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MECHANICS

Large Bending Strains in an Orthotropic Beam with a Preliminarily Stretched or Compressed Layer: Exact Solution

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Abstract—The model of an orthotropic nonlinear elastic compressible material generalizing the known model of an isotropic semilinear (harmonic) material and admitting a number of exact solutions at large strains is proposed. For this model of the material, an exact solution of the plane problem on large bending strains of a compound rectangular beam consisting of two layers, one of which is preliminarily deformed, is obtained. When formulating and solving the problem, the theory of superimposed large strains is used. On the basis of the calculations carried out, the effect of anisotropy on a strong bending of the compound beam with a preliminarily deformed layer is analyzed.

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1. MODEL OF THE ORTHOTROPIC NONLINEARLY ELASTIC MATERIAL

Hooke's law in the classical anisotropic theory of elasticity [4] has the form of $\boldsymbol{\varepsilon} = \mathbf{B} : \boldsymbol{\sigma}$, where $\boldsymbol{\varepsilon}$ is the linear strain tensor, $\boldsymbol{\sigma}$ is the stress tensor, the colon means the double convolution of tensors, and the tensor $\mathbf{B}$ of fourth rank is called the elastic compliance tensor.

We extend this constitutive relation to the case of large strains replacing the tensor $\boldsymbol{\varepsilon}$ with the tensor $\mathbf{U} - \mathbf{E}$, and replacing the tensor $\boldsymbol{\sigma}$ with the symmetric Biot stress tensor $\mathbf{S}$. We obtain

$$\mathbf{U} - \mathbf{E} = \mathbf{B} : \mathbf{S}, \qquad (1)$$

where $\mathbf{E}$ is the unit tensor, $\mathbf{U} = \mathbf{G}^{1/2}$ is the symmetric positively defined stretch tensor, $\mathbf{G} = \mathbf{F} \cdot \mathbf{F}^T$ is the Cauchy strain measure, and $\mathbf{F}$ is the strain gradient. The symmetric Biot tensor is related to the symmetric Piola–Kirchhoff stress tensor $\boldsymbol{\Sigma}^0$ and the asymmetrical Piola stress tensor $\mathbf{P}$ as follows:

$$\mathbf{S} = \frac{1}{2}(\boldsymbol{\Sigma}^0 \cdot \mathbf{U} + \mathbf{U} \cdot \boldsymbol{\Sigma}^0) = \frac{1}{2}(\mathbf{P} \cdot \mathbf{A}^T + \mathbf{A} \cdot \mathbf{P}^T). \qquad (2)$$

Here, $\mathbf{A} = \mathbf{U}^{-1} \cdot \mathbf{F}$ is the proper orthogonal rotation tensor [1].

In the case of an orthotropic material, the components of the tensor $\mathbf{B}$ in the basis of the principal axes of orthotropy are determined from the formulas

$$B_{iiii} = \frac{1}{E_i}, \quad B_{iijj} = B_{jjii} = -\frac{\nu_{ij}}{E_i}, \quad B_{ijij} = B_{ijji} = B_{jiij} = B_{jiji} = \frac{1}{4G_{ij}}. \qquad (3)$$

The components of the tensor $\mathbf{B}$ not written in formulas (3) on the basis of the principal axes of orthotropy are zero. In these formulas, there is no summation over repeating subscripts. The values of E_k, G_{sk}, and ν_{sk} are the elastic constants. They are connected by the relations $E_1\nu_{21} = E_2\nu_{12}, E_2\nu_{32} = E_3\nu_{23}, E_3\nu_{13} = E_1\nu_{31}$.

This model of an orthotropic nonlinear elastic material is correct from the thermodynamic viewpoint because the Biot stress tensor $\mathbf{S}$ is conjugated in energy to the tensor $\mathbf{U}$. If all three Young's moduli E_k coincide among themselves and are equal to E, all Poisson ratios ν_{sk} are equal to ν, all shear moduli G_{sk} are equal to G, and the equality $E = 2G(1 + \nu)$ is fulfilled, then this model of the material passes into the known model of an isotropic semi-linear (harmonic) material [1]. In the case of small strains, this model passes into Hooke's law for an orthotropic material [4]. For this reason, the material with constitutive relations (1)–(3) can be called an orthotropic semi-linear material.

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Similar to the isotropic semi-linear material, the model of this material admits a number of exact analytical solutions for large strains of anisotropic elastic compressible solids.

Because of linearity, constitutive relations (1) can be transformed to the form

$$\mathbf{S} = \mathbf{c} : (\mathbf{U} - \mathbf{E}). \tag{4}$$

Here, $\mathbf{c}$ is the elastic modulus tensor of the fourth rank.

