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PARTIAL DIFFERENTIAL EQUATIONS

Three-Dimensional Quasiconformal Mappings and Axisymmetric Problems

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Abstract—Quasiconformal mappings of axisymmetric domains are considered as a special case of three-dimensional transformations. For a three-dimensional steady irrotational flow of an inviscid incompressible fluid, two stream functions are introduced along with the velocity potential. Any solenoidal vector can be represented as the cross product of the gradients of two stream functions. As a result, a relationship between the velocity components and the stream functions is obtained for determining the velocity potential. On the one hand, these transformations underlie Lavrentiev-harmonic mappings. On the other hand, these conditions can be treated as a generalization of the Cauchy–Riemann conditions to the three-dimensional case. In this work, the generalized three-dimensional Cauchy–Riemann conditions for harmonic mappings are reduced to the usual Cauchy–Riemann conditions in polar coordinates of complex variable functions. Lavrentiev-harmonic conditions are used to construct an analogue of quasiconformal mapping of axisymmetric domains and to generalize mappings of axisymmetric domains to arbitrary domains. Examples of visualization of quasiconformal mappings of axisymmetric domains and their generalizations are given.

Keywords: conformal mappings, Lavrentiev-harmonic mappings, generalized Cauchy–Riemann conditions, axisymmetric potential flows

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1. INTRODUCTION

Two-dimensional conformal mappings are used to calculate and visualize harmonic vector fields in fluid dynamics, elasticity, filtration theory, electromagnetism, etc. The application of two-dimensional conformal mappings is associated with solving boundary value problems for the Laplace equation, to which numerous stationary problems of mathematical physics reduce. These problems describe steady flows of an incompressible ideal fluid, wave propagation, diffusion processes, heat propagation, the theory of gravitation, electrostatics, etc.

Attempts to extend the methods of two-dimensional conformal mappings to the three-dimensional case have been made for many years. In general form, the properties of plane conformal mappings are not generalized to three-dimensional problems. In Euclidean space for $n > 2$, the class of conformal mappings is covered with a finite number of compositions of mappings of four types: translations, similarities, orthogonal transformations, and inversions (see [1]).

The theory of functions of several complex variables has been covered in numerous monographs (see, e.g., [2–5]). A good theory is strong in its applications, as is evident from the two-dimensional theory of functions of a complex variable. For functions of several complex variables, the application of a powerful mathematical apparatus has led to significant complications. In the three-dimensional case, the solvability of problems is associated with the topological and analytical properties of complex manifolds. The main issues of the theory of three-dimensional conformal mappings have been studied insufficiently, and the theory of multidimensional conformal mappings has not found practical applications.

If some general restrictions are dropped, the properties of two-dimensional conformal mappings can be generalized to the three-dimensional case (see [6]). An analogue of three-dimensional quasiconformal mappings was obtained by sequentially using two usual functions of a complex variable. Examples of mesh generation by applying the theory of quasiconformal mappings were given. Attempts to use conformal mappings for mesh generation were made earlier in [7, 8]. A class of quasiconformal mappings for compositions of plane mappings was constructed in [9].

Below, an axisymmetric problem is considered from the point of view of the theory of three-dimensional quasiconformal mappings. Three-dimensional (axisymmetric) problems have symmetry that makes it possible to study them as, in fact, two-dimensional problems. In contrast to the plane case, for which the theory was developed with applications in various fields, a similar theory for axisymmetric flows requires further development. At the same time, spatial axisymmetric problems are essentially three-dimensional.

This paper is organized as follows. The mathematical background of Lavrentiev-harmonic mappings is introduced in Section 2. In Section 3, we consider the equations of motion for the axisymmetric case. Three-dimensional quasiconformal mappings with polar coordinates of two usual functions of a complex variable are considered in Section 4. Examples of visualization are given in Section 5.

2. LAVRENTIEV-HARMONIC MAPPINGS

1. The most interesting quasiconformal mappings of three-dimensional domains are obtained using a hydrodynamic analogy (see [10]). Consider a steady incompressible inviscid flow. Assume that the flow is irrotational. The equations of motion in a Cartesian coordinate system have the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,\quad \frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}=0,\quad \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=0,\quad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0. \tag{1}$$

The vector field u, v, w is potential and solenoidal.

Since $\operatorname{curl}\mathbf{u}=0$, which means no vorticity, in the entire flow region, there exists a function $\xi(x,y,z)$ in the case of steady flow or a function $\xi(x,y,z,t)$ of coordinates and time in the case of unsteady flow such that $\mathbf{u}=\nabla\xi$, $\left(u=\frac{\partial\xi}{\partial x}, v=\frac{\partial\xi}{\partial y}, w=\frac{\partial\xi}{\partial z}\right)$. Here, the operator ∇ is defined as $\nabla=\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}$. The function ξ is called the velocity potential. Assume that ξ is continuous, together with its first two derivatives with respect to time and coordinates. In contrast to the two-dimensional case, system (1) is overdetermined: it consists of four equations for three variables. The theory of overdetermined systems of differential equations with constant coefficients relies heavily on the theory of functions of several complex variables.

The problem of determining the function ξ satisfying the Laplace equation in a domain D from the values of its normal derivative given on the surface S is called the Neumann problem. A harmonic function is sought given the value of its normal derivative on the domain boundary. To such a problem, it is possible to reduce the problem of determining the velocity potential of an incompressible fluid flowing over a given geometry. If the flow region contains a point at infinity, then we require the existence of the limit of $\operatorname{grad}\xi$—the free-stream velocity $\mathbf{u}_\infty$—as $x^2+y^2+z^2\to\infty$ and assume that this vector is given. For domains D with a sufficiently smooth boundary, a harmonic function ξ in D that satisfies the boundary condition $\frac{\partial\xi}{n\partial n}=0$ and condition at infinity if D contains an infinite point always exists and is defined up to a real constant.

Flow problems reduce to a mapping of the flow domain to a domain in the space of the potential. Steady inviscid flows without vortices or sources can be regarded as mappings of the flow domain to the range of the velocity vector. Mappings satisfying the conditions $u=\xi_x$, $v=\xi_y$, and $w=\xi_x$ or the conditions $\operatorname{div}\mathbf{u}=0$, $\operatorname{curl}\mathbf{u}=0$ are called harmonic. Vector functions defining harmonic mappings have a number of properties similar to those of analytic functions.

