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Spectral Estimates for the Fourth-Order Operator with Matrix Coefficients

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Abstract—The fourth-order differential operator with matrix coefficients with the domain determined by quasi-periodic boundary conditions is considered. For this operator, the asymptotics of the arithmetic mean of the eigenvalues is found. Moreover, for various special cases, the asymptotics of the eigenvalues is also obtained. The spectral characteristics in the case of periodic and antiperiodic boundary conditions are studied separately. The results are better than those known before.

Keywords: fourth-order differential operator, asymptotics of eigenvalues, matrix coefficients, arithmetic mean of eigenvalues

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INTRODUCTION

Consider the Hilbert space $L_2[0,1]$ of measurable square integrable complex functions defined on the interval $[0,1]$. Denote by $L_2^k[0,1] = L_2([0,1],\mathbb{C}^k)$ the space $L_2^k[0,1] = \underbrace{L_2[0,1]\times\ldots\times L_2[0,1]}_{k \text{ times}}$ with the scalar product

$$(f,g) = \sum_{j=1}^{k}(f_j,g_j), \quad f = (f_1,f_2,\ldots,f_k)\in L_2^k[0,1], \quad g = (g_1,g_2,\ldots,g_k)\in L_2^k[0,1],$$

where $(f_j,g_j) = \int_0^1 f_j(t)\overline{g_j(t)}dt$. Then, the norm induced by this scalar product is defined by

$$\|f\|_{L_2} = \left(\sum_{j=1}^{k}\int_0^1 |f_j(t)|^2 dt\right)^{1/2}.$$

The aim of this paper is to study the spectral characteristics of the fourth-order differential operator $L_\theta : D(L_\theta)\subset L_2^k[0,1]\to L_2^k[0,1]$ defined by the differential expression

$$l(y) = y^{IV} - \mathfrak{A}(t)y'' - \mathfrak{B}(t)y,$$

where $\mathfrak{A}(t) = (a_{pj}(t))_{p,j=1}^k$ and $\mathfrak{B}(t) = (b_{pj}(t))_{p,j=1}^k$ are $k\times k$ matrices and the elements of these matrices a_{pj} and b_{pj} belong to the space $L_2[0,1]$.

Denote by $\mathfrak{A}_0$ the matrix $\mathfrak{A}_0 = (a_{0,pj})$, where $a_{0,pj} = \int_0^1 a_{pj}(t)dt$, $p,j = 1,\ldots,k$. Below, we assume that the matrix $\mathfrak{A}_0$ is similar to a diagonal matrix, i.e., it is a matrix of simple structure (see [1, Chapter III, § 8]). The domain $D(L_\theta) = \left\{y\in W_2^4([0,1],\mathbb{C}^k)\right\}\subset L_2^k[0,1]$ of the operator L_θ is determined by the boundary conditions

$$y^{(j)}(1) = e^{i\pi\theta}y^{(j)}(0), \quad j = 0,1,2,3,$$

where $\theta\in(0,2)$, $\theta\neq 1$.

Let us now describe the history of studies of this class of operators. The study of asymptotic formulas for high-order differential operators with matrix coefficients and integrable elements seems to be started in [2]. In that paper, regular boundary conditions were defined and theorems about the eigenvalue expansion of functions belonging to the domain of regular operators were proved. In a modified form, these results were described in [3, Chapter III]. In the class of regular boundary conditions, a class of strongly regular conditions is distinguished. In the scalar case, it was proved in [4] that the system of eigenfunctions and associated functions of the regular differential operator form an unconditional basis. It was proved in [5] that for the regular operators only the basis property with parentheses can be guaranteed. Moreover, only the terms corresponding to the asymptotically mutually approaching eigenvalues should be parenthesized. General results on the Riesz basis property of ordinary high-order differential operators and more complicated boundary value problems with a nonlinear occurrence of the spectral parameter in the equation and the boundary conditions were obtained in [6]. It was also noted in [7] that the method proposed in [6] makes it possible to prove the unconditional basis property with parentheses for the system of eigenfunctions and associated functions of the regular differential equations also in the matrix case. In addition, for the strongly regular case, the unconditional basis property is guaranteed. A generalization of these results was obtained in [8].

