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MATHEMATICAL PHYSICS

Determination of the Attenuation Coefficient for the Nonstationary Radiative Transfer Equation

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Abstract—For the nonstationary radiative transfer equation, the inverse problem of determining the attenuation coefficient from a known solution at the domain boundary is considered. The structure and the continuous properties of the solution to an initial-boundary value problem for the radiative transfer equation are studied. Under special assumptions about the radiation source, it is shown that the inverse problem has a unique solution and a formula for the Radon transform of the attenuation coefficient is derived. The quality of the reconstructed tomographic images of the sought function is analyzed numerically in the case of various angular and time flux density distributions of the external source.

Keywords: nonstationary radiative transfer equation, radiation sources, inverse problems, attenuation coefficient, tomography

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INTRODUCTION

There is a wide variety of inverse problem formulations for radiative transfer equations and, despite the rather long history of research in this field, much attention is still given to inverse problems in radiative transfer theory (see [1–20]). In most problems, the task is to find the attenuation coefficient from incident and transmitted radiation. This formulation is natural and traditional for tomography and corresponds to a model in which scattering and internal sources in the medium can be neglected. In this case, the attenuation coefficient in the equation remains the only unknown quantity determining the properties of the medium. Especially valuable in practice are inverse problem formulations in which at least one of the coefficients of the equation is determined and knowledge of the others is not required. For example, problems of determining the attenuation coefficient with special-type external sources suppressing the influence of scattering and internal sources in the irradiated medium, and problems of finding discontinuity surfaces of equation coefficients from information on only transmitted radiation were studied in [5–14]. Specifically, in [7, 8] the attenuation coefficient in a stationary transfer equation was determined using special radiation source having a jump discontinuity in the angular variable on a subset of the unit sphere, for example, in a horizontal cross section. It was proved that the solution can be represented as a sum of ballistic and scattered components, where the former has a jump discontinuity and the latter is a continuous function. In recent works [19, 20], similar statements were proved in the case when the incident flux density has discontinuities in space variables on some curve belonging to the boundary of the exposed domain.

By using the indicated representation, the Radon transform of the attenuation coefficient can be expressed in terms of the jump sizes in the incident and transmitted radiation flux densities. Thus, the inverse problem for an integro-differential equation is reduced to a traditional integral geometry problem, namely, to Radon transform inversion in a cross section of the three-dimensional object under study.

In this paper, the results of [7, 8] obtained for the stationary radiative transfer equation are generalized to the nonstationary case. We examine the structure of the nonstationary solution to the direct problem and derive a formula for the Radon transform of the attenuation coefficient. To be fair, we note that, in the nonstationary radiative transfer model, the discrimination of the scattered component in a measured signal can be achieved by reducing the duration of the sounding pulse. Indeed, pulsed radiation sources have been successfully used in tomography for a long time. Such sources are more widespread in optical tomography, since the generation of ultrashort pulses in the X-ray range imposes more severe physical and technological restrictions. Another important advantage of pulsed sources over conventional continuously radiating X-ray generators is associated with reduced radiation exposure in tomography of biological objects.

In this work, an algorithm for solving an inverse problem is numerically tested as applied to a well-known phantom (see [21]). It is shown that the quality of reconstructed images is improved with reducing pulse duration. Note that the filtration of the scattered field relies on the fact that the ballistic and scattered radiation components have different smoothness, so there is no urgent need to use ultrashort momenta in the proposed tomography method (see [22, 23]). To obtain synthetic data on transmitted radiation, we numerically implement a Monte Carlo solution of the nonstationary transfer equation with radiation sources of special type.

1. FORMULATION OF THE INVERSE PROBLEM

Consider an integro-differential equation of the following form (see [17, 22, 24]):

$$\left(\frac{1}{v}\frac{\partial}{\partial t} + \omega\cdot\nabla_r + \mu(r)\right) I(r,\omega,t) = \sigma(r)\int_\Omega p(r,\omega\cdot\omega')I(r,\omega',t)d\omega' + J(r,\omega,t). \quad (1)$$

Equation (1) describes the unsteady interaction of radiation with substance, and the function $I(r,\omega,t)$ is interpreted as the particle flux density at the time $t\in[0,T]$ at the point $r\in\mathbb{R}^3$; here, the particles move with the velocity v in the direction of the unit vector $\omega\in\Omega=\{\omega\in\mathbb{R}^3 : |\omega|=1\}$. The functions μ and σ are interpreted as the attenuation and scattering coefficients, p denotes the scattering phase function, and J is the density of internal sources.

Radiation transfer occurs in a multicomponent system G representing the union of a finite number of bounded and pairwise disjoint subdomains $G_1, \dots, G_m$, and the closure $\bar{G}$ is a convex set in $\mathbb{R}^3$. The surface $\partial\bar{G}$ is called the outer boundary of the set G, and $\gamma = \partial G\backslash\partial\bar{G}$, the inner boundary of G.

Additionally, the following assumption is made about G: any straight line having a common point with G intersects ∂G in a finite number of points. This condition, called the generalized convexity condition, is typical of radiation transfer theory and is not restrictive from an application point of view. The apparent inconsistency associated with the presence of line sections on ∂G can generally be overcome by continuing these sections with a possible increase in the number of domains G_i.

In what follows, the symbol $\mathrm{mes}_m(X)$ denotes the Lebesgue measure of a set X in $\mathbb{R}^m$. The generalized convexity condition implies that $\mathrm{mes}_3(\partial G)=0$ (see [11]).

Let $L_{r,\omega}$ denote the ray starting at the point $r\in\mathbb{R}^3$ in the direction ω, $L_{r,\omega}=\{r+\omega t : t>0\}$, and $d(r,\omega)$ denote the distance from the point $r\in\bar{G}$ to the boundary $\partial\bar{G}$ in the direction ω, i.e., $d(r,\omega)=\mathrm{mes}_1(L_{r,\omega}\cap\bar{G})$. Suppose that the set $\Gamma_{\pm\omega}$ consists of points $z\in\partial\bar{G}$ representable in the form

$$z = r \pm d(r,\pm\omega)\omega, \quad \text{where} \quad r\in G, \quad \omega\in\Omega.$$

Using Γ_ω, we construct the sets $\Gamma^\pm=\Gamma_{\pm\omega}\times\Omega$, $\Gamma=\Gamma^+\cup\Gamma^-$ and, for brevity, introduce some notation:

$$X = G\times\Omega\times[0,T], \quad X_0 = G\times\Omega\times\{t=0\}, \quad Y^\pm = \Gamma^\pm\times[0,T], \quad X^- = Y^-\cup X_0.$$

Equation (1) is subject to the initial and boundary conditions

$$I|_{X_0} = h_0(r,\omega), \quad (2)$$

$$I|_{Y^-} = h_{\mathrm{ext}}(z,\omega,t). \quad (3)$$

For convenience, we define the function

$$h(z,\omega,t) = \begin{cases} h_0(z,\omega) & \text{if} \quad (z,\omega,t)\in X_0, \\ h_{\mathrm{ext}}(z,\omega,t) & \text{if} \quad (z,\omega,t)\in Y^- \end{cases}$$

and write initial and boundary conditions (2), (3) in the form

$$I|_{X^-} = h(r,\omega,t). \quad (4)$$

Problem 1 (direct). Given μ, σ, J, p, v, and h, determine the function I from Eq. (1) and condition (4).