2. Along with the velocity potential $\xi(x,y,z)$, we introduce two stream functions, $\zeta(x,y,z)$ and $\eta(x,y,z)$, such that the surfaces $\zeta(x,y,z)=\text{const}$ and $\eta(x,y,z)=\text{const}$ intersect along a streamline. For irrotational flow, $\nabla\times\mathbf{u}=0$. Each solenoidal vector can be represented as the cross product of the gradients of two functions:

$$\mathbf{u}=\nabla\zeta\times\nabla\eta.$$

The velocity is tangent to two families of surfaces $\nabla\zeta$, $\nabla\eta$, which are stream surfaces. The velocity vector is represented in the form

$$\mathbf{u}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}=\nabla\zeta\times\nabla\eta.$$

Recall that

$$\nabla\zeta\times\nabla\eta=\begin{bmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \zeta_x & \zeta_y & \zeta_z\\ \eta_x & \eta_y & \eta_z\end{bmatrix}=\mathbf{i}(\zeta_y\eta_z-\zeta_z\eta_y)+\mathbf{j}(\zeta_z\eta_x-\zeta_x\eta_z)+\mathbf{k}(\zeta_x\eta_y-\zeta_y\eta_x). \quad (2)$$

The relationships between the new variables can be written as

$$\xi_x=\zeta_y\eta_z-\zeta_z\eta_y,\quad \xi_y=\eta_x\zeta_z-\eta_z\zeta_x,\quad \xi_z=\zeta_x\eta_y-\zeta_y\eta_x. \quad (3)$$

Relationships (3) connect geometric and hydrodynamic relations. For determining the velocity potential, we have two stream functions ζ, η. System (3) is a Cauchy–Kovalevskaya type system. Transformations (3) underlie Lavrentiev-harmonic mappings (see [10]). Conditions (3) can be treated as a generalization of the Cauchy–Riemann conditions to the three-dimensional case (see [10]), from which three-dimensional quasiconformal mappings follow.

Consider the inverse transform. In system (3), the variables x, y, z are made dependent, while the variables ξ, η, ζ are made independent. For this purpose, the formulas for transforming the derivatives in the transition from the coordinates ξ,ζ,η to x, y, z are substituted into system (3).

As a result, we obtain

$$x_\xi=y_\zeta z_\eta-z_\zeta y_\eta,\quad y_\xi=z_\zeta x_\eta-x_\zeta z_\eta,\quad z_\xi=x_\zeta y_\eta-y_\zeta x_\eta. \quad (4)$$

Mapping (4) can be treated as a transformation of some flow in D to translational motion in another domain.

3. GOVERNING EQUATIONS OF MOTION OF AN INVISCID INCOMPRESSIBLE FLUID IN THE AXISYMMETRIC CASE

The axisymmetric case will be considered as a special case of three-dimensional transformations. The axis of symmetry is used as the z axis, and the distance to the axis is denoted by ρ. Consider three-dimensional flows such that all velocity vectors lie in half-planes passing through a straight line, which is called the axis of symmetry. In any of these half-planes, flow is described by a plane field. The velocity components are denoted by u, v, respectively. The equations of motion have the form (see [11])

$$\frac{\partial\rho u}{\partial z}+\frac{\partial\rho v}{\partial\rho}=0,\quad \frac{\partial u}{\partial\rho}-\frac{\partial v}{\partial z}=0. \quad (5)$$

The first relation is the continuity one ($\operatorname{div}\mathbf{u}=0$), and the second is the no-vorticity condition ($\operatorname{curl}\mathbf{u}=0$). The latter condition is necessary and sufficient for the velocity field to be potential. The velocity components of an axially symmetric flow are expressed as

$$u=\frac{\partial\xi}{\partial z},\quad v=\frac{\partial\xi}{\partial\rho}. \quad (6)$$

The velocity potential ξ satisfies the relation

$$\frac{\partial^2\xi}{\partial z^2}+\frac{\partial^2\xi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\xi}{\partial\rho}=0,$$

which is the Laplace equation in cylindrical coordinates. Note that ξ is a harmonic function of Cartesian coordinates.

According to the first equation, the expression $-\rho v dz+\rho u d\rho$ is the exact differential of the function ζ. We obtain

$$\mathrm{u}=\frac{1}{\rho}\frac{\partial\zeta}{\partial\rho},\quad v=-\frac{1}{\rho}\frac{\partial\zeta}{\partial z}. \quad (7)$$

The function $\zeta(z,\rho)$ is called the stream function. It remains constant on each streamline and remains constant on the surface obtained by rotation of this streamline around the axis of symmetry. Note that the stream function depends on the choice of the coordinate system and the character of the motion. The notation accepted in this paper may differ, up to the sign, from those used in other works.

The stream function ζ satisfies the equation

$$\frac{\partial^2\zeta}{\partial z^2}+\frac{\partial^2\zeta}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial\zeta}{\partial\rho}=0.$$

This equation is not the Laplace one, and the function ζ is not harmonic in Cartesian coordinates.

It follows from (6) and (7) that the stream function and the potential are related by the equalities

$$\xi_z=\zeta_\rho/\rho,\quad \xi_\rho=-\zeta_z/\rho. \tag{8}$$

The equations for axisymmetric flows are similar in many respects to the equations for plane motions. The vector lines of the velocity field coincide with the lines $\zeta(z,\rho)=\mathrm{const}$ and, as in the two-dimensional case, are streamlines. It follows from the equations that the lines $\xi(z,\rho)=\mathrm{const}\,z$ and $\zeta(x,\rho)=\mathrm{const}$ are orthogonal. Indeed,

$$\xi_z\zeta_z+\xi_\rho\zeta_\rho=0.$$

If the velocity potential is given, then the stream function in the axisymmetric case can be found using well-known formulas.

System (5) is a system of elliptic equations with a singularity on the axis of rotation. Although the system has a singularity, the Riemann theorem on the existence and uniqueness of mappings holds for it. The mappings have the basic properties of quasiconformal mappings (see [11]).

4. QUASICONFORMAL MAPPINGS OF THREE-DIMENSIONAL AXISYMMETRIC DOMAINS

1. Let us return to the generalized Cauchy–Riemann conditions (3). If the stream function $\eta(x,y,z)$ depends on two variables, $\eta(x,y)$, i.e., $\eta_z=0$, then system (3) reduces to

$$\xi_x=-\zeta_z\eta_y,\quad \xi_y=\eta_x\zeta_z,\quad \xi_z=\zeta_x\eta_y-\zeta_y\eta_x. \tag{9}$$

In the three-dimensional case, each problem is determined by the topological and analytical properties of the considered domains. Let C^2 be the space of two independent complex variables $z=(z_1,z_2)$. In what follows, by the space C^2 of complex variables z_1, z_2, we mean the usual Euclidean space R^4 of real variables (x,y,z,t). In passing from the 4-dimensional Euclidean space R^4 to the complex space, there appears a certain asymmetry.

Let us show that the solution of system (9) can be represented using two independent complex variable functions $\zeta_1=f_1(z_1)$ and $\zeta_2=f_2(z_2)$ defined in their domains. Assume that a univalent analytic function $\zeta_1=f_1(z_1)$ is given in the domain D. Every complex number $z_1=x+iy$ from some domain D_1 ($D_1\subset C^2$) is put in correspondence with a complex number $\zeta_1=\tau e^{i\eta}$. Similarly, $\zeta_2=f_2(z_2)=\xi(z,t)+i\zeta(z,t)$ is a univalent analytic function of the complex variable $z_2=z+it$. Every complex number $z_2=z+it$ from some domain D_2 ($D_2\subset C^2$) is put in correspondence with a complex number $\zeta_2=\xi+i\zeta$.

