civil engineering, have singular components with unbounded energy and can be solved by topological methods with pointwise convergence, in particular, by the block element method. Numerical methods that are based on using the energy integral are not applicable to such problems in view of its divergence.

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1. Let us consider applying the integral factorization method (which is a part of the block element method) to studying a half-strip rigid stamp acting statically without friction on an elastic layer of finite thickness, which can be both isotropic and anisotropic. It is well known that the mixed problem is reduced to an integral equation of the following form:

$$\iint_{\Omega} k(x-\xi, y-\eta) g(\xi,\eta) d\xi d\eta = f(x,y),$$
$$x, y \in \Omega,$$
$$k(x,y) = \frac{1}{4\pi^2}\iint_{R^2} K(\alpha,\beta) e^{-i(\alpha x+\beta y)} d\alpha d\beta \qquad (1)$$
$$\equiv \mathbf{F}_2^{-1}(x,y) K(\alpha,\beta),$$
$$\Omega(-\infty < x < c;\ -s \le y \le s),$$
$$K(\alpha,\beta) = O(A^{-1}), \quad A = \sqrt{\alpha^2+\beta^2} \to \infty.$$
$$K(\alpha,\beta) = C(\beta)\left[\frac{1}{|\alpha|} + O\left(\frac{1}{\alpha^3}\right)\right], \quad |\alpha| \to \infty.$$

Here, $K(\alpha,\beta)$ is an analytical function of two complex variables α and β; both variables are even, in particular, meromorphic, and positive when both variables are real; numerous examples were presented in [1–3]. Function $g(x,y)$ means contact stresses under the stamp. $f(x,y)$ is the vertical displacement of the stamp when it penetrates into the layer, and $\mathbf{F}_2(\alpha,\beta)$, $\mathbf{F}_1(\alpha)$ are operators of the two- and one-dimensional Fourier transform, respectively. They have the form

$$\Phi_n(\alpha,\beta) = \mathbf{F}_2(\alpha,\beta)\varphi_n(x,y),$$
$$\Phi_n(\alpha) = \mathbf{F}_1(\alpha)\varphi_n(x), \quad \Phi_n(\beta) = \mathbf{F}_1(\beta)\varphi_n(y).$$

It is supposed that the function $f(x,y)$ is twice continuously differentiable and, for simplicity, exponentially decreases as $x \to -\infty$.

There are two analytical methods for studying integral equation (1) in the anisotropic case; numerical methods are not effective. These are reduction to the theory of normed rings, which leads to the method of fictitious absorption [3, 4], and the direct factorization method, i.e., the integral method of factorization which is part of the block element method going back to N. Wiener's works [2, 5–9]. Applying the second method, we introduce the topology induced by the Euclidean space and consider the integral equation as a topological object on a compact, namely, on the whole plane. We extend the equation to the whole plane by introducing new variables $\varphi_n(x,y)$:

$$\varphi_1(x,y) \in \Omega_1(c \le x < \infty;\ s < y < \infty),$$
$$\varphi_2(x,y) \in \Omega_2(-\infty < x < c;\ s < y < \infty),$$
$$\varphi_6(x,y) \in \Omega_6(-\infty < x < c;\ -\infty < y < -s),$$
$$\varphi_7(x,y) \in \Omega_7(c \le x < \infty;\ -\infty < y < -s),$$
$$\varphi_8(x,y) \in \Omega_8(c < x < \infty;\ -s \le y \le s).$$

Let us agree upon the following notation for the Fourier transforms and their properties.

