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PHYSICS

Escape Velocity in General Relativity

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Abstract—The problem of determination of the escape velocity in terms of Newton's gravitational theory and general relativity (GR) is considered. The analysis is performed based on the solution of the equation describing the particle motion along the radial geodesic line. The classical result is derived from the linearized equation for a weak gravitational interaction. A similar result is derived for metric coefficients of space corresponding to the Schwarzschild solution. The relationship determining the escape velocity for the generalized solution leading to metric coefficients having no singularity is also presented.

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It is known that the escape velocity for a spherical body with mass m and radius of the outer surface R has the form

$$v_2 = c\sqrt{\frac{r_g}{R}}, \tag{1}$$

$$r_g = \frac{2mG}{c^2}, \tag{2}$$

where c is the velocity of light and r_g is the so-called gravitational radius (G is the gravitational constant). Expression (2) was derived by Michel in 1783 and by Laplace in 1796 and was interpreted as the radius of a "dark star," for which the escape velocity equals the velocity of light. Subsequently, this interpretation based on Newton's gravitational model and the corpuscular model of light was rejected [1]. It follows from the results presented below that this conclusion can turn out to have been anticipatory.

To determine the escape velocity, let us introduce a spherical system of coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, and $x^4 = ct$ in the GR and corresponding metric form

$$ds^2 = g_{11}dr^2 + g_{22}(d\theta^2 + \sin^2\theta d\varphi^2) - g_{44}d(ct)^2, \tag{3}$$

in which the components of metric tensor g_{ij} depend on r and use the equation determining the trajectory of the geodesic line [2]

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0. \tag{4}$$

To investigate the motion of the particle, which moves away from the body along the radius, let us write Eq. (4) for $x^1 = r$. Using the known transformation, which allows us to replace variable s by variable t [2] and expressing the Christoffel symbols in Eq. (4) through components of the metric tensor, we finally derive

$$\frac{d^2r}{dt^2} + c^2\frac{g'_{11}}{2g_{11}} - \left(\frac{g'_{44}}{g_{44}} - \frac{g'_{11}}{2g_{11}}\right)\left(\frac{dr}{dt}\right)^2 = 0, \tag{5}$$

where $(\cdot)' = \frac{d(\cdot)}{dr}$.

The coefficients of metric tensor g_{ii}, which enter this equation, are the solution of GR equations. For a spherically symmetric problem, these equations have the form [2, 3]

$$\chi T^1_1 = \frac{1}{g_{22}} - \frac{g'_{22}}{2g_{11}g_{22}}\left(\frac{g'_{22}}{2g_{22}} + \frac{g'_{44}}{g_{44}}\right), \tag{6}$$

$$\chi T^4_4 = \frac{1}{g_{22}} - \frac{1}{g_{11}}\left[\frac{g''_{22}}{g_{22}} - \frac{1}{4}\left(\frac{g'_{22}}{g_{22}}\right)^2 - \frac{g'_{11}g'_{22}}{2g_{11}g_{22}}\right], \tag{7}$$

where T^j_i is the tensor of the pulse energy and $\chi = \frac{8\pi G}{c^4}$ is the gravitational GR constant. We note that there are only two mutually independent Eqs. (6) and (7) in the GR for three coefficients g_{11}, g_{22}, and g_{44}. Attempts to add these equations by the so-called harmonic coordinate conditions are known [4].

The classical solution of the GR spherically symmetric problem derived by Schwarzschild in 1916 is the solution of Eqs. (6) and (7) at $g_{22} = r^2$. For the outer empty space ($r \geq R$), in which $T^j_i = 0$, it has the form [3]

$$g_{11} = \left(1 - \frac{r_g}{r}\right)^{-1}, \quad g_{44} = 1 - \frac{r_g}{r}, \tag{8}$$

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where r_g is determined by equality (2). Allowing for equalities (8), Eq. (5) is reduced to the form

$$\frac{d^2r}{dt^2}+\frac{c^2r_g}{2r^2}\left(1-\frac{r_g}{r}\right)+\frac{3r_g}{2r^2\left(1-\frac{r_g}{r}\right)}\left(\frac{dr}{dt}\right)^2=0. \quad (9)$$

This equation is integrated under the initial conditions, according to which we have at $t = 0$:

$$r = R, \quad v=\frac{\sqrt{g_{11}}dr}{\sqrt{g_{44}}dt}=v_0, \quad (10)$$

where v is the physical velocity of the particle [5] and v_0 is its initial velocity.