For the functions $f_1(z_1)$, $f_2(z_2)$ to be analytic in a domain of C^2, it is necessary that the Cauchy–Riemann conditions hold in this domain. If the function $\zeta_1=f_1(z_1)$ of complex variable $z_1=x+iy$ can be represented in exponential form $\zeta_1=\tau e^{i\eta}$, then the Cauchy–Riemann conditions for the differentiability of the function with respect to x, y have the form (see [12])

$$\tau_x/\tau=\eta_y,\quad \tau_y/\tau=-\eta_x. \tag{10}$$

For the function $\zeta_2=f_2(z_2)=\xi(z,t)+i\zeta(z,t)$, the Cauchy–Riemann conditions are given by

$$\xi_z=\zeta_t/t,\quad \xi_t=-\zeta_z/t. \tag{11}$$

A solution of system (9) is sought in the form $\xi=\xi(z,\tau(x,y))$, $\zeta=\zeta(z,\tau(x,y))$, $\eta=\eta(x,y)$. Here, as t, we choose the function $t=\tau(x,y)$. For the first two equations in (9), we obtain

$$\xi_\tau\tau_x=-\zeta_z\eta_y,\quad \xi_\tau\tau_y=\eta_x\zeta_z. \tag{12}$$

System (12) decouples and, taking into account (10), the first and second equations of system (12) take the same form:

$$\xi_\tau = -\zeta_z/\tau, \quad \xi_\tau = -\zeta_z/\tau. \tag{13}$$

The third equation of system (9), i.e., $\xi_z = \zeta_x\eta_y - \zeta_y\eta_x$, reduces to the form

$$\xi_z = \zeta_\tau(\tau_x\eta_y - \tau_y\eta_x) = J_1^*\zeta_\tau/\tau, \quad J_1 = J_1^*/\tau = \tau_x\eta_y - \tau_y\eta_x, \quad J_1^* > 0. \tag{14}$$

By using the changes of variables $z^* = \sqrt{J_1^*}z$, $\zeta^* = \sqrt{J_1^*}\zeta$, and $J_1^* = \tau J_1$, system (13), (14) reduces to

$$\xi_{z^*} = \zeta_\tau^*/\tau, \quad \xi_\tau = -\zeta_{z^*}^*/\tau. \tag{15}$$

Obviously, the solution of (15) can be represented in the form

$$\zeta_2^* = f_2(z_2^*) = \xi(z^*, t) + i\zeta^*(z^*, t).$$

In the special case when $J_1^* = 1$ and $t = \tau(x, y)$, Eqs. (13) and (14) become

$$\xi_z = \zeta_\tau/\tau, \quad \xi_\tau = -\zeta_z/\tau. \tag{16}$$

Thus, the three-dimensional quasiconformal mapping (9) can be represented in the form of a sequence of two two-dimensional conformal mappings: $\zeta_1 = f_1(z_1)$, $\zeta_2^* = f_2(z_2^*)$. Here, $\zeta_1 = \tau e^{i\eta}$, $z_1 = x + iy$, and $\zeta_2^* = \xi + i\zeta^*$, $z_2 = z^* + it$. Using relations (10) and (15) in the domain D of three-dimensional real space R^3, it is possible to find the velocity potential ξ and the stream function, as well as the second stream function η and the quantity τ. If the function $\tau = \tau(x, y)$ is used as t in system (11), then we obtain

$$\xi = \xi(z^*, \tau(x, y)), \quad \zeta^* = \zeta^*(z^*, \tau(x, y)), \quad \eta = \eta(x, y), \quad t = \tau(x, y).$$

Recall that $z^* = \sqrt{J_1^*}z$, $\zeta^* = \sqrt{J_1^*}\zeta$, and $J_1 = J_1^*/\tau = \tau_x\eta_y - \tau_y\eta_x$.

2. Note that, in the axisymmetric case, conditions (16) and (8) for the stream function and the velocity potential coincide if the function $\rho = (x^2 + y^2)^{1/2}$ is used as τ. Indeed,

$$\eta_x = -y/\rho^2, \quad \eta_y = x/\rho^2, \quad \rho_x = x/\rho, \quad \rho_y = y/\rho.$$

It follows that

$$\rho_x^2 + \rho_y^2 = 1, \quad \eta_x^2 + \eta_y^2 = 1/\rho^2 \tag{17}$$

and

$$\rho_x/\rho = \eta_y, \quad \rho_y/\rho = -\eta_x.$$

Straightforward calculations, taking into account (17), yield $J_1^* = 1$ and $J_1 = 1/\rho$.

Thus,

$$\zeta_2 = f_2(z_2), \quad \zeta_1 = f_1(z_1)$$

or

$$\xi = \xi(z, \rho), \quad \zeta = \zeta(z, \rho), \quad \eta = \arctan(y/x), \quad \rho = (x^2 + y^2)^{1/2}.$$

3. Consider the inverse transformation (4). Since $z_\eta = 0$ in the case under consideration, system (4) becomes

$$x_\xi = -z_\zeta y_\eta, \quad y_\xi = z_\zeta x_\eta, \quad z_\xi = x_\zeta y_\eta - y_\zeta x_\eta. \tag{18}$$

Let us show that the solution of system (18) can be represented using complex variable functions $z_1 = F_1(\zeta_1)$ and $z_2 = F_2(\zeta_2)$. Here, $z_1 = x + iy$ and $\zeta_1 = \tau e^{i\eta}$. Assume that a univalent analytic function $z_1 = F_1(\zeta_1)$ is given in a domain G_1. Each complex number $\zeta_1 = \tau e^{i\eta}$ from G_1 ($G_1 \subset C^2$) is assigned a complex number $z_1 = x + iy$. Similarly, $z_2 = F_2(\zeta_2)$ is an univalent analytic function of the complex variable $\zeta_2 = \xi + i\zeta$. Each complex number $\zeta_2 = \xi + i\zeta$ in a domain G_2 ($G_2 \subset C^2$) is assigned a complex number $z_2 = z + it$.

If the independent variable ζ_1 can be represented in exponential form $\zeta_1 = \tau e^{i\eta}$, then sufficient conditions for the differentiability of the function $z_1 = x(\rho,\eta) + iy(\rho,\eta)$ have the form (see [12])

$$x_\tau \tau = y_\eta, \quad x_\eta = -\tau y_\tau. \tag{19}$$

Similarly, the Cauchy–Riemann conditions for the function $z_2 = z(\xi,\zeta) + it(\xi,\zeta)$ are given by

$$z_\xi = tt_\zeta, \quad z_\zeta = -t_\xi/t. \tag{20}$$

Consider the first and second equations of system (18). A solution is sought in the form $x = x(\tau(\xi, \zeta), \eta)$, $y = y(\tau(\xi, \zeta), \eta)$, $z = z(\xi,\zeta)$, $\tau = t(\xi,\zeta)$. In view of $z_\zeta = -t_\xi/t$, the first and second equations in (18) can be represented as

$$x_\tau \tau = y_\eta, \quad x_\eta = -\tau y_\tau. \tag{21}$$

Thus, relation (21) coincides with (19).