We consider one important special case of finite strain of the orthotropic medium. We assume that the principal axes of the tensor $\mathbf{U}$ coincide with the axes of material symmetry, i.e., with the principal axes of orthotropy. Then, from constitutive relations (1), (3), it follows that the tensors $\mathbf{U}$ and $\mathbf{S}$ are coaxial. With use of the solution of Eq. (2) with respect to $\mathbf{\Sigma}^0$ obtained in [5], it is possible to show that the Piola–Kirchhoff stress tensor $\mathbf{\Sigma}^0$ is coaxial to the tensor $\mathbf{U}$ and, thus, commutes with it. In this case, according to Eq. (2), the tensor $\mathbf{\Sigma}^0$ and the Piola stress tensor $\mathbf{P}$ are expressed through $\mathbf{S}$ reasonably simply:

$$\mathbf{\Sigma}^0 = \mathbf{S} \cdot \mathbf{U}^{-1} = \mathbf{U}^{-1} \cdot \mathbf{S}, \mathbf{P} = \mathbf{S} \cdot \mathbf{A}. \tag{5}$$

In the case of the plane strain, which is implemented in a plane orthogonal to one of the axes of symmetry of the material, the stress S_{33} is excluded from relations (1)–(3) by means of the condition $U_{33} = 1$, which results in the following dependences in the basis of the principal orthotropic axes

$$\begin{gathered} U_{11} = 1 + pS_{11} - tS_{22}, \\ U_{22} = 1 - tS_{11} + qS_{22}, \quad U_{12} = \frac{S_{12}}{2G_{12}}, \end{gathered} \tag{6}$$

$$\begin{gathered} p = \frac{1 - \nu_{31}\nu_{13}}{E_1}, \quad q = \frac{1 - \nu_{32}\nu_{23}}{E_2}, \\ t = \frac{\nu_{21} + \nu_{31}\nu_{23}}{E_2} = \frac{\nu_{12} + \nu_{32}\nu_{13}}{E_1}. \end{gathered} \tag{7}$$

2. NONLINEAR BENDING OF A COMPOUND RECTANGULAR BEAM FROM AN ORTHOTROPIC MATERIAL

The problem on bending of a compound beam is formulated on the basis of the theory of superimposed large strains [2, 3] and is similar to that of the corresponding problem for an isotropic material [6–8]. Further, we use the designations and terminology of this theory [6, 7]. For the sake of brevity, the subscripts at the tensors describing the transition from the initial to final state are omitted, for example, $\mathbf{F}_{0,2} = \mathbf{F}$. The problem is solved in the coordinates of an intermediate state into which the lower part of the beam passes after the preliminary strain. The Cartesian coordinates of beam's points in this state are designated as x, y, z.

We consider the plane strain of the rectangular beam occupying the region $0 \le x \le l$, $0 \le y \le h$, in the reference configuration. The size on the coordinate z is of no importance. The region $0 \le y \le h_1$ $(0 \le h_1 \le h)$ is occupied with a preliminarily stressed layer. The initial (preliminary) strain of this layer with respect to the initial state is homogeneous and is set by the following deformation gradient:

$$\mathbf{F}_{0,1} = \alpha \mathbf{i}_1 \otimes \mathbf{i}_1 + \gamma \mathbf{i}_2 \otimes \mathbf{i}_2 + \mathbf{i}_3 \otimes \mathbf{i}_3, \tag{8}$$

where $\mathbf{i}_k$ are the coordinate unit vectors and α and γ are positive constants. The material of both layers of the beam is assumed as orthotropic, the axes of the material symmetry coinciding with the axes x, y, z.

Let the constitutive relation of an elastic material be known in the coordinates of the initial state

$$\mathbf{P} = \Phi(\mathbf{F}). \tag{9}$$

Here, $\mathbf{F}$ is the deformation gradient, $\mathbf{P}$ is the Piola stress tensor, and Φ is the tensor function. The constitutive relation of the same material in the intermediate-state coordinates has the form

$$\mathbf{P}^1 = (\det \mathbf{F}_{0,1})^{-1} \mathbf{F}_{0,1}^T \cdot \Phi(\mathbf{F}), \mathbf{F} = \mathbf{F}_{0,1} \cdot \mathbf{F}_{1,2}. \tag{10}$$

Further, we use the model constructed above for an orthotropic semi-linear compressible material.