Ratio $\frac{r_g}{r}$ for a weak gravitational interaction can be considered small compared with unity. Then we have $g_{11} = g_{44} = 1$ from equalities (8), and Eq. (9) is simplified as follows:

$$\frac{d^2r}{dt^2}+\frac{c^2r_g}{2r^2}=0.$$

The first integral of this equation has the form

$$v=\frac{dr}{dt}=\sqrt{C_1+c^2\frac{r_g}{r}}. \quad (11)$$

Determining integration constant C_1 from second condition (10), we will find

$$C_1=v_0^2-\frac{r_gc^2}{R}. \quad (12)$$

Integration of Eq. (11) allowing for first condition (10) gives

$$t=r\sqrt{r_gc^2-rC_1}-R\sqrt{r_gc^2-RC_1}$$
$$+\frac{r_gc^2}{2\sqrt{C_1}}\ln\frac{r_gc^2+2C_1R+2\sqrt{RC_1(r_gc^2+RC_1)}}{r_gc^2+2C_1r+2\sqrt{rC_1(r_gc^2+rC_1)}}. \quad (13)$$

Equation (13) determines the time dependence of the radial coordinate. This dependence is meaningful if $C_1 \geq 0$. Finally, it follows from equality (12) that

$$v_0 \geq v_2 = c\sqrt{\frac{r_g}{R}}. \quad (14)$$

The expression found for v_2 coincides with formula (1); i.e., the solution of the GR problem for a weak gravitational interaction leads to the escape velocity following from Newton's gravitational theory. We note that this velocity can be found without applying Eq. (13). Indeed, substituting C_1 from equality (12) into expression (11) for velocity, we find

$$v=\sqrt{v_0^2-r_gc^2\left(\frac{1}{R}-\frac{1}{r}\right)}.$$

From here, we find at $r \to \infty$ that

$$v_\infty=\sqrt{v_0^2-\frac{r_gc^2}{R}}.$$

Determining v_2 as the minimal initial velocity, at which $v_\infty = 0$, we derive equality (14). If $v_0 = v_2$, then $C_1 = 0$, and the solution of Eq. (11) determines the following time dependence of the radial coordinate:

$$r=R\sqrt[3]{\left(1+\frac{3v_2t}{2R}\right)^2}.$$

Let us find the escape velocity corresponding to the Schwarzschild solution. The first integral of Eq. (9)

$$\frac{dr}{dt}=c\left(1-\frac{r_g}{r}\right)\sqrt{\frac{r_g}{r}+C_2\left(1-\frac{r_g}{r}\right)} \quad (15)$$

determines the velocity of motion, which has the form according to second equality (10)

$$v=c\sqrt{\frac{r_g}{r}+C_2\left(1-\frac{r_g}{r}\right)}. \quad (16)$$

Assuming $v(R) = v_0$, we find the integration constant

$$C_2=\frac{1}{1-\frac{r_g}{R}}\left(\frac{v_0^2}{c^2}-\frac{r_g}{R}\right). \quad (17)$$

The equation similar to Eq. (13) is rather cumbersome and is not presented here. However, as before, it follows from it that in order for the solution to be meaningful, the fulfillment of condition $C_2 \geq 0$ is necessary. Finally, we have expression (14) from equality (17). A similar result follows from condition $v_\infty = 0$. Thus, the Schwarzschild solution leads to the same expression for the escape velocity as Newton's gravitational theory [1]. At $R = r_g$, we have $v_2 = c$.

Let us assume that $v_0 = v_2$. Then accepting $C_2 = 0$ in Eq. (15) and integrating this equation allowing for first condition (10), we can derive the following dependence between time t and radial coordinate r:

$$t=\frac{r_g}{c}\left\{\frac{2}{3}\left[\sqrt{\frac{r}{r_g}}\left(3+\frac{r}{r_g}\right)\right.\right.$$
$$\left.\left.-\sqrt{\frac{R}{r_g}}\left(3+\frac{R}{r_g}\right)\right]+\ln\frac{\left(\sqrt{R}+\sqrt{r_g}\right)\left(\sqrt{r}-\sqrt{r_g}\right)}{\left(\sqrt{R}-\sqrt{r_g}\right)\left(\sqrt{r}+\sqrt{r_g}\right)}\right\}.$$

We have $R = r_g$ at $t \to \infty$; i.e., an infinitely long time is required at the escape velocity equal to the velocity of light.