The third equation of system (18) becomes

$$z_\xi = x_\tau \tau_\zeta y_\eta - y_\tau \tau_\zeta x_\eta = \tau\tau_\zeta J_2^*, \quad J_2 = x_\tau y_\eta - y_\tau x_\eta, \quad J_2 = \tau J_2^*, \quad J_2^* > 0.$$

Under the conditions $J_2 = J_2^* \tau$ and $z_\eta = 0$, the generalized Cauchy–Riemann conditions (18) reduce to the equations

$$x_\tau = y_\eta/\tau, \quad x_\eta = -\tau y_\tau \quad \text{and} \quad z_\xi = \tau_\zeta \tau J_2^*, \quad z_\zeta = -\tau_\xi/\tau. \tag{22}$$

With the changes of variables $z^* = z/\sqrt{J_2^*}$, $\zeta^* = \zeta/\sqrt{J_2^*}$, and $J_2^* = J_2/\tau$, the last two equations in (22) can be reduced to a form satisfying the usual Cauchy–Riemann conditions. As a result,

$$z_\xi^* = \tau_{\zeta^*}\tau, \quad z_{\zeta^*}^* = -\tau_\xi/\tau. \tag{23}$$

Conditions (23) are ones for the differentiability of the function

$$z_2^* = F_2(\zeta_2^*) = z^*(\xi,\zeta^*) + it(\xi,\zeta^*), \quad z^* = z/\sqrt{J_2^*}, \quad \zeta^* = \zeta/\sqrt{J_2^*}.$$

Therefore, system (18) can be solved using arbitrary univalent analytic functions $z_1 = F_1(\zeta_1)$ and $z_2^* = F_2(\zeta_2^*)$. Here, $z_1 = x + iy$, $\zeta_1 = \tau e^{i\eta}$, $z_2^* = z^* + it$, and $\zeta_2 = \xi + i\zeta^*$.

The solution of system (18) has the form

$$x = x(t(\xi,\zeta^*),\eta), \quad y = y(t(\xi,\zeta^*), \eta), \quad z = z(\xi,\zeta^*), \quad \tau = t(\xi,\zeta^*).$$

For the axisymmetric case, direct verification yields ($\tau = \rho(\xi,\zeta)$)

$$x_\rho = y_\eta/\rho, \quad y_\rho = -x_\eta/\rho \quad \text{and} \quad z_\xi = \rho_\zeta\rho, \quad z_\zeta = -\rho_\xi/\rho, \quad J_2^* = 1.$$

Thus, in the axisymmetric case, we obtain

$$x = \rho\cos\eta, \quad y = \rho\sin\eta, \quad z = z(\xi,\zeta), \quad \rho = \rho(\xi,\zeta).$$

The equations imply that the lines $z(\xi,\zeta) = \text{const}$, $\rho(\xi,\zeta) = \text{const}$ are orthogonal. Indeed,

$$z_\xi z_\zeta + \rho_\xi\rho_\zeta = 0.$$

4. The covariant components of the metric coefficients are given by the formulas

$$g_{ij}^* = \frac{\partial x}{\partial x^i}\frac{\partial x}{\partial x^j} + \frac{\partial y}{\partial x^i}\frac{\partial y}{\partial x^j} + \frac{\partial z^*}{\partial x^i}\frac{\partial z^*}{\partial x^j}, \quad i,j = 1,2,3 \quad (\xi = x^1, \ \zeta^* = x^2, \ \eta = x^3).$$

Straightforward calculations show that

$$g_{11}^* = (x_\tau^2 + y_\tau^2)(z_\xi^{*2} + \tau_\xi^2), \quad g_{22}^* = z_{\zeta^*}^{*2} + \tau_{\zeta^*}^2 = (z_\xi^{*2} + \tau_\xi^2)/\tau^2, \quad g_{33}^* = x_\eta^2 + y_\eta^2 = (x_\tau^2 + y_\tau^2)/\tau^2,$$

whence

$$g_{11}^* = g_{22}^* g_{33}^*,$$

and

$$g_{12}^* = g_{13}^* = g_{23}^* = g_{21}^* = g_{31}^* = g_{32}^* = 0.$$

Thus, we have obtained a triple orthogonal system of coordinates. However, by the Cotton–Darboux theorem, the metric in three-dimensional space can always be brought to a diagonal form by a local transformation.

5. A necessary condition for the mapping defined by analytic functions $z_1 = F(\zeta_1)$ and $z_2 = F(\zeta_2)$ to be quasiconformal is that it is one-to-one, i.e., the functions $F_1(\zeta_1)$, $F_2(\zeta_2)$ have to be univalent in G. Recall that, in the two-dimensional case, a necessary condition for an analytic function to be univalent in G is that $F'(\zeta) \neq 0$, i.e., the derivative has to be nonzero everywhere in this domain, except for nonisolated essential singularities. The transformation is quasiconformal and continuous everywhere, except for domains where the derivatives $F_1'(\zeta_1)$, $F_2'(\zeta_2)$ or $1/F_1'(\zeta_1)$, $1/F_2'(\zeta_2)$ do not exist. The inverse of a quasiconformal mapping is also quasiconformal. In other words, if the functions $z_1 = F(\zeta_1)$ and $z_2 = F(\zeta_2)$ quasiconformally map G onto D, then the inverse function is a quasiconformal mapping of D onto G.

For a mapping to be one-to-one, the Jacobian of the transformation has to be finite and nonzero. The local homeomorphicity of the mapping can be violated only at points where the Jacobian of the mapping vanishes. Finding zeros of the Jacobian of the mapping reduces to studying the properties of critical points. To find the Jacobian of the transformation, $J = \partial(\xi,\eta,\zeta)/\partial(x,y,z)$, in the case $\eta_z = 0$, we use Cauchy–Riemann-type conditions of the form (9). As a result,

$$J = (\tau_x\eta_y - \tau_y\eta_x)(\xi_z\zeta_\tau - \xi_\tau\zeta_z) = (\xi_{z^*}^2 + \xi_\tau^2)(\tau_x^2 + \tau_y^2) > 0.$$

The Jacobian of the transformation is positive. This guarantees that the mapping of the parametric domain to the given domain is one-to-one.

Along with the case $\eta_z = 0$, we might similarly consider the cases $\eta_x = 0$ and $\eta_y = 0$. Note that the theory of two-dimensional conformal mappings is completely described by the theory of univalent analytic functions of one complex variable, $z_1 = x + iy$, $\bar{z}_1 = x - iy$ and $z_2 = z + it$, $\bar{z}_2 = z - it$. Therefore, quasiconformal mappings are implemented by holomorphic functions of complex variable or by anti-holomorphic functions. Thus, in addition to the case considered above, the solution of (17) can be represented in the form of anti-holomorphic functions or a combination of functions.