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The regularity signs for functions in half-planes of the complex plane are the plus sign above in the upper half-plane and the minus sign above in the lower half-plane. The subscripts indicate the parameter in which the regularity takes place (α or β). As a result, we have

$$\mathbf{F}_2(\alpha,\beta)\varphi_1(x,y) = \Phi_1(\alpha,\beta) = \Phi^+_{1\alpha}(\alpha,\beta) = \Phi^+_{1\beta}(\alpha,\beta),$$
$$\mathbf{F}_2(\alpha,\beta)\varphi_2(x,y) = \Phi_2(\alpha,\beta) = \Phi^+_{2\beta}(\alpha,\beta),$$
$$\mathbf{F}_2(\alpha,\beta)\varphi_6(x,y) = \Phi_6(\alpha,\beta) = \Phi^-_{6\beta}(\alpha,\beta),$$
$$\mathbf{F}_2(\alpha,\beta)\varphi_7(x,y) = \Phi_7(\alpha,\beta) = \Phi^+_{7\alpha}(\alpha,\beta) = \Phi^-_{7\beta}(\alpha,\beta),$$
$$\mathbf{F}_2(\alpha,\beta)\varphi_8(x,y) = \Phi_8(\alpha,\beta) = \Phi^+_{8\alpha}(\alpha,\beta).$$

In what follows, to avoid frequent use of the symbol of the Fourier transform and inversion operator, we take upper case letters $K(\alpha,\beta)$, $F(\alpha,\beta)$, and $\Phi(\alpha,\beta)$ for the Fourier transform of functions having lower case designations $k(x,y)$, $f(x,y)$, and $\varphi(x,y)$, respectively. We also have to perform operations of function factorization in the form of sums and products. These operations will be performed on functions depending on two complex variables α and β. The operation of factorization in the form of a sum will be carried out using the notation taken in [2] with curly brackets. For example, if a function $G(\alpha,\beta)$ is factorized as a sum in the parameter α with respect to the real axis, the functions obtained as a result of this operation are denoted by the formulas

$$G(\alpha,\beta) = \{G(\alpha,\beta)\}^+_\alpha + \{G(\alpha,\beta)\}^-_\alpha. \qquad (2)$$

The first function in the right-hand side is regular in the upper complex plane; the second one is regular in the lower complex plane. In the case in which the functions are also subjected to factorization in the parameter β, these factorized functions are denoted as

$$\{G(\alpha,\beta)\}^+_\alpha = \{\{G(\alpha,\beta)\}^+_\alpha\}^+_\beta + \{\{G(\alpha,\beta)\}^+_\alpha\}^-_\beta.$$

In the case in which the functions have indices, e.g., denote quadrants that are supporters of the functions, factorization is denoted by the rule according to which the indices follow the designation of factorization, i.e.,

$$G^+_{m\alpha} \equiv \{G^+_{m\alpha}\}^+_\beta + \{G^+_{m\alpha}\}^-_\beta,$$
$$\{G^+_{m\alpha}\}^+_\beta = \{\{G^+_{m\alpha}\}^+_\beta\}^+_\alpha + \{\{G^+_{m\alpha}\}^+_\beta\}^-_\alpha.$$

Below, expressions of operators implementing factorization in the form of a sum are presented [2]:

$$\{G(\alpha,\beta)\}^+_\alpha = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{G(\xi,\beta)}{\xi-\alpha}d\xi, \quad \operatorname{Im}\alpha > 0,$$
$$\{G(\alpha,\beta)\}^-_\alpha = -\frac{1}{2\pi i}\int_{\Gamma_1}\frac{G(\xi,\beta)}{\xi-\alpha}d\xi, \quad \operatorname{Im}\alpha < 0. \qquad (3)$$

The inequalities following the integral expressions mean that the parameter α is above the contour Γ_1 coinciding with the real axis (the first case) or below it (the second case). Factorization of functions in the form of a product in the sequel is implemented only for the function $K(\alpha,\beta)$ and, when performed in α or β, has respectively the form

$$K(\alpha,\beta) = K_{+\alpha}(\alpha,\beta)K_{-\alpha}(\alpha,\beta),$$
$$K(\alpha,\beta) = K_{+\beta}(\alpha,\beta)K_{-\beta}(\alpha,\beta). \qquad (4)$$