We search for the solution of equilibrium equations for a compound beam in a form corresponding to nonlinear pure bending [9]:

$$\begin{gathered} X = \rho(y)\sin\beta x, \quad Y = \rho(y)\cos\beta x, \\ Z = z, \quad \beta = \text{const}, \end{gathered} \tag{11}$$

where X, Y, Z are the Cartesian coordinates of the beam points in the final state. For strain (11), each straight line $y = \text{const}$ is transformed into an arch of a circle with radius $\rho(y)$. Therefore, $\rho(y) > 0$. The geometrical meaning of the parameter β consists in the fact that βx is the angle of rotation of the cross section $x = \text{const}$ around the unit vector $\mathbf{i}_3$. If $\beta > 0$, the beam is bent with convexity upward, and, if $\beta < 0$, it is bent with convexity downward.

The deformation gradient responsible for map (11) has the form

$$\begin{gathered} \mathbf{F}_{1,2} = \beta\rho\, \mathbf{i}_1 \otimes \mathbf{e}_1 + \frac{d\rho}{dy} \mathbf{i}_2 \otimes \mathbf{e}_2 + \mathbf{i}_3 \otimes \mathbf{i}_3, \\ \mathbf{e}_1 = \mathbf{i}_1 \cos\beta x - \mathbf{i}_2 \sin\beta x, \\ \mathbf{e}_2 = \mathbf{i}_1 \sin\beta x + \mathbf{i}_2 \cos\beta x. \end{gathered} \tag{12}$$

Hence, $\det \mathbf{F}_{1,2} = \beta\rho\left(\dfrac{d\rho}{dy}\right)$ should be positive and the values of β and $\dfrac{d\rho}{dy}$ have identical signs. Taking into account this fact, we obtain from Eq. (12)

$$\mathbf{U} = \operatorname{sgn}\beta\left(\beta\rho\,\mathbf{i}_1\otimes\mathbf{i}_1 + \frac{d\rho}{dy}\mathbf{i}_2\otimes\mathbf{i}_2\right) + \mathbf{i}_3\otimes\mathbf{i}_3,\qquad \mathbf{A} = \operatorname{sgn}\beta(\mathbf{i}_1\otimes\mathbf{e}_1 + \mathbf{i}_2\otimes\mathbf{e}_2) + \mathbf{i}_3\otimes\mathbf{i}_3. \quad (13)$$

According to Eqs. (13), the eigenvectors of the tension tensor in the bending problem under consideration are directed along the axes of symmetry of the orthotropic material, which provides the validity of relations (5). On the basis of Eqs. (2) and (13), we have now

$$\mathbf{S} = S_{11}\mathbf{i}_1\otimes\mathbf{i}_1 + S_{22}\mathbf{i}_2\otimes\mathbf{i}_2 + S_{33}\mathbf{i}_3\otimes\mathbf{i}_3,\qquad \mathbf{P} = P_1(y)\,\mathbf{i}_1\otimes\mathbf{e}_1 + P_2(y)\,\mathbf{i}_2\otimes\mathbf{e}_2 + P_3(y)\,\mathbf{i}_3\otimes\mathbf{i}_3, \quad (14)$$

$$P_1 = (\operatorname{sgn}\beta)S_{11},\quad P_2 = (\operatorname{sgn}\beta)S_{22},\quad P_3 = S_{33}. \quad (15)$$

Introducing the designations $\beta\rho = C_1(y)$, $\dfrac{d\rho}{dy} = C_2(y)$ and taking into account the constitutive relations (6), (7) of the material, we obtain the following expressions for the upper layer of the compound beam $(h_1 < y \le h)$ with the help of Eqs. (13)–(15):

$$C_1 = \operatorname{sgn}\beta + pP_1 - tP_2,\quad C_2 = \operatorname{sgn}\beta - tP_1 + qP_2. \quad (16)$$

The Piola stress tensor for the preliminarily deformed lower layer has the form of

$$\mathbf{P}^1 = P_1'(y)\,\mathbf{i}_1\otimes\mathbf{e}_1 + P_2'(y)\,\mathbf{i}_2\otimes\mathbf{e}_2 + P_3'(y)\,\mathbf{i}_3\otimes\mathbf{i}_3,\quad 0 \le y < h_1. \quad (17)$$

Hereinafter, the prime marks the characteristics of the stress-strain state of the preliminarily stressed portion of the beam.