Direct problems for stationary and nonstationary radiative transfer equations of this type have been extensively studied. We note some well-known monographs on this subject, namely, [24–29].

Problem 2 (inverse). Given h, H, and v, determine the function μ from relations (1), (4) and the additional condition

$$I(z,\omega,t) = H(z,\omega,t), \quad (z,\omega,t)\in Y^+. \tag{5}$$

For the stationary radiative transfer equation, Problem 2 with special conditions imposed on the external radiation source was considered in [5, 7, 8, 11, 14, 19, 20]. Below, the results of [7, 8, 11] are generalized to the nonstationary case and the quality of attenuation coefficient tomograms in the case of a decreasing source pulse duration is numerically analyzed.

2. FUNCTION SPACES AND BASIC CONSTRAINTS

Let $C_b(X)$ denote the space of continuous and bounded functions on X. The functions $\mu(r)$, $\sigma(r)$, $J(r,\omega,t)$, and $p(r,\omega\cdot\omega')$ are assumed to be nonnegative and such that $\mu \ge \text{const} > 0$, $\mu,\sigma\in C_b(G)$, $J\in C_b(X)$, $p(r,\omega\cdot\omega')\in C_b(G\times[-1,1])$, and, for all r, $p(r,\omega\cdot\omega')$ satisfies the normalization condition

$$\int_\Omega p(r,\omega\cdot\omega')d\omega' = 1.$$

By the differential expression on the left-hand side of Eq. (1), we mean the derivative at the point (r,t) in the direction of the vector $(\omega_1,\omega_2,\omega_3,1/v)$:

$$\left(\frac{1}{v}\frac{\partial}{\partial t} + \omega\cdot\nabla_r\right) f(r,\omega,t) = \frac{\partial}{\partial\tau} f(r+\tau\omega,\omega,t+\tau/v)\Big|_{\tau=0},$$

and $d^\pm(r,\omega,t)$ is used to denote

$$d^-(r,\omega,t) = \min\{d(r,-\omega), vt\}, \quad d^+(r,\omega,t) = \min\{d(r,\omega), v(T-t)\}.$$

The function $f(r,\omega,t)$ is said to belong to $D(X)$ if the following conditions are satisfied:

(i) for all $(r,\omega,t)\in X$, the function $f(r+\tau\omega,\omega,t+\tau/v)$ is absolutely continuous with respect to τ, $\tau\in[-d^-(r,\omega,t), d^+(r,\omega,t)]$;

(ii) $f(r-d^-(r,\omega,t)\omega,\omega,t-d^-(r,\omega,t)/v) = 0$;

(iii) $\left(\frac{1}{v}\frac{\partial}{\partial t} + \omega\cdot\nabla_r + \mu\right) f \in C_b(X)$.

The operator $\mathcal{L} : D(X)\to C_b(X)$ is defined as

$$\mathcal{L}f = \left(\frac{1}{v}\frac{\partial}{\partial t} + \omega\cdot\nabla_r\right) f + \mu f. \tag{6}$$

Let Ω_0 denote a measure zero subset of Ω. Consider the operator $\mathcal{S} : C_b(G\times(\Omega\backslash\Omega_0)\times[0,T]) \to C_b(X)$ defined by the relation

$$\mathcal{S}f = \sigma(r)\int_\Omega p(r,\omega\cdot\omega')f(r,\omega',t)d\omega'. \tag{7}$$

Since $\sigma(r)\in C_b(G)$, $p(r,\omega\cdot\omega')\in C_b(G\times[-1,1])$, we conclude that, for $f\in C_b(G\times\Omega\backslash\Omega_0\times[0,T])$, the function $\mathcal{S}f$ belongs to the space $C_b(G\times\Omega\times[0,T])$ (see [11]). Therefore, the operator $\mathcal{S}$ is well defined.

In the next section, we consider the problem of finding a function $f\in D(X)$ satisfying the equation

$$\mathcal{L}f = \mathcal{S}f + J \tag{8}$$

on the set X. The definition of the function space $D(X)$ implies that the function f is the solution of the direct problem (1), (4) with homogeneous boundary and initial conditions ($h = 0$), and the solution remains continuous in the direction ω in passing through the material interface ($r\in\partial G\backslash\partial\bar{G}$).

3. SOLVABILITY OF THE DIRECT PROBLEM WITH HOMOGENEOUS INITIAL AND BOUNDARY CONDITIONS

Preliminarily, we prove several auxiliary lemmas. Let $G' \times \Omega$ be a set consisting of points $(r, \omega) \in \bar{G} \times \Omega$ such that the straight line $\{r + \tau\omega, -\infty < \tau < +\infty\}$ has a nonempty intersection with G. By the generalized convexity condition, the set G is a subset of G'.

Lemma 1. *Suppose that* $\mu(r) \in C_b(G)$ *and* $\phi(r, \omega, t) \in C_b(X)$. *Then the function*

$$a(r, \omega, \tau) = \int_0^\tau \mu(r + \tau'\omega) d\tau' \tag{9}$$

belongs to the space $C_b(G' \times \Omega \times [0, d(r, \omega)])$ *and the function*

$$\phi^\pm(r, \omega, t) = \int_0^{d^\pm(r,\omega,t)} \exp(-a(r, \pm\omega, \tau'))\phi(r \pm \tau'\omega, \omega, t \pm \tau'/v) d\tau' \tag{10}$$

belongs to the space $C_b(X')$, *where* $X' = G' \times \Omega \times [0, T]$.

Proof. Let us prove the first assertion of the lemma. First, we note that, since $\tau \le d(r, \omega) \le d$, where d is the diameter of $\bar{G}$, the boundedness of μ on the set G implies that the function a is bounded as well. Now we prove that $a(r, \omega, \tau)$ is continuous on $G' \times \Omega \times [0, d(r, \omega)]$.