The operators that implement the factorization operation in the form of a product are represented by the formulas

$$K_{+\alpha}(\alpha,\beta) = \exp\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\ln K(\xi,\beta)}{\xi-\alpha}d\xi, \quad \operatorname{Im}\alpha > 0,$$
$$K_{-\alpha}(\alpha,\beta) = \exp\left(-\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\ln K(\xi,\beta)}{\xi-\alpha}d\xi\right), \qquad (5)$$
$$\operatorname{Im}\alpha < 0.$$

In cases of factorization in the parameter β, it is necessary to replace α in all formulas by this parameter, together with the replacement of the contour Γ_1 by Γ_2, which also coincides with the real axis of the complex plane β. This leads to the following expressions:

$$K_{+\beta}(\alpha,\beta) = \exp\frac{1}{2\pi i}\int_{\Gamma_2}\frac{\ln K(\alpha,\xi)}{\xi-\beta}d\xi, \quad \operatorname{Im}\beta > 0,$$
$$K_{-\beta}(\alpha,\beta) = \exp\left(-\frac{1}{2\pi i}\int_{\Gamma_2}\frac{\ln K(\alpha,\xi)}{\xi-\beta}d\xi\right), \qquad (6)$$
$$\operatorname{Im}\beta < 0.$$

The conditions imposed on the functions when applying formulas of factorization in the form of a sum or a product have been described in the literature in sufficient detail, e.g., in [2, 3], and are not presented here. Let us apply operator $\mathbf{F}_2(\alpha,\beta)$ to integral equation (1) and represent it as a functional equation of the form

$$KQ = \Phi_1 + \Phi_2 + \Phi_6 + \Phi_7 + \Phi_8 + F. \qquad (7)$$

We factorize functions K in (7) in the form of products

$$K = K_{+\alpha}K_{-\alpha}, \quad K = K_{+\beta}K_{-\beta}$$

and transform the functional equation (7) by applying factorizations (2)–(6) with the aim to obtain a system of integral equations for finding the new unknowns introduced. As a result, the following theorem is valid.

Theorem. *For the fulfillment of integral equation* (1), *it is sufficient that the functions* Φ_n *satisfy the system of integral equations of the form*

$$\begin{aligned}
\Phi^+_{8\alpha} &= -\{e^{is\beta}\{e^{-is\beta}K_{+\alpha}e^{ic\alpha}\{K^{-1}_{+\alpha}e^{-ic\alpha}(\Phi^+_{2\beta} \\
&\qquad + \Phi^-_{6\beta} + F)\}^+_\alpha\}^-_\beta\}^+_\beta, \\
\Phi^+_{2\beta} &= -\{e^{-ic\alpha}K_{+\beta}e^{is\beta}\{K^{-1}_{+\beta}e^{-is\beta}(\Psi^-_{4\beta} + \Phi^+_{8\alpha} + F)\}^+_\beta\}^-_\alpha, \\
\Phi^-_{6\beta} &= -\{e^{-ic\alpha}K_{-\beta}e^{-is\beta}\{K^{-1}_{-\beta}e^{is\beta}(\Psi^+_{3\beta} + \Phi^+_{8\alpha} + F)\}^-_\beta\}^-_\alpha, \\
\Psi^+_{3\beta} &= -K_{+\beta}e^{is\beta}\{K^{-1}_{+\beta}e^{-is\beta}(\Psi^-_{4\beta} + \Phi^+_{8\alpha} + F)\}^+_\beta, \\
\Psi^-_{4\beta} &= -K_{-\beta}e^{-is\beta}\{K^{-1}_{-\beta}e^{is\beta}(\Psi^+_{3\beta} + \Phi^+_{8\alpha} + F)\}^-_\beta.
\end{aligned} \tag{8}$$

Other unknowns are determined from the relations

$$\Phi^+_{1\beta} = \Psi^+_{3\beta} - \Phi^+_{2\beta}, \quad \Phi^-_{6\beta} = \Psi^-_{4\beta} - \Phi^-_{7\beta}.$$