By using conditions (3) and (4) for the construction of Lavrentiev-harmonic mappings, it is possible to generalize three-dimensional quasiconformal mappings to the axisymmetric case. In the general three-dimensional case, the construction of quasiconformal mapping depends on the space topology. The number of possible classes of quasiconformal mappings increases and requires further research.

5. VISUALIZATION RESULTS FOR THREE-DIMENSIONAL QUASICONFORMAL MAPPINGS

Consider some results of visualizing three-dimensional vector fields, examples of which for the two-dimensional case are given, for instance, in [12].

With the help of quasiconformal mappings, we can introduce three-dimensional systems of curvilinear coordinates (see [13]). As was shown above, the solution of system (4) can be represented using two analytic functions of a complex variable: $z_1 = F_1(\zeta_1)$ and $z_2 = F_2(\zeta_2)$. Here, $z_1 = x + iy$ and $\zeta_1 = \rho e^{i\eta}$. Similarly, $z_2 = z + i\rho$ and $\zeta_2 = \xi + i\zeta$. Thus, in the axisymmetric case, we obtain $x = \rho\cos\eta$, $y = \rho\sin\eta$, and $z = z(\xi,\zeta)$, where $\rho = \rho(\xi,\zeta)$. These relations yield the following examples of curvilinear coordinates:

—cylindrical circular coordinates

$$z_1 = \rho e^{i\eta}, \quad z_2 = \zeta_2, \quad x = \rho\cos\eta, \quad y = \rho\sin\eta, \quad z = \xi, \quad \rho = \zeta;$$

—spherical coordinates

$$z_1 = \rho e^{i\eta}, \quad z_2 = e^{\xi + i\zeta}, \quad x = \rho\cos\eta, \quad y = \rho\sin\eta, \quad z = e^{\xi}\cos\zeta, \quad \rho = e^{\xi}\sin\zeta$$

(usually, different notation is used for spherical coordinates, namely, $r = e^{\xi}$, $x = r\cos\eta\sin\zeta$, $y = r\sin\eta\sin\zeta$, $z = r\cos\zeta$).

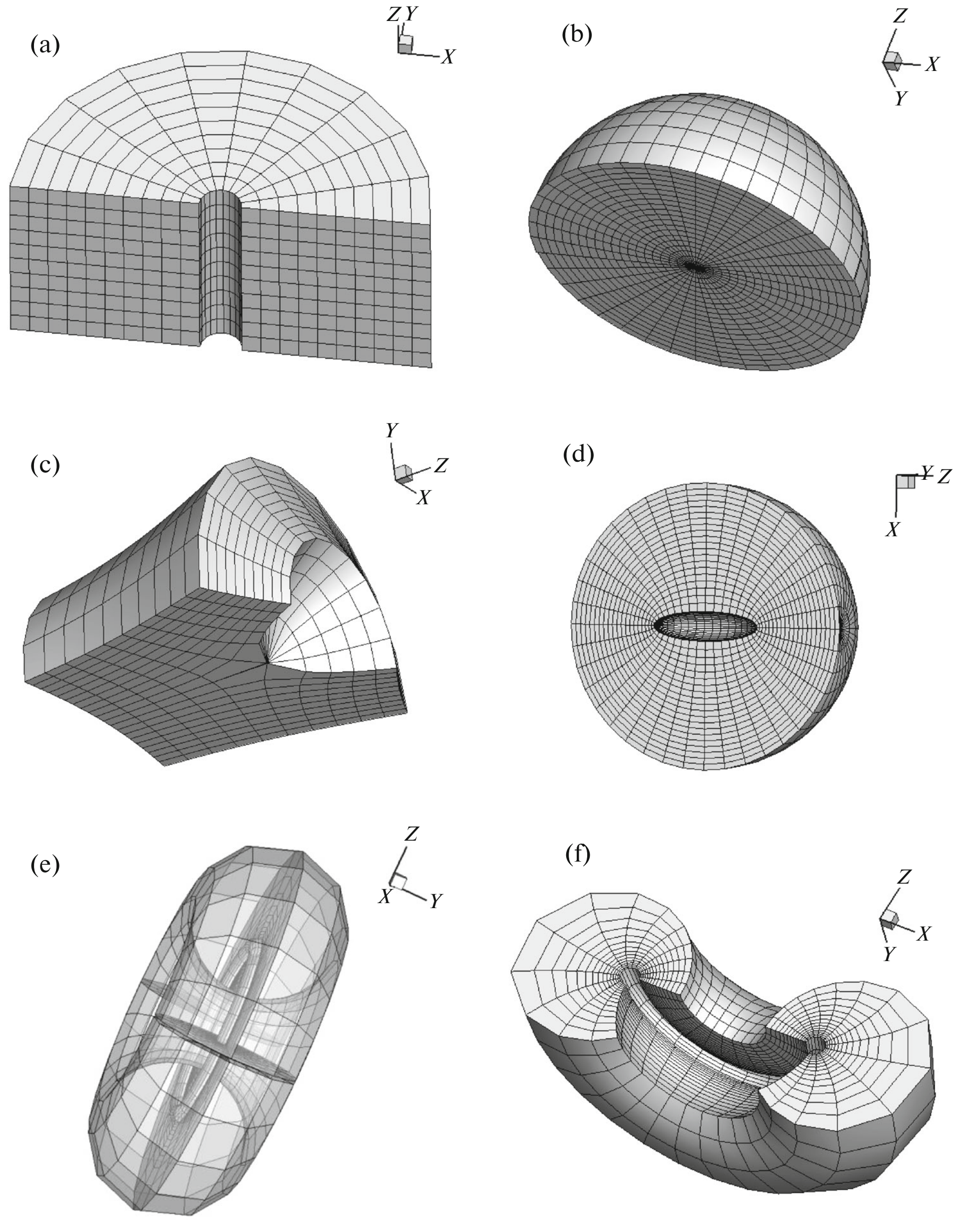

Fig. 1.

—parabolic coordinates of rotation

$$z_1 = \rho e^{i\eta}, \quad z_2 = \zeta_2^2/2, \quad x = \rho\cos\eta, \quad y = \rho\sin\eta, \quad z = (\xi^2 - \zeta^2)/2, \quad \rho = \xi\zeta;$$

—coordinates of a prolate ellipsoid of rotation

$$z_1 = \zeta_1, \quad z_2 = \alpha\sin\zeta_2, \quad x = \rho\cos\eta, \quad y = \rho\sin\eta, \quad z = \alpha\sin\xi\cosh\zeta, \quad \rho = \alpha\cos\xi\sinh\zeta;$$

—toroidal coordinates

$$z_1 = \rho e^{i\eta}, \quad z_2 = \tanh(\zeta_2/2), \quad x = \rho\cos\eta, \quad y = \rho\sin\eta,$$
$$z = \alpha\sin\zeta/(\cosh\xi + \cos\zeta), \quad \rho = \alpha\sinh\xi/(\cosh\xi + \cos\zeta).$$

Software programs in FORTRAN and C^{++} were developed to visualize three-dimensional mappings. Figure 1 presents visual representations of systems of curvilinear coordinates: (a) cylindrical, (b) spherical, (c) parabolic, (d) ellipsoidal, and (e) and (f) toroidal.