Fix an arbitrary point $(r, \omega, \tau) \in G' \times \Omega \times [0, d(r, \omega)]$ and, for any sequence of points $(r_n, \omega_n, \tau_n) \in G' \times \Omega \times [0, d(r, \omega)]$ converging to the point (r, ω, τ) as $n \to \infty$, consider the expressions

$$a(r_n, \omega_n, \tau_n) = \int_0^{\tau_n} \mu(r_n + \tau'\omega_n) d\tau' = \int_0^1 \psi_n(\tau') d\tau',$$

$$a(r, \omega, \tau) = \int_0^\tau \mu(r + \tau'\omega) d\tau' = \int_0^1 \psi(\tau') d\tau',$$

where $\psi_n(\tau') = \tau_n \mu(r_n + \tau'\tau_n\omega_n)$ and $\psi(\tau') = \tau\mu(r + \tau'\tau\omega)$.

Note that $|\psi_n(t')| \le \text{const}$ for any n. Let us prove that the sequence $\psi_n(t')$ tends to $\psi(t')$ almost everywhere on $[0, 1]$ as $n \to \infty$. By the generalized convexity condition, for any n, the ray $L_{r_n,\omega_n} = \{r_n + \tau'\omega_n, \tau' \ge 0\}$ crosses the boundary ∂G in a finite number of points. Let $r_n + \tau_{i,n}\tau_n\omega_n$ $(i = 1, 2, \ldots, q_n,\ q_n < \infty)$ be the points of intersection of the interval $\{r_n + \tau'\tau_n\omega_n, 0 \le \tau' \le 1\}$ with the boundary ∂G. Then the set

$$\Pi = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{q_n} \tau_{i,k}$$

is a countable subset of [0, 1], since it represents a countable union of finite sets.

Thus, since $\mu(r) \in C_b(G)$, all functions $\psi_j(\tau')$ in the sequence $\{\psi_n(\tau')\}$, $n = 1, 2, \ldots$, are continuous with respect to τ', $\tau' \in [0, 1] \backslash \Pi$. Therefore, $\psi_n(\tau') \to \psi(\tau')$ almost everywhere on [0, 1]. Then, by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty} a(r_n, \omega_n, \tau_n) = \lim_{n\to\infty} \int_0^1 \psi_n(\tau') d\tau' = \int_0^1 \lim_{n\to\infty} \psi_n(\tau') d\tau' = \int_0^1 \psi(\tau') d\tau' = a(r, \omega, \tau).$$

As a result, we have proved the continuity of the function $a(r, \omega, \tau)$ on the set $G' \times \Omega \times [0, d(r, \omega)]$.

Let us prove the second assertion of the lemma. We show that the function $d^\pm(r, \omega, t)$ is in $C_b(X')$. Since the set $\bar{G}$ is convex, the function $d(r, \omega)$ on this set can be defined as

$$d(r, \omega) = \text{mes}_1 \left(L_{r,\omega} \cap \bar{G}\right) = \int_0^d \chi(r + t\omega) dt,$$

where $\chi(r)$ is the characteristic function of the set $\bar{G}$, whose diameter is equal to d. Since $\chi(r) \in C_b(G)$, by the proved assertion of the first part of the lemma, we have $d(r, \omega) \in C_b(G' \times \Omega) \subset C_b(G \times \Omega)$.

By definition, $d^+(r,\omega,t) = \min\{d(r,\omega), v(T-t)\}$ and $d^-(r,\omega,t) = \min\{d(r,-\omega), vt\}$; therefore, the functions $d^\pm(r,\omega,t)$ belong to $C_b(X')$.

Let us show the continuity of the function Φ^+; the continuity of Φ^- can be proved similarly. The boundedness of Φ^+ follows from the boundedness of $a(r,\omega,t)$, $\phi(r,\omega,t)$, and $d^+(r,\omega,t)$. To prove the continuity of Φ^+ on the set X', fix an arbitrary point $(r,\omega,t)\in X'$ and, for any sequence of points $(r_n,\omega_n,t_n)\in X'$ such that $(r_n,\omega_n,t_n)\to(r,\omega,t)$ as $n\to\infty$, consider the expressions

$$\Phi^+(r_n,\omega_n,t_n) = \int_0^{d^+(r_n,\omega_n,t_n)} \exp(-a(r_n,\omega_n,\tau))\phi(r_n+\tau\omega_n,\omega_n,t_n+\tau/v')d\tau = \int_0^1 \psi_n(\tau')d\tau',$$

$$\Phi^+(r,\omega,t) = \int_0^{d^+(r,\omega,t)} \exp(-a(r,\omega,\tau))\phi(r+\tau\omega,\omega,t+\tau/v)d\tau = \int_0^1 \psi(\tau')d\tau',$$

where

$$\psi_n(\tau') = d_n^+\exp(-a(r_n,\omega_n,\tau' d_n^+))\phi(r_n+\tau' d_n^+\omega_n,\omega_n,t_n+\tau' d_n^+/v), \quad d_n^+ = d^+(r_n,\omega_n,t_n),$$

and

$$\psi(\tau') = d^+\phi(r+\tau' d^+\omega,\omega,t+\tau' d^+/v), \quad d^+ = d^+(r,\omega,t).$$

Since $d^+(r,\omega,t)\in C_b(X')$, in a similar manner to what was proved in the first part of the assertion, it can be shown that $\psi_n(t')\to\psi(t')$ as $n\to\infty$ almost everywhere on [0, 1]. By the Lebesgue theorem, we conclude that the function $\Phi^+(r,\omega,t)$ is continuous on the set X'. The assertions of the lemma are proved.

Lemma 2. *For any function $F\in C_b(X)$, there exists a unique solution $f\in D(X)$ of the equation*

$$\mathcal{L}f = F, \tag{11}$$

and, for all $(r,\omega,t)\in X$, this solution is given by the formula

$$f(r,\omega,t) = \int_0^{d^-(r,\omega,t)} \exp\left[-\int_0^\tau \mu(r-\tau'\omega)d\tau'\right]F(r-\tau\omega,\omega,t-\tau/v)d\tau \tag{12}$$

and satisfies the condition

$$\|f\| = \sup_{(r,\omega,t)\in X}|f(r,\omega,t)| \le (1-e^{-\bar{\mu}d})\left\|\frac{F}{\mu}\right\|, \tag{13}$$

where $\bar{\mu} = \inf_{r\in G}|\mu(r)|$.

Proof. It can be directly verified that the function f given by (12) satisfies Eq. (11) and conditions (i) and (ii) from the definition of the space D. By Lemma 1, the function $a(r,\omega,\tau)$ defined by (9) belongs to $C_b(G'\times\Omega\times[0,d(r,\omega)])$; therefore, $a(r,-\omega,\tau)$ belongs to $C_b(G'\times\Omega\times[0,d(r,-\omega)])$. The function $F(r,\omega,t)$ belongs to $C_b(X)$ by assumption; therefore, by Lemma 1, the function f in (12) belongs to $C_b(X)$. The function $\left(\frac{1}{v}\frac{\partial}{\partial t}+\omega\cdot\nabla_r+\mu\right)f$ belongs to $C_b(X)$, since f satisfies Eq. (11) with a right-hand side $F\in C_b(X)$.