It follows from equality (8) for metric coefficient g_{11} that the Schwarzschild solution is singular if the minimal radial coordinate $r = R$ turns out to be equal to r_g. In connection with this, r_g is called the event horizon radius of the "black hole." The authors of [6, 7] derived the solution of the GR spherically symmetric problem for generalized metric form (3), in which $g_{22} = \rho^2(r)$. The need for such a solution is determined by the following reason.

In 1916, Schwarzschild derived the solution for the internal ($0 \le r \le R$) region of the spherical body with constant density μ. This solution follows from Eq. (7) if we accept $g_{22} = r^2$ and $T_4^4 = \mu c^2$ in it. Finally, we have (index i corresponds to the internal space)

$$g_{11}^i = \frac{1}{1 - \frac{\chi\mu\rho c^2}{3} r^2}. \quad (18)$$

Solutions (8) and (18) should coincide on the body surface ($r = R$), which is provided by equality

$$\frac{\chi\mu c^2 R^3}{3} = r_g. \quad (19)$$

This resulted in the fact that solution (18) accepts the following final form [3]:

$$g_{11}^i = \frac{1}{1 - \frac{r_g r^2}{R^3}}. \quad (20)$$

If we now substitute expression χ and r_g into equality (19), then we will find the mass

$$m_0 = \frac{4}{3}\pi\mu R^3, \quad (21)$$

corresponding to the Euclidian space. This contradicts the main GR idea, according to which the space inside the body is the Riemann space. Indeed, we have for metric form (3) at $g_{22} = r^2$ and g_{11}^i determined by equality (20):

$$m = 2\mu \int_0^{2\pi} d\varphi \int_0^{\pi/2} \sin\theta d\theta \int_0^R \sqrt{g_{11}^i} r^2 dr$$
$$= \frac{2\pi R^4}{r_g}\mu \left[\frac{1}{\sqrt{\frac{r_g}{R}}} \arcsin\sqrt{\frac{r_g}{R}} - \sqrt{1 - \frac{r_g}{R}} \right]$$
$$\approx m_0 \left(1 + \frac{3r_g}{10R} + \frac{9r_g^2}{56R^2} + \cdots \right).$$

The last part of this relationship comprises the expansion in regards to parameter $\frac{r_g}{R}$, from which it follows that $m = m_0$ only if $r_g = 0$.

The contradiction found between masses m_0 and m can be rejected if we accept $g_{22} = \rho^2$ in equality (3) and consider ρ as the unknown function of radial coordinate r. Accepting $T_i^j = 0$ and $g_{22} = \rho^2(r)$ in Eqs. (6) and (7) and assuming $\rho(r \to \infty) = r$, we can find the following metric coefficients for the external field:

$$g_{11} = \frac{(\rho')^2}{1 - \frac{r_g}{\rho}}, \quad g_{44} = 1 - \frac{r_g}{\rho}. \quad (22)$$

Accepting $g_{22} = \rho^2$ and $T_4^4 = \mu c^2$ in Eq. (7) and assuming $\rho(r = 0) = 0$, we can find the radial metric coefficient for the internal field

$$g_{11}^i = \frac{(\rho')^2}{1 - \frac{\chi\mu c^2}{3}\rho^2}. \quad (23)$$

Here, $0 \le \rho \le \rho_R$ and $\rho_R = \rho(r = R)$ corresponds to the outer sphere surface. The expression for the body mass has the form

$$m = 2\mu \int_0^{2\pi} d\varphi \int_0^{\pi/2} \sin\theta d\theta \int_0^R \sqrt{g_{11}^i}\rho^2 dr. \quad (24)$$

Let us determine functions $\rho(r)$ for the external and internal fields from conditions

$$\sqrt{g_{11}}\rho^2 = r^2, \quad \sqrt{g_{11}^i}\rho^2 = r^2. \quad (25)$$

Substitution of the second of these conditions into equality (24) gives $m = m_0$, where m_0 is determined by equality (21). Thus, the body mass turns out to be Euclidian. This result can be interpreted physically: gravitation, when varying the geometry inside the body according to the GR equations, does not vary the body mass. Since $m = m_0$, we can use equality (19), and coefficient (23) takes the following form:

$$g_{11}^i = \frac{(\rho')^2}{1 - \frac{r_g \rho^2}{R^3}}. \quad (26)$$

It also follows from conditions (25) that continuity of metric coefficient g_{11} on the body surface is provided if function ρ is continuous.