To obtain the expressions for components of the strain gradient $\mathbf{F}$ through the components of the Piola stress tensor in the lower layer, it is necessary to make the following replacements in representations (17) because of Eqs. (8)–(10):

$$C_1 \to \alpha C_1,\quad C_2 \to \gamma C_2,\qquad P_1 \to \gamma P_1',\quad P_2 \to \alpha P_2'. \quad (18)$$

Transformation (18) results in the following dependences valid in the region $0 \le y < h_1$:

$$C_1 = (\operatorname{sgn}\beta)\alpha^{-1} + \alpha^{-1}\gamma p' P_1' - t' P_2',\qquad C_2 = (\operatorname{sgn}\beta)\gamma^{-1} - t' P_1' + \alpha\gamma^{-1} q' P_2'. \quad (19)$$

Here we took into account that the orthotropic material of the lower layer, as well as the material of the upper layer, has the axes of material symmetry directed along the unit vectors $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$. The elastic constants of the lower layer p', q', and t', generally speaking, do not coincide with the elastic constants of the upper layer p, q, and t.

The vector equilibrium equation for the Piola stress $\nabla^1\cdot\mathbf{P}^1 = 0$ in the problem of bending according to Eqs. (14), (17) is reduced to one scalar equation

$$\frac{dP_2}{dy} = \beta P_1,\quad h_1 < y \le h,\qquad \frac{dP_2'}{dy} = \beta P_1',\quad 0 \le y < h_1. \quad (20)$$

Excluding the function $\rho(y)$ from the expressions for C_1 and C_2, we come to the compatibility equation

$$\frac{dC_1}{dy} = \beta C_2,\quad 0 \le y \le h. \quad (21)$$

On the basis of Eqs. (16), (19)–(21), we come to solving the equations of the problem of strong bending of a compound rectangular beam

$$\frac{d^2P_2}{dy^2} - \frac{\beta^2 q}{p}P_2 = \frac{(\operatorname{sgn}\beta)\beta^2}{p},\quad h_1 < y \le h, \quad (22)$$

$$\frac{d\,P_2'}{dy^2} - \frac{\alpha^2\beta^2 q'}{\gamma^2 p'}P_2' = \frac{(\operatorname{sgn}\beta)\alpha\beta^2}{\gamma^2 p'},\quad 0 \le y < h_1. \quad (23)$$

After finding the functions $P_2(y)$ and $P_2'(y)$, the stresses $P_1(y)$ and $P_1'(y)$ are expressed from Eq. (20), the functions $C_1(y)$ and $C_2(y)$ are determined with the help of Eqs. (16), (19), and the function $\rho(y)$ is expressed as $\rho(y) = \beta^{-1}C_1(y)$.

The boundary conditions for Eqs. (22), (23) are as follows:

$$(1)\ P_2(h) = 0;\quad (2)\ P_2'(0) = 0;\quad (3)\ P_2(h_1) = P_2'(h_1). \quad (24)$$

The fourth boundary condition expresses the requirement of continuity of the function $\rho(y)$ at the point $y = h_1$ and is written in the form

$$\operatorname{sgn}\beta + \beta^{-1}p\frac{d\,P_2}{dy} - tP_2 = (\operatorname{sgn}\beta)\alpha^{-1} + \alpha^{-1}\beta^{-1}\gamma p'\frac{d\,P_2'}{dy} - t' P_2' \quad (25)$$

at $y = h_1$.

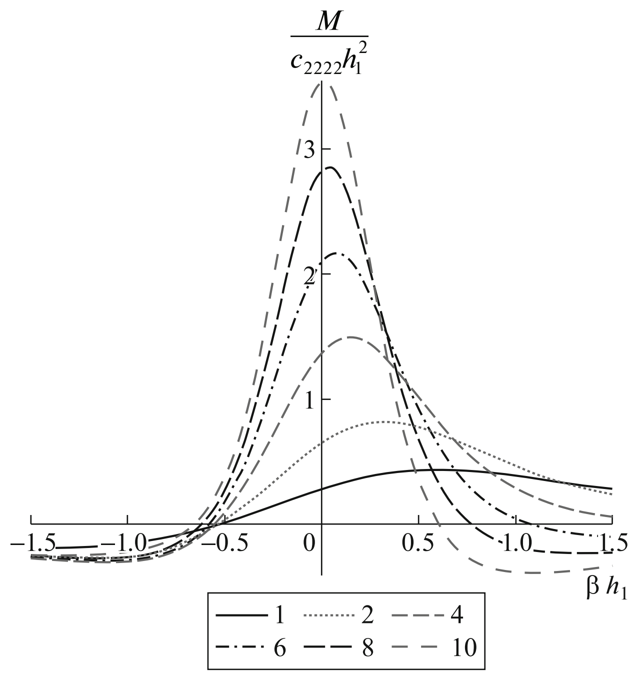

Fig. 1. Dependence of M on β for different values of the ratio c_{1111}/c_{2222} (under figure) at $\alpha = 0.5$.