To prove estimate (13), we use representation (12):

$$\begin{aligned}\|f(r,\omega,t)\| &\le \left\|\int_0^{d^-(r,\omega,t)} \exp\left[-\int_0^\tau \mu(r-\tau'\omega)d\tau'\right]\mu(r-\tau\omega)\left\|\frac{F}{\mu}\right\|d\tau\right\| \\ &\le \left\|\frac{F}{\mu}\right\|\left\|1-\exp\left[-\int_0^{d^-(r,\omega,t)}\mu(r-\tau'\omega)d\tau'\right]\right\| \le (1-e^{-\bar{\mu}d})\left\|\frac{F}{\mu}\right\|.\end{aligned} \tag{14}$$

The lemma is proved.

By Lemma 2, the operator $\mathcal{L}^{-1} : C_b(X) \to D(X)$ defined by the formula

$$\mathcal{L}^{-1}F = \int_0^{d^-(r,\omega,t)} \exp\left[-\int_0^{\tau} \mu(r - \tau'\omega)d\tau'\right] F(r - \tau\omega, \omega, t - \tau/v)d\tau, \tag{15}$$

exists and is bounded in $C_b(X)$.

The space $D(X)$ is equipped with the norm

$$\|f\|_D = \left\|\frac{\mathcal{L}f}{\mu}\right\|. \tag{16}$$

Condition (13) yields the obvious inequality

$$\|f\| \le (1 - e^{-\bar{\mu}d})\|f\|_D, \tag{17}$$

based on which we conclude that the convergence in $D(X)$ implies convergence in $C_b(X)$. It is easy to show that the linear set of functions D equipped with norm (16) is a Banach space.

By Lemma 2, the solution of Eq. (8) in D is equivalent to the solution of the equation

$$f = \mathcal{L}^{-1}J + \mathcal{L}^{-1}\mathcal{S}f. \tag{18}$$

Theorem 1. *Equation* (8) *is uniquely solvable in the space* D.

Proof. Since the function p is nonnegative and satisfies the normalization condition and, additionally, inequality (17) holds, we obtain $\|\mathcal{L}^{-1}\mathcal{S}f\|_D$

$$\|\mathcal{L}^{-1}\mathcal{S}f\|_D = \left\|\frac{\mathcal{S}f}{\mu}\right\| = \left\|\frac{\sigma(r)}{\mu(r)}\int_\Omega p(r, \omega\cdot\omega')f(r, \omega', t)d\omega'\right\| \le \bar{\lambda}\|f\| \le \bar{\lambda}(1 - e^{-\bar{\mu}d})\|f\|_D, \tag{19}$$

where $\bar{\lambda} = \left\|\frac{\sigma}{\mu}\right\|$. Since $\mu(r) \ge \sigma(r)$, we conclude that $\bar{\lambda} \le 1$ and (19) implies that

$$\|\mathcal{L}^{-1}\mathcal{S}\|_{D\to D} \le \bar{\lambda}(1 - e^{-\bar{\mu}d}) < 1.$$

Since the norm of the operator $\mathcal{L}^{-1}\mathcal{S}$ acting in a Banach space is less than unity, Eq. (18) is uniquely solvable and the solution can be found by the method of successive approximations

$$f_n = \mathcal{L}^{-1}\mathcal{S}f_{n-1} + f_0, \quad n = 1, 2, \ldots,$$

where $f_0 = \mathcal{L}^{-1}J$. Theorem 1 is proved.

Note that the condition $\bar{\lambda} \le 1$ is excessive for the existence of a unique bounded solution on the finite time interval $t \in [0, T]$. Indeed, if $\sigma > \mu$, then, in Eq. (8), we can make the substitution $f = f_\lambda e^{\lambda t}$, where λ is an arbitrary number satisfying the conditions $\mu(r) + \lambda/v > 0$ and $\frac{\sigma(r)}{\mu(r) + \lambda/v} \le 1$ for all $r \in G$. Then, with a suitable choice of the norm in D, for example,

$$\|f_\lambda\|_D = \left\|\frac{\mathcal{L}f_\lambda}{\mu + \lambda/v}\right\|,$$

Theorem 1 implies the existence of a unique solution f_λ and, hence, a solution f of Eq. (8). Nevertheless, the constraint $\bar{\lambda} \le 1$ for the unbounded time interval $t \in [0, \infty)$ guarantees that the nonstationary solution f stabilizes as $t \to \infty$.

4. STUDY OF THE INVERSE PROBLEM FOR THE RADIATIVE TRANSFER EQUATION

In Section 3, we considered an initial-boundary value problem for the radiative transfer equation with homogeneous initial and boundary conditions. The solution of the inhomogeneous initial-boundary value problem (1), (4) can be represented in the form $I = f + I_0$, where

$$I_0(r,\omega,t) = h(r - d^-(r,\omega,t)\omega,\omega,t - d^-(r,\omega,t)/v)\exp\left[-\int_0^{d^-(r,\omega,t)} \mu(r-\tau\omega)d\tau\right] \tag{20}$$

and the function f belongs to $D(X)$ and satisfies the equation

$$\mathcal{L}f = \mathcal{S}f + \mathcal{S}I_0 + J. \tag{21}$$

Since the operator $\mathcal{S} : C_b(G\times(\Omega\backslash\Omega_0)\times[0,T]) \to C_b(X)$ has smoothing properties and $J \in C_b(X)$, the function $\mathcal{S}I_0 + J$ belongs to $C_b(X)$ even if h has discontinuities with respect to ω. Therefore, according to the results of Section 3, Eq. (21) is uniquely solvable in $D(X)$. Thus, the function I can be represented in the form of a sum of two functions, f and I_0. The function f, which is interpreted as the scattered field, is continuous on the set $X' \subset \bar{G}\times\Omega\times[0,T]$, while the unscattered radiation I_0 may contain discontinuities with respect to ω on X'. This structural feature of the solution to the radiative transfer equation underlies the method used to solve the inverse problem.