Although the system of integral equations (8) seems to be complicated, it can be reduced to a system of algebraic equations by use of the theory of residues in the case of the boundary problem for a layered medium, and approximate solutions are obtained rather easily. The solution of the integral equation after finding all the unknowns can be represented in the form

$$\begin{aligned} &g(x, y) \\ &= \mathbf{F}_2^{-1}(x, y)K^{-1}(\Phi_1 + \Phi_2 + \Phi_6 + \Phi_7 + \Phi_8 + F). \end{aligned} \tag{9}$$

Endowing the kernel of the integral equation with properties suggested by the mixed boundary problem, one can easily study distinctive features of the solution obtained, such as the behavior of the solutions in the neighborhood of the boundary and at corner points which are known sufficiently well.

2. In problems of seismology, lithospheric plates have a more complex form. In spite of this, lithospheric plates with extended boundaries make contact with each other just at the edges of their boundaries. In connection with the fact that an initial earthquake [10] arises only in the contact zone of boundaries of lithospheric plates, the role of other boundaries, in relation to the scales of the plates themselves, is insignificant. To understand the feature of the interaction between lithospheric plates in more detail, we consider the behavior of contact stresses of two semi-infinite rigid stamps approaching contact but not interacting with each other (Fig. 1). Therefore, let us consider the case in which in the previous problem it is taken that $c = 0, \quad s \to \infty$; i.e., a semi-infinite stamp acts on an elastic medium from the left. The second stamp which is constructed by analogy with the first one occupies the half-plane $x > 0$ to the right from the oy axis; i.e., it is situated oppositely to the first stamp. Then, the integral equation of the boundary problem takes the form of a convolution equation:

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} k(x - \xi, y - \eta)g(\xi, \eta)d\xi d\eta = f(x, y), \tag{10}$$

$$x, y \in \Omega(-\infty < x < \infty; -\infty \le y \le \infty).$$

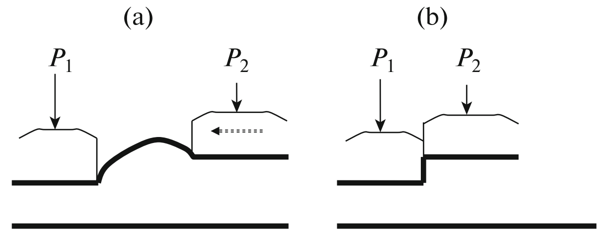

Fig. 1. Two stamps (thin contour) are pressed in an elastic layer (thick contour) by the forces P_1 and P_2 in the cases of (a) the presence of a distance between them and motion of the right stamp to the left one (dashed arrow); (b) in the absence of a distance between them. In this case, the layer boundary acquired a step and the contact stresses acquired a singularity in its zone.

The fact of the presence of two noninteracting stamps is emphasized only by the right-hand side of the integral equation, i.e., function $f(x, y)$ which can vary separately in the right and left half-planes and, in particular, have a discontinuity at $x = 0$. This discontinuity testifies about the free vertical displacements of each stamp. To resolve this integral equation in L_1 with kernel properties (1) and the requirement that the solution must belong to the energy space [1, 2], it is sufficient to satisfy the inequalities

$$\begin{aligned} &\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \left|K^{-1}(\alpha, \beta)F(\alpha, \beta)\right| d\alpha d\beta < \infty, \\ &\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \left|K^{-1}(\alpha, \beta)F^2(\alpha, \beta)\right| d\alpha d\beta < \infty. \end{aligned} \tag{11}$$