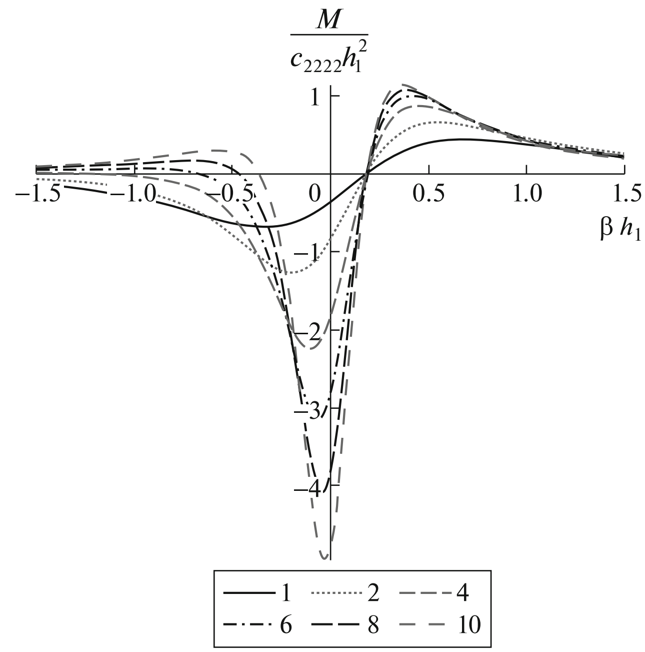

Fig. 2. Dependence of M on β for different values of the ratio c_{1111}/c_{2222} (under figure) at $\alpha = 1.6$.

The general solution of Eqs. (22), (23) has the form

$$P_2 = A_1 e^{\lambda_1 y} + A_2 e^{-\lambda_1 y} - (\operatorname{sgn}\beta) q^{-1},$$
$$P_2' = B_1 e^{\lambda_2 y} + B_2 e^{-\lambda_2 y} - (\operatorname{sgn}\beta)(\alpha q')^{-1}, \quad (26)$$
$$\lambda_1 = \beta\sqrt{\frac{q}{p}}, \quad \lambda_2 = \frac{\alpha\beta}{\gamma}\sqrt{\frac{q'}{p'}}.$$

The constants A_1, A_2, B_1, B_2 are found from the linear set of equations following from boundary conditions (24), (25).

The parameters α and γ setting the preliminary strain of the lower layer are not independent. They are connected by the relation

$$\gamma = 1 + \frac{(1-\alpha)t'}{p'}, \quad (27)$$

following from the absence of stresses on the area elements $y = \text{const}$ in the intermediate state of the lower layer.

To satisfy the boundary conditions at the beam end faces $x = 0$, $x = l$, in the integral sense, we determine the force resultant vector $\mathbf{f}$ and the moment resultant vector $\mathbf{M}$ operating in the cross section $x = \text{const}$ under strain of the type in Eq. (11). Using the same approach, as in [6–8], it is possible to show that $\mathbf{f} = 0$ and the bending moment $\mathbf{M} = M\,\mathbf{i}_3$ is identical in all cross sections of the beam and calculated from the formula

$$M = M(\beta) = \int_0^{h_1} P_1' \rho\, dy + \int_{h_1}^{h} P_1\, \rho\, dy.$$

The numerical calculations were fulfilled for the case when the materials of both layers are identical, at $h = 2.5h_1$, $c_{1122} = 0.5c_{2222}$ (c_{mnsk} are the constants of material in relations (4)). We investigated the dependences of the bending moment M on the parameter β at different values of the ratio c_{1111}/c_{2222} (Figs. 1, 2). In Fig. 1, these dependences are given for the case of $\alpha = 0.5$ (when the material of the lower layer is initially compressed in the direction of the axis x) and, in Fig. 2, they are for the case of $\alpha = 1.6$ (when the material of the lower layer is initially stretched in the direction of the axis x). At $\alpha = 0.5$, an increase in c_{1111} leads to an increase in the highest value of the bending moment and in the displacement of the point of the peak to the left (towards smaller values of β). At $c_{1111}\,/\,c_{2222} \geq 6$, there is a change in the sign of the moment at reasonably high positive values of β. At $\alpha = 1.6$ (Fig. 2), we observed the similar effects with the difference being that the peak value of the moment is replaced by the minimum.

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