Let the size of the jump in h with respect to ω as $\omega \to \omega^0 = (\omega_1,\omega_2,0)$ be denoted by

$$[h](\xi,\omega^0,t) = \lim_{\varepsilon\to 0}\left[h(\xi,(\omega_1,\omega_2,\varepsilon),t) - h(\xi,(\omega_1,\omega_2,-\varepsilon),t)\right].$$

The following constraint is imposed on the function h:

$$[h](\xi,\omega^0,t) \neq 0, \quad (\xi,\omega^0,t)\in Y^-. \tag{22}$$

Then, for boundary points $(\eta,\omega^0,t)\in Y^+$ satisfying the constraint $d(\eta,-\omega^0) \le vt \le vT - d(\eta,-\omega^0)$, we obtain

$$\begin{gathered}[H](\eta,\omega^0,t) = [f](\eta,\omega^0,t)\\ + [h](\eta - d(\eta,-\omega^0)\omega^0,\omega^0,t - d(\eta,-\omega^0)/v)\exp\left[-\int_0^{d(\eta,-\omega^0)} \mu(\eta-\tau\omega^0)d\tau\right].\end{gathered} \tag{23}$$

Since $f \in C_b(X')$, we have $[f](\eta,\omega^0,t) = 0$. Then, under condition (22), relation (23) implies that

$$\int_0^{d(\eta,-\omega^0)} \mu(\eta-\tau\omega^0)d\tau = \ln\frac{[h](\eta - d(\eta,-\omega^0)\omega^0,\omega^0,t - d(\eta,-\omega^0)/v)}{[H](\eta,\omega^0,t)}. \tag{24}$$

The left-hand side of (24) involves the Radon transform of the function μ in any horizontal plane $r_3 = \text{const}$. After changing variables, relation (24) can be rewritten as

$$\int_{d(r,-\omega^0)}^{d(r,\omega^0)} \mu(r+\tau\omega^0)d\tau = \ln\frac{[h](r - d(r,-\omega^0)\omega^0,\omega^0,t - d(r,-\omega^0)/v)}{[H](r + d(r,\omega^0)\omega^0,\omega^0,t + d(r,\omega^0)/v)}, \tag{25}$$

where r is any point of the domain G and the variable t belongs to the time interval $[d(r,-\omega^0)/v,\ T - d(r,\omega^0)/v]$ for a sufficiently large T.

The following theorem on the uniqueness of a solution of the inverse problem is proved by analogy with the stationary case (see [8, 11]).

Theorem 2. *Given some* $t \in (0,T)$, *suppose that*

$$I'(\eta,\omega,t) = I''(\eta,\omega,t) \quad \textit{on} \quad \Gamma^+, \tag{26}$$

where I' and I'' are solutions of the initial-boundary value problems (1)–(3) *with two sets of coefficients* $\{\mu'(r), \sigma'(r), p'(\omega\cdot\omega'), J'(r,\omega,t)\}$ *and* $\{\mu''(r), \sigma''(r), p''(\omega\cdot\omega'), J''(r,\omega,t)\}$ *and with the same function* $h(\xi,\omega,t)$ *satisfying condition* (22). *Then relation* (25) *holds and* $\mu'(r) = \mu''(r)$ *almost everywhere in G.*

A feature of the theorem is that it states that, within two sets of coefficients of Eq. (1), namely,

$$\{\mu'(r), \sigma'(r), p'(\omega\cdot\omega'), J'(r,\omega,t)\}, \quad \{\mu''(r), \sigma''(r), p''(\omega\cdot\omega'), J''(r,\omega,t)\},$$

only μ' and μ'' coincide with each other, while nothing is said about the other coefficients. It follows from (25) that the unknown functions σ, p, and J do not influence the procedure for determining the function μ. In terms of physics, we can say that the effects exerted by scattering and the presence of radioactive sources in the medium are suppressed by choosing a suitable external source. Since the function h has a discontinuity at the point $\omega^0 = (\omega_1, \omega_2, 0)$, the function $\mu(r)$ is recovered layerwise in horizontal planes $r_3 = \text{const}$, which is conventional for X-ray tomography (see [30]).

Time t in relations (25) and (26) can be arbitrary as long as condition (22) holds at this point, which guarantees that the function h has a nonzero discontinuity. However, when the inverse problem is solved numerically (and the method is implemented in practice), it is desirable that the jumps in h and H be maximal at the times t and $t + d(\eta, -\omega^0)/v$, respectively. For example, if the medium is probed by a pulse with a ballistic component of maximum intensity at the time $t = t_0$, then it is preferable to measure the signal going out of the medium at the point $(\eta, \omega^0) \in \Gamma^+$ at the time $t = t_0 + d(\eta, -\omega^0)/v$. At this time, the ratio of the unscattered radiation component to the scattered field is maximum at the point (η, ω^0).

5. CONSTRUCTION OF A MONTE CARLO ALGORITHM FOR SOLVING THE INITIAL-BOUNDARY VALUE PROBLEM

To test the algorithm for solving the inverse problem 2, we need to know the function H, which can be measured in physical experiments. In mathematical modeling, to find H, it is necessary to solve the direct problem 1 for all given coefficients of Eq. (1) and for the function h. Among the wide variety of numerical methods intended to solve radiative transfer equations in the multidimensional case, there is hardly an alternative to Monte Carlo methods. We use a Monte Carlo technique known as the conjugate walk method, which is appropriate for finding the radiant flux density at a fixed point of the phase space X (see [31, 32]).

Let $\mathcal{P}$ denote the operator $\mathcal{P} : C_b(Y^-) \to D(X)$ defined by the formula

$$(\mathcal{P}I)(r,\omega,t) = I(r - d^-(r,\omega,t)\omega, \omega, t - d^-(r,\omega,t)/v)\exp\left[-\int_0^{d^-(r,\omega,t)} \mu(r - \tau\omega)d\tau\right]. \tag{27}$$

Then an approximate solution of problem (1)–(3) with an inhomogeneous boundary condition can be written in the form of a truncated Neumann series:

$$I_N = \sum_{n=0}^{N} (\mathcal{L}^{-1}\mathcal{S})^n I_0, \quad I_0 = \mathcal{P}h + \mathcal{L}^{-1}J. \tag{28}$$

The function I_N approximates the solution I of the problem, and representation (28) underlies the construction of the Monte Carlo algorithm. Each element of the sequence I_n, $n = 1, 2, \ldots, N$, specifies the contribution made by the radiation after 1 to n scattering events, while the term I_0 takes into account the contribution of the unscattered radiation.

The truncated Neumann series (28), which is a sum of multidimensional integrals, is computed using the Monte Carlo method. We construct a particle trajectory

$$(r_0, \omega_0, t_0), \ldots, (r_n, \omega_n, t_n), \tag{29}$$

where the points of Markov chain (29) are determined according to the following rule:

$$r_{i+1} = r_i - \tau_{i+1}\omega_i, \quad t_{i+1} = t_i - \tau_{i+1}/v, \quad i = \overline{0, n-1}, \quad r_0 = r, \quad \omega_0 = \omega, \quad t_0 = t. \tag{30}$$

In (30) the random variable τ_{i+1} is distributed on the interval $[0, d^-(r_i, \omega_i, t_i)]$ with probability density

$$\mu(r_i - \tau_{i+1}\omega_i)\exp\left[-\int_0^{\tau_{i+1}} \mu(r_i - \tau\omega_i)d\tau\right]\left\{1 - \exp\left[-\int_0^{d^-(r_i,\omega_i,t_i)} \mu(r_i - \tau\omega_i)d\tau\right]\right\}^{-1}, \tag{31}$$

and the random vector ω_{i+1} is distributed over the unit sphere Ω with transition probability density $p(r_{i+1}, \omega_i \cdot \omega_{i+1})$.