Certainly, the most popular object of research in the spectral theory of differential operators is the non-self-adjoint Sturm–Liouville operator. In the matrix case, quasi-periodic boundary conditions were considered in [9], and sufficient conditions for the matrix coefficients that guarantee the strong regularity were found. In that paper, asymptotic formulas for the eigenvalues in the case of nonsmooth (integrable) matrix coefficients were obtained and the Riesz basis property of the eigenfunctions and associated functions in the space $L_2^k(0,1)$ for strongly regular operators was proved. Later, the results of [9] for the bound on the remainder term in asymptotic formulas for the eigenvalues were strengthened in [10] for the case of square integrable matrix coefficients. Further improvements and generalizations of the results concerning the asymptotics of eigenvalues and the unconditional basis property of eigenfunctions and associated functions were made in [11–14].

In this paper, we derive asymptotic formulas for the eigenvalues of the differential operator L_θ for $\theta \in (0,2)$, $\theta \neq 1$, and compare these results with the earlier known ones. The case $k = 1$ is of special interest. The main method of investigation used in this paper is a version of the method of similar operators (see [15–17] and [10]). However, here we develop an adapted scheme of this method that differs from the schemes used in the works cited above. This modification makes it possible to strengthen earlier results.

Before formulating the main results of this paper, we give some notation. Recall that $\theta \in (0,2)$ and $\theta \neq 1$. Let us represent the operator L_θ in the form $L_\theta = L_\theta^0 - B$, where $L_\theta^0 : D(L_\theta^0) = D(L_\theta) \subset L_2^k[0,1] \to L_2^k[0,1]$, $L_\theta^0 y = y^{IV}$, and $B : D(B) \subset L_2^k[0,1] \to L_2^k[0,1]$, $(By)(t) = \mathfrak{A}(t)y''(t) + \mathfrak{B}(t)y(t)$. The operator L_θ^0 plays the role of unperturbed operator, and the operator B plays the role of perturbation. The spectrum of L_θ^0 is discrete, and its eigenvalues are $\lambda_{n,j} = \pi^4(2n+\theta)^4$, $n \in \mathbb{Z}$, $j = 1,\dots,k$. The corresponding eigenvectors are the functions $e_{n,j}(t) = e^{i\pi(2n+\theta)t} f_j(t)$ ($n \in \mathbb{Z}$, $j = 1, \dots, k$, $t \in [0,1]$), where the vectors f_j ($j = 1, \dots, k$) form an orthonormal basis in $\mathbb{C}^k$. In addition, for any $x \in L_2^k[0,1]$, we define the Riesz projection P_n ($n \in \mathbb{Z}$) by

$$P_n x = \sum_{j=1}^{k} (x, e_{n,j}) e_{n,j}, \quad n \in \mathbb{Z}. \tag{0.1}$$

Definition 1. For every bounded matrix A acting in $\mathbb{C}^k$, the arithmetic mean of its eigenvalues is defined by

$$\hat{\lambda} = \frac{1}{k}\sum_{j=1}^{k} \lambda_j,$$

where λ_j are the eigenvalues of A.

Note that in some works (e.g., in [10]) it was called the weighted mean of eigenvalues. In this paper, the more adequate name will be used.

Theorem 1. *There exists a number $m \in \mathbb{Z}_+$ for which the spectrum of the operator L_θ can be represented in the form*

$$\sigma(L_\theta) = \sigma_{(m)} \cup \left(\bigcup_{|n| \geq m+1} \sigma_n \right), \tag{0.2}$$

where $\sigma_{(m)}$ is a finite set and σ_n contains no more than k points. Moreover, each set σ_n coincides with the spectrum of the restriction of the operator L_θ to the subspace $\operatorname{Im} P_n$, $|n| \geq m+1$. Then, for $\hat{\lambda}_n$ we have the asymptotic representation

$$\hat{\lambda}_n = \pi^4 (2n+\theta)^4 + \frac{\pi^2 (2n+\theta)^2}{k} \sum_{j=1}^{k} \mu_j + \mathbb{O}(|n|), \quad |n| \geq m+1, \tag{0.3}$$

where μ_j $(j = 1, \ldots, k)$ are the eigenvalues of the matrix $\mathfrak{A}_0$.

Since the eigenvalues of $\mathfrak{A}_0$ may be multiple, here we may speak only about asymptotic formulas for the arithmetic mean of the eigenvalues.

Theorem 1 describes the most general situation, and below we consider various special cases.

Theorem 2. *There exists a number $m \in \mathbb{Z}_+$ for which the spectrum of the operator L_θ can be represented in form* (0.2). *If the eigenvalues μ_j $(j = 1, \ldots, k)$ of the matrix $\mathfrak{A}_0$ are simple, then we have the asymptotics*

$$\tilde{\lambda}_{n,j} = \pi^4 (2n+\theta)^4 + \pi^2 (2n+\theta)^2 \mu_j + \mathbb{O}(|n|), \quad j = 1, \ldots, k, \quad |n| \geq m+1.$$

The asymptotic formulas presented in Theorems 1 and 2 elaborate the results [11, Theorems 1, 2].