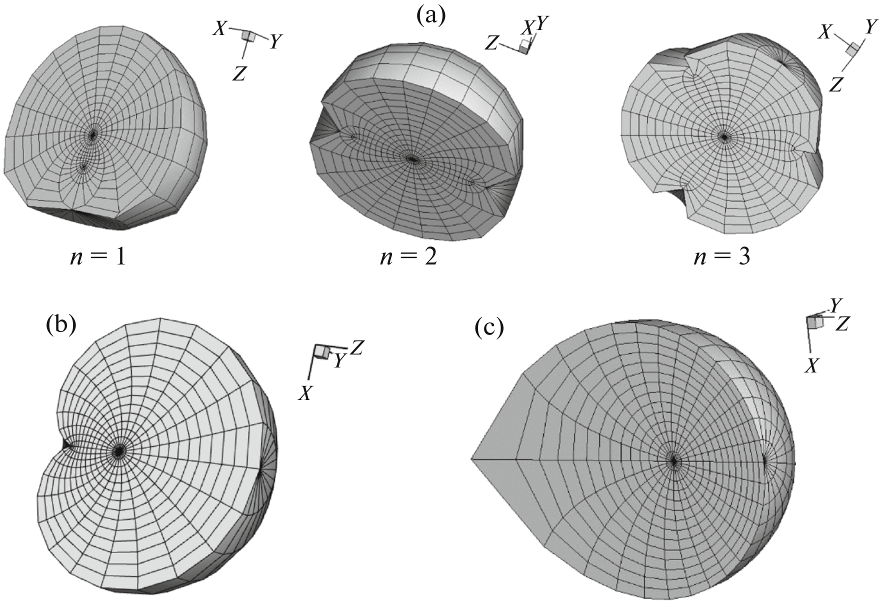

Fig. 2.

For finite two-dimensional simply connected domains, a canonical domain is the unit disk. Figure 2 shows the results of visualization of quasiconformal mappings for a finite domain. An analytic continuation is obtained from the corresponding plane coordinates z, ρ by rotation through the angle η around the axis of symmetry $\rho = 0$. The coordinates z, ρ vary in a half-plane. The functions $z_1 = \rho e^{i\eta}$ and $z_2 = F_2(\zeta_2)$ define a mapping of the interior of the domain ($|\zeta_2| < 1$) to the considered domain. For the complex number in exponential form, the argument $\eta = \arg z_1$ is a multivalued function. We choose η in the interval $0 \le \eta < 2\pi$. For domains symmetric about the origin, the function $z_2 = F_2(\zeta_2)$ satisfies the conditions $F_2(0) = 0$ and $F_2'(0) > 0$, which uniquely define $z_2 = F_2(\zeta_2)$. The cases presented in Fig. 2 correspond to the following mappings:

(a) $z_1 = \rho e^{i\eta}$, $z_2 = \zeta_2 t^{-2/n}$, $t = r + \sqrt{r^2 - \zeta_2^n}$, $r = 0.5p(1 + \zeta_2^n)$, $p > 1$, $n > 1$;

(b) $z_1 = \rho e^{i\eta}$, $z_2 = t^2 + 2t$, $t = \zeta_2/p$, $p > 1$;

(c) $z_1 = \rho e^{i\eta}$, $z_2 = \sqrt{t+1} - 1$, $t = \zeta_2/p$, $p > 1$.

Note that Fig. 2a shows a domain with n symmetric radial cuts, where $n = 1, 2, 3$. At the points of intersection of a cut with the ball, there appears a singularity, which is shown by saddle points. Note that the mapping is not isogonal at these points.

For the exterior of finite plane contours, a canonical domain is the exterior of the unit disk with a point at infinity included. For the exterior of finite domains, Fig. 3 presents the functions $z_1 = \rho e^{i\eta}$ and $z_2 = F_2(\zeta_2)$ mapping the exterior of a domain to the considered domain. For each infinite simply connected domain with a finite boundary, the mapping of the disk's exterior $\zeta_2 > 1$ to the considered domain $z_2 = F_2(\zeta_2)$ is uniquely defined under the conditions $F_2(\infty) = \infty$ and $F_2'(\infty) > 0$. The solution in the coordinates z, ρ is continued to the domain $x = \rho\cos\eta$, $y = \rho\sin\eta$ by rotation through the angle η around the axis of symmetry $\rho = 0$. The given visualization correspond to the following cases:

(a) $z_1 = \rho e^{i\eta}$, $z_2 = 0.5(a + b)\zeta_2 + 0.5(a - b)/\zeta_2$ (exterior of an ellipsoid with semiaxes a, $b > 0$);

(b) $z_1 = \rho e^{i\eta}$, $z_2 = \pi i/s(1 + p)$, $s = \ln r$, $r = (\zeta_2 - 1/r_0)(\zeta_2 - r_0)$, $r_0 = e^{i\pi p/(1+p)}$, $p > 0$ (exterior of a ball with cuts);

(c) $z_1 = \rho e^{i\eta}$, $z_2 = t\sqrt{1 - 1/t^2}$, $t = 0.5[s(p_1 + p_2) + p_1 - p_2]$, $s = 0.5(\zeta_2 + 1/\zeta_2)$, $p_1, p_2 \ge 1$ (exterior of a symmetric cross-shaped cut);