Let $\mathbb{E}[\Theta]$ denote the expectation of the random variable Θ. Then, according to the Monte Carlo method, the value of the function I_N at the point (r, ω, t) can be expressed as

$$I_N(r, \omega, t) = \mathbb{E}[\Theta_N], \quad \Theta_N = \sum_{n=0}^{N} \theta_n, \tag{32}$$

where $\theta_0 = I_0(r_0, \omega_0, t_0)$ and the random variables θ_n for $n > 0$ are determined recurrently as

$$\theta_{n+1} = \frac{\sigma(r_{n+1})}{\mu(r_{n+1})}1 - \exp\left[-\int_0^{d^-(r_n,\omega_n,t_n)} \mu(r_n - \omega_n\tau)d\tau\right]\theta_n, \quad n = 1, 2, \ldots, N-1. \tag{33}$$

Repeating this procedure M times yields a sample of size M for the random variable Θ_N. The sample mean gives an estimate for the expectation of Θ_N and, hence, an approximate value of I_N at the point (r, ω, t). A simple analysis of recurrence relations (33) shows that the described weighted Monte Carlo method takes into account the absorption and escape of particles from the domain. Allowance for these effects causes a relatively small increase in the complexity of the method, while leading to a considerable reduction in the variance of the estimated expectation of Θ_N and to a significant improvement in the accuracy of the solution to the radiative transfer equation (see [32]).

6. NUMERICAL EXPERIMENTS

In this section, we describe the numerical results concerning the recovery of an internal medium structure. The algorithm was tested in two steps. At the first step, for given medium parameters, the transmitted radiation H was computed by applying the Monte Carlo method described in Section 5. In solving the direct problem, the nonstationary radiation source h of the pulse type (of various duration) was assumed to have a jump discontinuity with respect to the angular variable in the plane $\omega_3 = 0$. At the second step, the inverse problem was solved using formula (25), which filtered out the scattered radiation, or using the classical tomography formula

$$\int_{d(r,-\omega^0)}^{d(r,\omega^0)} \mu(r + \tau\omega^0)d\tau = \ln\frac{h(r - d(r,-\omega^0)\omega^0, \omega^0, t - d(r,-\omega^0)/v)}{H(r + d(r,\omega^0)\omega^0, \omega^0, t + d(r,\omega^0)/v)}. \tag{34}$$

Formula (34) differs from (25) in that the logarithm involves the values of the functions h and H at domain boundary points, rather than the sizes of the jumps in these functions. To find the function μ from Eqs. (25) and (34), we can use various algorithms for Radon transform inversion. An example is the convolution back projection algorithm, which is popular in the modern literature (see [30]).

The solution algorithm proposed for the inverse problem was tested using the phantom designed in [21] for performance evaluation of computed tomography algorithms. The phantom is a cylinder 10 cm in height and diameter. The internal volume of the cylinder is divided into five cylindrical layers (compartments), each playing its own role in testing certain qualities of the reconstruction algorithm.

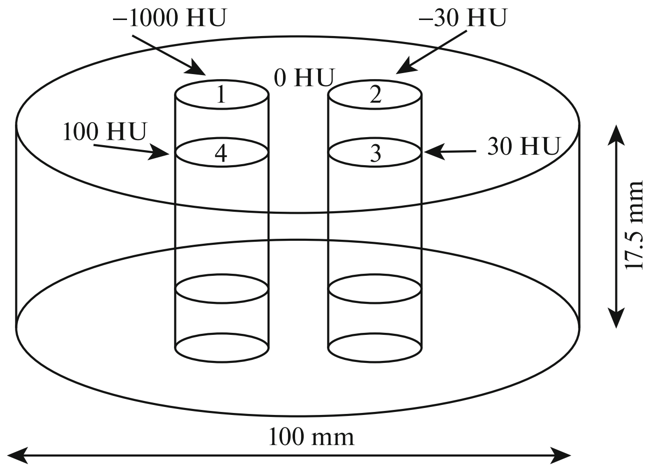

Fig. 1. Schematic view of the phantom for the tested algorithm.

In the experiments, we used only the first compartment, which is intended to test the linearity of reconstructed attenuation coefficients (see [33]). This is a major characteristic used in diagnosis and treatment planning (see [34]), and it is defined as follows. In computed tomography, attenuation coefficients for various materials are traditionally given in Hounsfield units (HU). The HU value is related to the attenuation coefficient μ of substance by the linear law

$$\mathrm{HU} = \frac{\mu - \mu_{\mathrm{water}}}{\mu_{\mathrm{water}} - \mu_{\mathrm{air}}} \times 1000, \tag{35}$$

where μ is the attenuation coefficient of the substance and μ_{water}, μ_{air} are the attenuation coefficients for water and air, respectively. The Hounsfield scale has been successfully used in classical computed tomography as applied to soft tissue diagnosis. In the case of denser tissues, such as bones, due to the relatively low energy of incident radiation and the increased fraction of the scattered field, inclusions of identical density yield different attenuation coefficients on the Hounsfield scale depending on their locations. As a result, the linear relation (35) between Hounsfield units and the attenuation coefficient of the medium is violated. This problem is particularly acute in cone-beam computed tomography (see [35]).

The used phantom is schematically shown in Fig. 1. The given section is a cylinder 100 mm in diameter and 17.5 mm in height made of plastic with an attenuation coefficient equivalent to that of water (HU = 0). The section contains four cylindrical inclusions 13 mm in diameter and 17.5 mm in height with different attenuation coefficients (–1000, 30, – 30, 100 HU). Following the technique proposed in [21], linearity was testes by computing the average attenuation coefficient for each of the materials included in the phantom, after which the results were compared with reference values in Hounsfield units. The radiation source was specified as a Gaussian distribution with respect to time having the form

$$h(r,\omega,t) = \chi(\omega) I_0 \exp\left[-4\ln 2 \frac{(t-t_0)^2}{t_p}\right], \tag{36}$$

where I_0 is the pulse amplitude, t_0 is the time corresponding to the maximum power of the signal, t_p is the pulse duration at half maximum, and

$$\chi(\omega) = \begin{cases} 1, & \omega_3 > 0, \\ 0, & \omega_3 \le 0. \end{cases} \tag{37}$$

In the experiments, we used $I_0 = 32\,000$, $t_0 = 3$ ns, and $t_p = 3, 30, 300, 3000$ ps.