With the help of expressions (22) and (26), conditions (25) are reduced to the following equations for function $\rho(r)$ for the internal and external spaces:

$$\rho'\rho^2 = r^2\sqrt{1 - \frac{r_g\rho^2}{R^3}}, \quad \rho'\rho^2 = r^2\sqrt{1 - \frac{r_g}{\rho}}. \quad (27)$$

For the internal space, the solution of first equation (27) satisfying condition $\rho(r = 0) = 0$ has the form

$$F_i(\rho) = \frac{2}{3}\frac{r_g r^3}{R^4},$$
$$F_i(\rho) = \sqrt{\frac{R}{r_g}}\arcsin\left(\frac{\rho}{R}\sqrt{\frac{r_g}{R}}\right) - \frac{\rho}{R}\sqrt{1 - \frac{r_g\rho^2}{R^3}}. \tag{28}$$

Taking into account $r = R$ in solution (28), we derive the following equation associating ρ_R, i.e., the value of ρ on the sphere surface, with r_g:

$$F_i(\rho_R) = \frac{2r_g}{3R}. \tag{29}$$

For the external space, the solution of second Eq. (27) satisfying the continuity condition of function ρ on the surface of the sphere can be written in the form

$$F_e(\rho) - F_e(\rho_R) = \frac{1}{3}\left(\frac{r^3}{R^3} - 1\right), \tag{30}$$

$$F_e(\rho) = \frac{1}{R^3}\left(\frac{\rho^2}{3} + \frac{5r_g\rho}{12} + \frac{5r_g^2}{8}\right)\sqrt{\rho(\rho - r_g)} + \frac{5r_g^{\ 3}}{8R^3}\ln\left[\sqrt{\frac{\rho}{R}} + \sqrt{\frac{\rho - r_g}{R}}\right]. \tag{31}$$

The asymptotic analysis of Eq. (30) shows that $\rho(r \to \infty) = r$. The numerical analysis of dependences $g_{11}(r)$ and $\rho(r)$ [6, 7] shows that the solution found at $r = 0$ and $r \to \infty$ coincides with the Schwarzschild solution. The greatest difference takes place on the surface of sphere $r = R$. In this case, the Schwarzschild solution for g_{11} diverges when approaching R to r_g, while the found solution remains finite.

It follows from equality (31) that the constructed solution is meaningful if minimal value $\rho = \rho_R$ is larger than r_g or equals it. Thus, we have $\rho_R = r_g$ in the limiting case. Assuming $\rho_R = r_g$ in Eq. (29), we find the minimal possible value of radius $R_g = 1.115r_g$. If $R > R_g$, the solution turns out to be imaginary. In this case, in contrast to the Schwarzschild solution, it is nonsingular at $R = R_g$. Taking into account formula (2) for r_g, we will finally derive

$$R_g = \frac{0.327c}{\sqrt{\mu G}}. \tag{32}$$

The definition of the escape velocity for the space with metric coefficients (22) is similar to the above-constructed solution for the Schwarzschild metric (8) (variable r in Eq. (9) is simply replaced by ρ), and the result has the form

$$v_2 = c\sqrt{\frac{r_g}{\rho_R}}.$$

For the sphere with a limiting minimal radius, we have $\rho_R = r_g$ and, consequently, $v_2 = c$. This result can be treated as the generalization of the Laplace concept of a "dark star," which has no singularity in contrast with a "black hole." The assumption on the occurrence of such starts makes it possible to explain the phenomenon of dark matter.

If "dark stars" occur, then the radii of visible stars should be smaller than R_g. For the largest of visible stars—the UY Scuti red supergiant—we have $R = 1.19 \times 10^{12}$ m and $m = 6.4 \times 10^{30}$ kg [8]. Determining the average density according to formula (21) and using equality (32), we will derive R_g exceeding the radius of this star by approximately 4000.

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