However, the concentrations of stresses under deformed lithosphere plates obtained for initial earthquakes by the topological method with pointwise convergence in [10], without the requirement that the solution must belong to the energy space, demonstrated that similar concentrations of stresses must also exist in the case of stamps. Indeed, they are easy to find. To do this, it is necessary to reject the fulfillment of conditions (11) and take the right-hand side $f(x, y)$ of the integral equations in the form

$$f(x, y) = f_0(y)\operatorname{sgn} x + f_1(x, y).$$

The function $f_1(x, y)$ satisfies conditions (11), and $f_0(y)$ is a finite function equal to zero together with the derivatives beyond a certain segment. The choice of a function means that one of the stamps penetrated into the layer deeper than the other and the surface underwent a discontinuity (Fig. 1b). Constructing the solution of Eq. (10) in generalized functions [11], we obtain its representation in the form

$$g(x,y) = g_0(y)\frac{2}{\pi x} + g_1(x,y),$$
$$g_0(y)\mathbf{F}_1^{-1}(y)K_0^{-1}(\infty,\beta)F_0(\beta),$$
$$g_1(x,y) = \mathbf{F}_2^{-1}(x,y)K^{-1}(\alpha,\beta)F_2(\alpha,\beta),$$
$$K_0^{-1}(\infty,\beta) = \lim|\alpha|^{-1}K^{-1}(\alpha,\beta), \quad |\alpha| \to \infty, \qquad (12)$$
$$F_2(\alpha,\beta) = F_1(\alpha,\beta) + F_0(\beta)\frac{2i}{\alpha}$$
$$-2iK_0^{-1}(\infty,\beta)K(\alpha,\beta)F_0(\beta)\operatorname{sgn}\alpha,$$
$$F_0(\beta) = \mathbf{F}_1(\beta)f_0(y), \quad F_1(\alpha,\beta) = \mathbf{F}_2(\alpha,\beta)f_1(x,y).$$

Figure 1 schematically shows the approach of rigid stamps. This process explains the formation of a step at the layer boundary.

Let us again return to integral equation (1). Consider the case in which $c \to \infty$ and s remains bounded. In this case, the stamp becomes striplike, $-\infty \le x \le \infty$, $|y| < s$, and the integral equation turns out to be given in this strip. Let the strip be a collection of $N+1$ rigid stamps with vertical boundaries passing through the points x_n, $n = 1, 2, \ldots, N$, which approached each other by boundaries. It means that the function $f(x,y)$ in the general case has a stepwise form at the points x_n. Functional equation (7) takes the representation

$$K(\alpha,\beta)Q_N(\alpha,\beta) = \Phi_2 + \Phi_6 + F,$$
$$Q_N(\alpha,\beta) = \mathbf{F}_2(\alpha,\beta)g_N(x,y).$$

Here, $g_N(x,y)$ denotes contact stresses under the whole collection of $N+1$ stamps. In order to obtain singular concentrations of stresses under the set of approaching stamps, we use the properties of solutions for a strip stamp; they have been obtained in many works [2, 3] in which energy contact stresses were constructed. Comparing it with the result obtained above and showing that singularities are generated only by a discontinuity of the function $f(x,y)$, we obtain the following representation for the concentration of singular contact stresses in neighborhoods of the points x_n in zones of approaching rigid stamps situated in the strip:

$$g_N(x,y) = \sum_{n=1}^{N} f_N(y)(x - x_n)^{-1}, \quad f_N(y) \in L_1(-s,s).$$

CONCLUSIONS

Thus, the solution of the integral equation shows that contact stresses in the zone of approaching stamps at the boundary of each of them have a singularity removing the solution from the energy space. This fact demonstrates that the zone of singular stresses in reality had broken before the stresses reached the singular value. The problem of elasticity theory was only an indicator that this process had occurred and had resulted in an initial earthquake [10]. It is important to note that, during an earthquake, the part of the Earth's surface that remains elastic almost exactly reflects the fault displacement obtained theoretically (Fig. 1b). Therefore, in spite of destruction that took place in the earthquake source zone, the increase in stresses from a certain value to the singular value has time to trigger a fault shear of the Earth's surface according to the right-hand side of the integral equation. Another important conclusion following from the result obtained is that study of the conditions for the appearance of an initial earthquake by numerical methods involving the energy integral, e.g., by the finite element method, will not be successful (as has already been observed in many models) in connection with the fact that the solution is not energetic.

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