Figure 2 presents the results of phantom reconstruction for a sounding pulse of various durations. Specifically, the source reconstruction results obtained by inverting the Radon transform with the help of formula (34) are shown on the left, while the results obtained after filtering out the scattered field based on

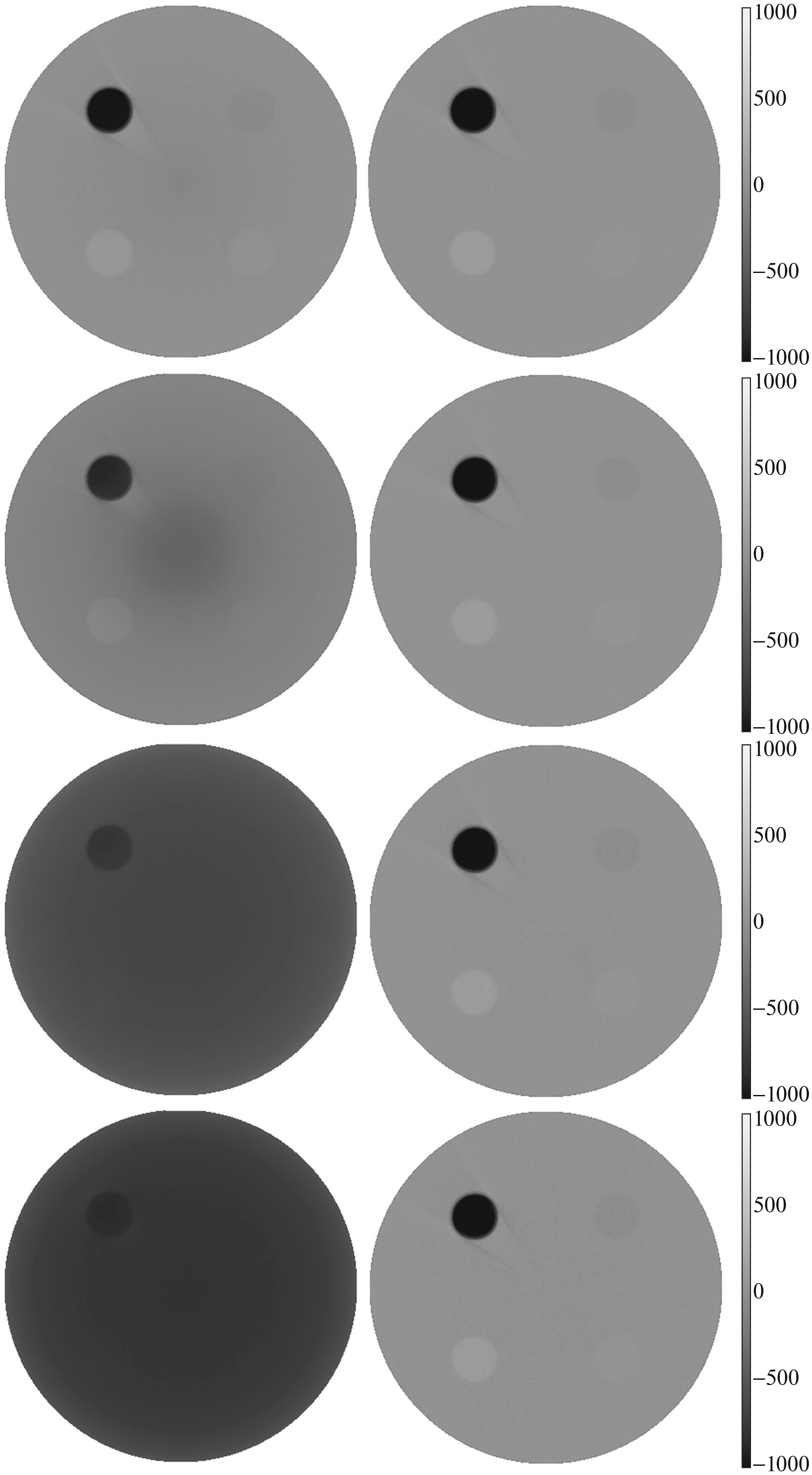

Fig. 2. Results of phantom reconstruction for an increasing duration of the sounding pulse: 3, 30, 300, 3000 ps in the case of no signal processing (left) and after filtering out the scattered field using formula (25) (right).

Table 1. Average values of the reconstructed attenuation coefficients μ in Hounsfield units for various durations t_p of the sounding pulse

Inclusion number	0		1		2		3		4	
Exact value, HU	0		–1000		–30		30		100	
t_p = 3 ps	–101	3	–980	–991	–60	–31	–6	25	60	97
t_p = 30 ps	–396	–5	–868	–998	–211	–32	–172	24	–128	96
t_p = 300 ps	–673	–7	–785	–996	–640	–34	–633	21	–624	93
t_p = 3000 ps	–812	–8	–866	–996	–787	–35	–784	21	–779	92

For each inclusion, the table presents the exact value of μ obtained by inverting the Radon transform without preprocessing the transmitted radiation (left) and obtained after filtering out the scattered field using formula (25) (right).

a special-type source with the help of formula (25) are shown on the right. The average values of the reconstructed attenuation coefficients in Hounsfield units are given in Table 1. More specifically, the average value computed using formula (34) without preliminarily filtering out the scattered field is presented on the left, while the value based on formula (25) is given on the right.

Inspection of Table 1 and the presented tomograms show that short-pulsed radiation makes it possible to filter out the scattered component in the transmitted radiation. Accordingly, when the phantom is exposed to short pulses, the reconstruction results are fairly good even without applying additional scattered field filtration based on formula (25). Overall, the error of the reconstructed attenuation coefficient was found to be at the level obtained in [21]. Note that, even in this "good" case, the additional processing of the signal with the help of formula (25) yields better correspondence between the reference and reconstructed values of the attenuation coefficient. As the pulse duration increases, the quality of the reconstruction without signal preprocessing degrades sharply. For a pulse duration of 3000 ps, the numerical values of the reconstructed coefficients differ significantly and three out of four inclusions are hardly visually distinguishable. On the contrary, the use of a special-type source with the subsequent signal processing yields good results. The right pictures in Fig. 2 show that the internal structure of the phantom is recovered with acceptable quality for any pulse duration. However, as the pulse duration grows, the quality of the reconstruction degrades, though insignificantly. Thus, a combination of a pulse and a special-type source is recommended for achieving the best quality of the reconstruction.