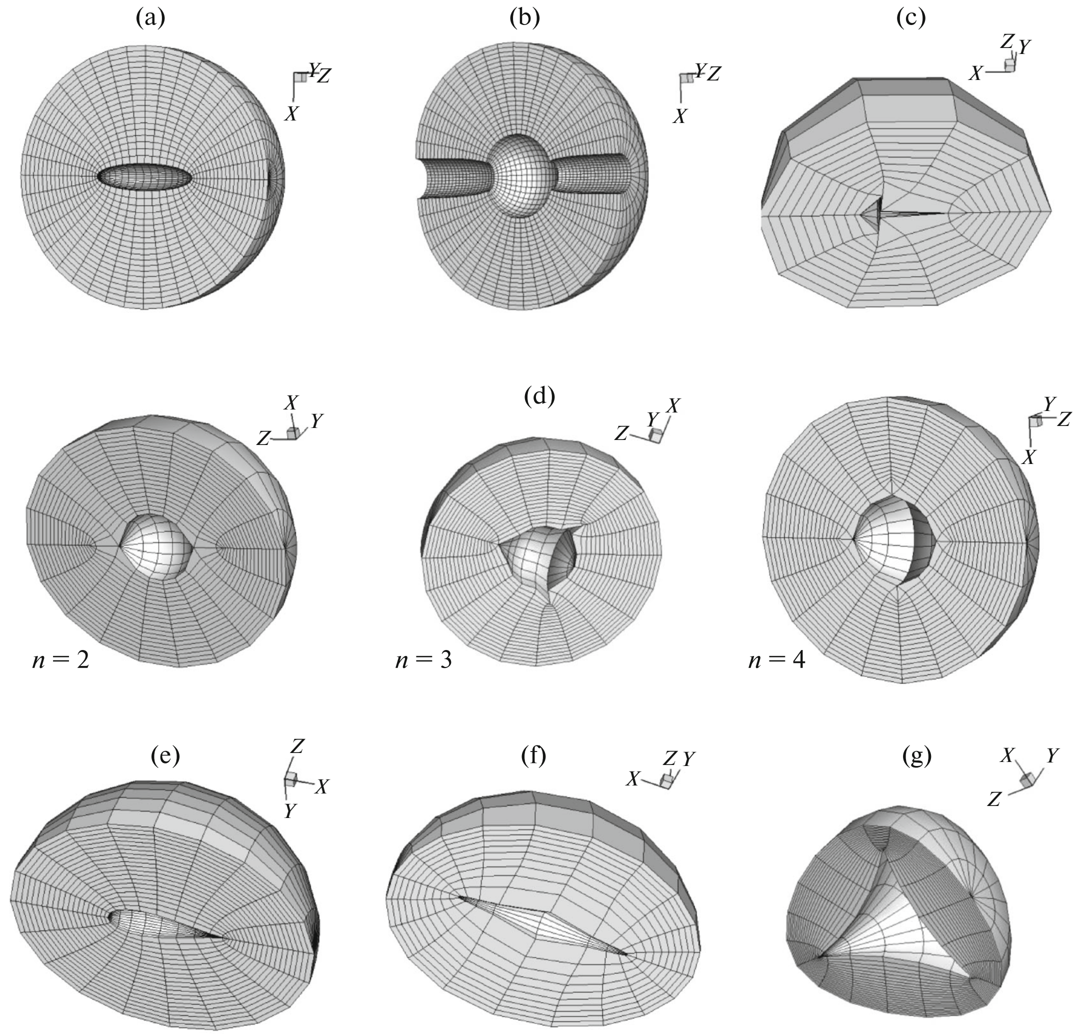

Fig. 3.

(d) $z_1 = \rho e^{i\eta}$, $z_2 = zt^{n/2}$, $t = s + \sqrt{s^2 - 1/\zeta_2^n}$, $s = p(1 + 1/\zeta_2^n)$, $p = 1.2451$ (exterior of a ball with n-symmetric cuts, $n = 2, 3, 4$);

(e) $z_1 = \rho e^{i\eta}$, $z_2 = 0.5(t + 1/t)$, $t = \zeta_2(p+1) - p$, $p = 0.3$ (a symmetric Joukowski airfoil);

(f) $z_1 = \rho e^{i\eta}$, $z_2 = t\sqrt{1 + 1/t^2}$, $t = p\zeta_2$, $p = 1.05$ (exterior of a single-contour cassian);

(g) $z_1 = \rho e^{i\eta}$, $z_2 = \zeta_2 + 1/(pn\zeta_2^n)$, $p = 1.1$; a truncated hypotrochoid for $p > 1$ and a hypocycloid for $p = 1$, $n \geq 2$.

Note the singularity presented in Fig. 3c, which appears at the origin under rotation of the symmetric cross-shaped cut. The solution is not isogonal near the origin and locally loses conformal properties.

For curvilinear plane half-planes, a canonical domain is the half-plane $\operatorname{Im}\zeta_2 > 0$ (see [12]). The mapping $z_2 = F_2(\zeta_2)$ of the angular domain taking the half-plane $\operatorname{Im}\zeta_2 > 0$ to the considered domain and satisfying the condition $F_2(\infty) = \infty$ is defined up to a linear transformation of ζ_2. An analytic continuation is obtained from the corresponding plane coordinates z, ρ by rotation through the angle η around the axis of symmetry. Figure 4 shows the results corresponding to the following cases, respectively:

(a) $z_1 = \rho e^{i\eta}$, $z_2 = (1 + s)/(1 - s)$, $s = r^p$, $r = (\zeta_2 - 1)/(\zeta_2 + 1)$, a hump for $0 < p < 1$ and a trough for $1 < p < 2$, $p = 1.6$;

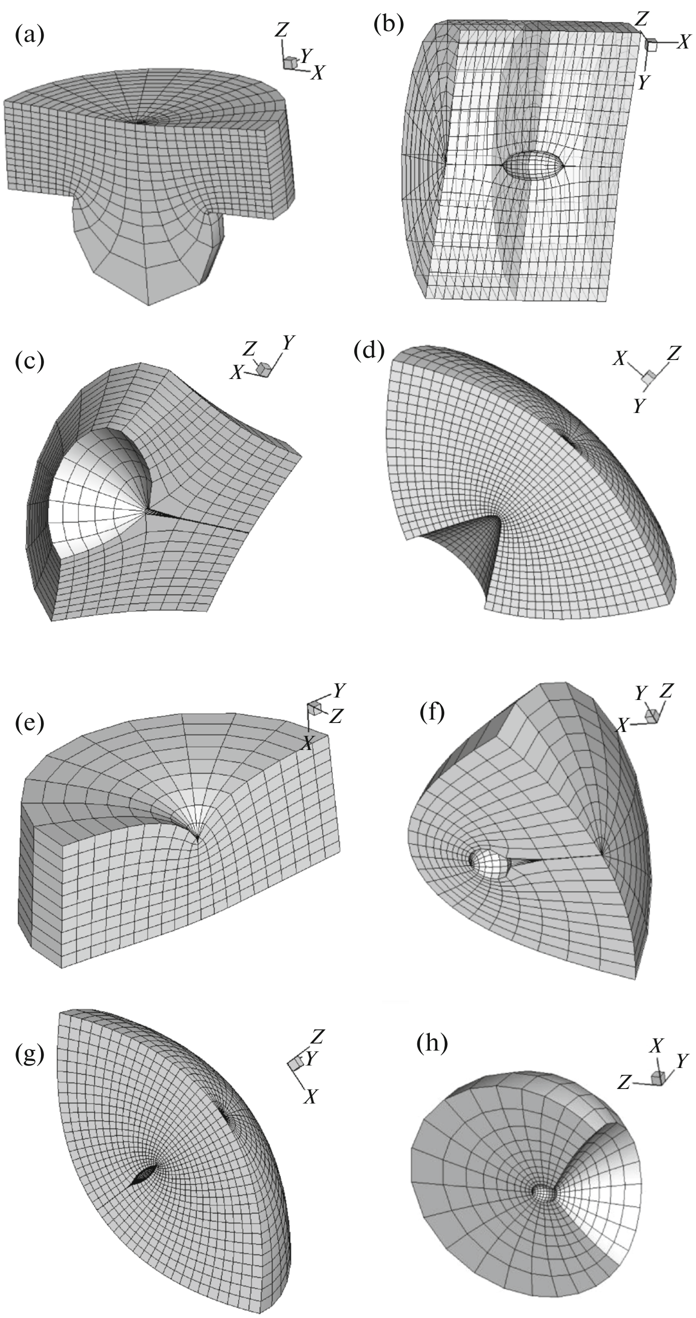

Fig. 4.