CONCLUSIONS

The structure of a nonstationary solution to the radiative transfer equation with a special-type external source having a jump discontinuity in the angular variable was studied. It was shown that the solution of the direct problem can be represented in the form of a sum of two components. The first component describes the unscattered radiation and transfers the discontinuity from the boundary into the domain, while the second component is a continuous function describing the scattered field. This representation was used to solve the inverse problem of determining the attenuation coefficient in the case when the solution of the equation is known at the boundary of the domain. A formula for the Radon transform of the sought function was derived, and the uniqueness of the solution to the inverse problem was proved. An unknown medium exposed to X-ray radiation with various time and angular distribution structures was numerically simulated. The numerical results showed that the best quality of the tomographic images is achieved in the case of irradiation by a combined pulsed source of special type.

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REFERENCES

1. G. I. Marchuk, “Formulations of some inverse problems,” Dokl. Akad. Nauk SSSR **156** (3), 503–506 (1964).
2. M. V. Maslennikov, “The Milne problem with anisotropic scattering,” Proc. Steklov Inst. Math. **97**, 1–161 (1968).
3. D. S. Anikonov, “On inverse problems for the transport equation,” Differ. Uravn. **2** (1), 7–17 (1974).
4. A. I. Prilepko and A. L. Ivankov, “Inverse problems of determining a coefficient, scattering phase function, and right side of a nonstationary multispeed transport equation,” Differ. Uravn. **21** (5), 870–885 (1985).
5. D. S. Anikonov, “Uniqueness of determining a coefficient of a transfer equation with a special-type source,” Dokl. Akad. Nauk SSSR **284** (5), 1033–1037 (1985).
6. D. G. Orlovskii and A. I. Prilepko, “Some inverse problems for the linearized Boltzmann equation,” USSR Comput. Math. Math. Phys. **27** (6), 58–65 (1987).
7. D. S. Anikonov and I. V. Prokhorov, “Determination of the coefficient of a transfer equation with energy and angular singularities of external radiation,” Dokl. Akad. Nauk **327** (2), 205–207 (1992).
8. D. S. Anikonov, I. V. Prokhorov, and A. E. Kovtanyuk, “Investigation of scattering and absorbing media by the methods of X-ray tomography,” J. Inverse Ill-Posed Probl. **1** (4), 259–282 (1993).
9. V. S. Antyufeev and A. N. Bondarenko, “X-ray tomography in scattering media,” SIAM J. Appl. Math. **56** (2), 573–587 (1996).
10. V. G. Romanov, “A stability estimate in the problem of determining the dispersion index and relaxation for the transport equation,” Sib. Math. J. **37** (2), 308–324 (1996).
11. D. S. Anikonov, A. E. Kovtanyuk, and I. V. Prokhorov, *Transport Equation and Tomography* (VSP, Boston, 2002).
12. D. S. Anikonov, V. G. Nazarov, and I. V. Prokhorov, *Poorly Visible Media in X-Ray Tomography* (VSP, Boston, 2002).
13. S. A. Tereshchenko, *Computed Tomography Methods* (Fizmatlit, Moscow, 2004) [in Russian].
14. A. E. Kovtanyuk and I. V. Prokhorov, “Numerical solution of the inverse problem for the polarized-radiation transfer equation,” Numer. Anal. Appl. **1** (1), 46–57 (2008).
15. I. V. Prokhorov, I. P. Yarovenko, and V. G. Nazarov, “Optical tomography problems at layered media,” Inverse Probl. **24** (2), 025019 (2008).
16. N. P. Volkov, “Solvability of certain inverse problems for the nonstationary kinetic transport equation,” Comput. Math. Math. Phys. **56** (9), 1598–1603 (2016).
17. G. Bal, “Inverse Transport Theory and Applications,” Inverse Probl. **25** (5), 053001 (2009).
18. M. Bellassoued and Y. Boughanja, “An inverse problem for the linear Boltzmann equation with a time-dependent coefficient,” Inverse Probl. **35** (8), 085003 (2019).
19. I. K. Chen and D. Kawagoe, “Propagation of boundary-induced discontinuity in stationary radiative transfer and its application to the optical tomography,” Inverse Probl. Imaging **13** (2), 337–351 (2019).
20. D. Kawagoe and I.-K. Chen, “Propagation of boundary-induced discontinuity in stationary radiative transfer,” J. Stat. Phys. **170** (1), 127–140 (2018).
21. C. Steiding, D. Kolditz, and W. A. Kalender, “A quality assurance framework for the fully automated and objective evaluation of image quality in cone-beam computed tomography,” Medic. Phys. **41**, 031901 (2014).
22. V. S. Kuznetsov, O. V. Nikolaeva, L. P. Bass, A. V. Bykov, and A. V. Priezzhev, “Modeling of ultrashort light pulse propagation in strongly scattering medium,” Mat. Model. **21** (4), 3–14 (2009).
23. G. V. Fetisov, “X-ray diffraction methods for structural diagnostics of materials: progress and achievements,” Phys.-Usp. **63** (1), 2–32 (2020).
24. Yu. I. Ershov and S. B. Shikhov, *Mathematical Fundamentals of Transfer Theory* (Energoatomizdat, Moscow, 1985) [in Russian].
25. V. S. Vladimirov, “Mathematical problems of the uniform-speed theory of transport,” Tr. Mat. Inst. im. V.A. Steklov Akad. Nauk SSSR **61**, 3–158 (1961).
26. C. Cercignani, *Theory and Application of the Boltzmann Equation* (Elsevier, New York, 1975).
27. V. M. Novikov and S. B. Shikhov, *Theory of Parametric Effect on Neutron Transport* (Energoizdat, Moscow, 1982) [in Russian].
28. T. A. Germogenova, *Local Properties of Solutions to Transport Equations* (Nauka, Moscow, 1986) [in Russian].
29. V. I. Agoshkov, *Boundary Value Problems for Transport Equations* (Birkhäuser, Boston, MA, 1998).
30. F. Natterer, *The Mathematics of Computerized Tomography* (Wiley, Chichester 1986).

31. G. I. Marchuk, G. A. Mikhailov, M. A. Nazaraliev, et al., *The Monte Carlo Methods in Atmospheric Optics* (Nauka, Novosibirsk, 1976; Springer-Verlag, Berlin, 1980).
32. G. A. Mikhailov and I. N. Medvedev, *Optimization of Weighted Algorithms for Statistical Modeling* (Omega Print, Novosibirsk, 2011) [in Russian].
33. W. A. Kalender, *Computed Tomography: Fundamentals, System Technology, Image Quality, Applications*, 3rd ed. (Publicis, Erlangen, 2011).
34. P. Mah, T. E. Reeves, and W. D. McDavid, "Deriving Hounsfield units using grey levels in cone beam computed tomography," Dentomaxillofac Radiol. **39**, 323–335 (2010).
35. R. Pauwels, R. Jacobs, S. R. Singer, and M. Mupparapu, "CBCT-Based bone quality assessment: Are Hounsfield units applicable?" Dentomaxillofac Radiol. **44** (1), 20140238 (2015).

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