(b) $z_1 = \rho e^{i\eta}$, $z_2 = a\zeta_2 + ib\sqrt{1-\zeta_2^2}$, $a = 1$, $b = 0.5$, an ellipsoid of revolution;

(c) $z_1 = \rho e^{i\eta}$, $z_2 = (\sqrt{\zeta_2} + i)^2$, a paraboloid of rotation;

(d) $z_1 = \rho e^{i\eta}$, $z_2 = \cos t$, $t = 2p\arccos s$, $s = -i\zeta_2$, $0 < p < 1$, a hyperboloid of rotation;

(e) $z_1 = \rho e^{i\eta}$, $z_2 = i(t + 1/t - 2)$, $t = 1 - i\zeta_2$, rotation of the pole of a cissoid;

(f) $z_1 = \rho e^{i\eta}$, $z_2 = t + \sqrt{t-1}\sqrt{t+1}$, $t = 1 - \zeta_2^2$, rotation of a half-plane with a rounded edge;

(g) $z_1 = \rho e^{i\eta}$, $z_2 = (1+t)/(1-t)$, $t = s^{2p}$, $s = (r-1)/(r+1)$, $r = u + \sqrt{u-1}\sqrt{u+1}$, $u = 1 - \zeta_2^2$, $0 < p < 1$, rotation of a half-plane with a finned edge, $p = 0.85$;

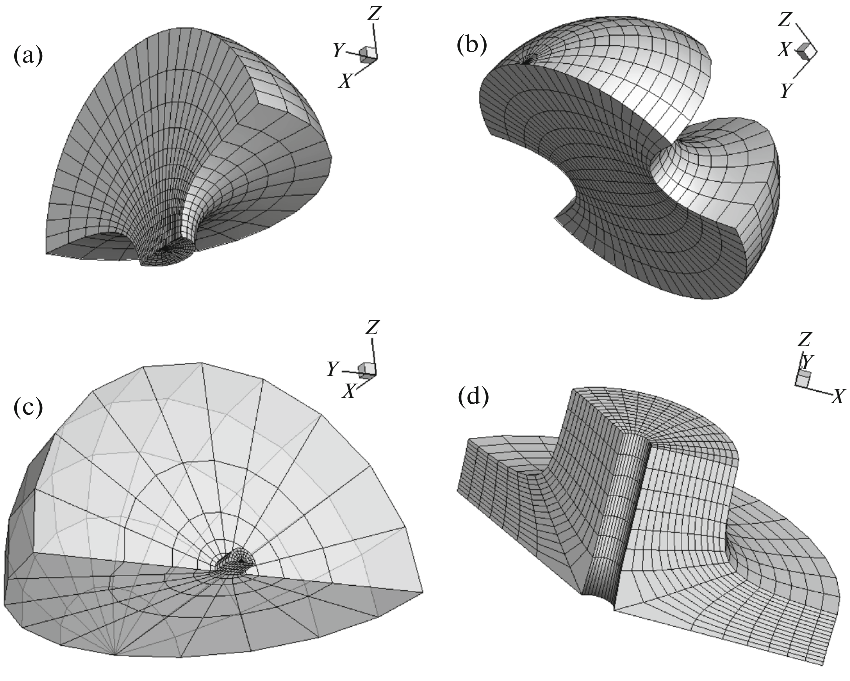

Fig. 5.

(h) $z_1 = \rho e^{i\eta}$, $z_2 = t^{2p}$, $t = \sqrt{1-\zeta_2^2} - i\zeta_2$, $0 < p < 1$, rotation of an angle with a rounded vertex.

For two-dimensional curvilinear strips, a canonical domain is specified as the straight horizontal strip $0 < \operatorname{Im}\zeta_2 < \pi$. Figure 5 plots the functions $z_1 = \rho e^{i\eta}$ and $z_2 = F_2(\zeta_2)$ mapping the strip $0 < \operatorname{Im}\zeta_2 < \pi$ to the considered curvilinear domain in such a way that the points $\zeta_2 = -\infty$, $\zeta_2 = \infty$ are reflected to the arms of the strip. The function $z_2 = F_2(\zeta_2)$ is defined up to a shift of the argument ζ_2. Figures 5a–5d present the results corresponding to the following cases:

(a) $z_1 = \rho e^{i\eta}$, $z_2 = t + e^t$, $t = \zeta_2 + i\pi/2$;

(b) $z_1 = \rho e^{i\eta}$, $z_2 = t + \sinh t$, $t = \zeta_2 - i\pi/2$;

(c) $z_1 = \rho e^{i\eta}$, $z_2 = t - \operatorname{arctanh}(1/t)$, $t = \sqrt{e^s + 1}$, $s = 2\zeta_2 - i\pi$;

(d) $z_1 = \rho e^{i\eta}$, $z_2 = \operatorname{parth} t - \arctan pt$, $t = \sqrt{(s-1)/(s+p^2)}$, $s = \exp(\zeta_2)$, $p = 1.5$.

These cases can be treated as corresponding to various channels formed by rotating a nonrectangular hyperbola, a chain line, etc.

The presented results correspond to the two-dimensional results given in catalogs 1–4 in [12]. Although the results correspond to the axisymmetric case, they are much wider. For example, the axisymmetric case can be generalized if we consider generalized coordinates of rotation: $z_1 = \zeta_1^\alpha$ and $z_2 = F_2(\zeta_2)$ $(\alpha > 0)$.

6. CONCLUSIONS

The application of transformations underlying Lavrentiev-harmonic mappings makes it possible to construct quasiconformal mappings of axisymmetric domains. A three-dimensional quasiconformal mapping of axisymmetric domains is obtained as a composition of arbitrary conformal mappings in polar coordinates. Quasiconformal mappings of axisymmetric domains have the basic properties of conformal mappings. Examples of constructing three-dimensional coordinate systems and their visualization are given.

In the three-dimensional case, depending on the topological and analytical properties of the considered domains, the number of possible classes of quasiconformal mappings increases, which requires further research. The study of mappings of three-dimensional domains that might replace plane conformal mappings in applications should be continued. As a result, quasiconformal mappings would be classified, catalogs of three-dimensional quasiconformal mappings would be compiled, etc.

The best proof of the obtained results is their visualization.

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CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.

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