Everywhere below we denote by the symbol $C > 0$ various positive constants.

Let the matrices $\mathfrak{A}$ and $\mathfrak{B}$ of the operator L_θ be of size 1×1, i.e., let each matrix consist of a single element. We denote these elements by a and b. Each of them belongs to the space $L_2[0,1]$; therefore, we have the expansions

$$a(t) = \sum_{s \in \mathbb{Z}} a_s e^{i2\pi st}, \quad b(t) = \sum_{s \in \mathbb{Z}} b_s e^{i2\pi st}, \quad t \in [0,1],$$

where a_s and b_s are the Fourier coefficients of the functions a and b, respectively. Thus, L_θ in this case is an ordinary fourth-order differential operator with nonsmooth complex coefficients. For this operator, the following results hold.

Theorem 3. *Let the elements a and b belong to the space $L_2[0,1]$. Then, L_θ is an operator with a discrete spectrum, and there exists a number $m \in \mathbb{Z}_+$ such that its spectrum can be represented in form* (0.2). *The eigenvalues $\tilde{\lambda}_{n,1}$ $(|n| \geq m+1)$ satisfy the bound*

$$\begin{aligned} &\left| \tilde{\lambda}_{n,1} - \pi^4 (2n+\theta)^4 - \pi^2 (2n+\theta)^2 a_0 + (2n+\theta)^2 \sum_{\substack{s \in \mathbb{Z} \\ s \neq n}} \frac{(2s+\theta)^2 a_{n-s} a_{s-n}}{(2s+\theta)^4 - (2n+\theta)^4} \right| \\ &\qquad \leq \frac{C |n| \gamma_n}{(\theta-1)^2 (1 - |\theta - 1|)^2}, \quad |n| \geq m+1, \end{aligned} \tag{0.4}$$

where (γ_n) is a summable sequence.

Remark 1. The asymptotic term with the summation sign in (0.4) satisfies the bound

$$\left| (2n+\theta)^2 \sum_{\substack{s \in \mathbb{Z} \\ s \neq n}} \frac{(2s+\theta)^2 a_{n-s} a_{s-n}}{(2s+\theta)^4 - (2n+\theta)^4} \right| \leq C |n| \alpha_n, \quad |n| \geq m+1,$$

where (α_n) is a square summable sequence. The exact form of this sequence will be given in the proof of Theorem 3. Thus, the term of the asymptotics written above is the second-order approximation.

Theorem 4. *Let the coefficients a and b be functions of bounded variation. Then L_θ is an operator with a discrete spectrum, and this spectrum can be represented in form* (0.2) *for a certain $m \in \mathbb{Z}_+$. The eigenvalues $\tilde{\lambda}_{n,1}$ ($|n| \geq m+1$) have the asymptotic representation*

$$\tilde{\lambda}_{n,1} = \pi^4(2n+\theta)^4 + \pi^2(2n+\theta)^2 a_0 - b_0$$
$$- (2n+\theta)^2 \sum_{\substack{s\in\mathbb{Z}\\ s\neq n}} \frac{(2s+\theta)^2 a_{n-s}a_{s-n}}{(2s+\theta)^4 - (2n+\theta)^4} + \mathbb{O}(|n|^{-1}), \quad |n| \geq m+1.$$

Finally, consider an important case that was not described above. If we set $\theta = 0$ or $\theta = 1$, then L_θ becomes an operator with periodic or antiperiodic boundary conditions. Since the analysis of the eigenvalue asymptotics in this case presents certain difficulties (see [13]), we here consider only the one-dimensional case. We will use the technique developed in the present paper (with certain modifications). To simplify the presentation, we continue to use the notation introduced above. Then, the matrices $\mathfrak{A}$ and $\mathfrak{B}$ have the size 1×1, and their elements are functions a and b from the space $L_2[0,1]$. The operator L_θ is a fourth-order differential operator with nonsmooth coefficients and periodic or antiperiodic boundary conditions. The analysis of the spectral properties of this operator is of independent interest due to applications in mechanics (see [18, Chapter I, § 2.3]), optics and acoustics (see [19]), and in the investigation of nanotube conductance (see [20]). This operator also describes vibrations of beams, plates, hulls, and compressed rod on an elastic base (see [21, 22]).

The spectrum of the self-adjoined fourth-order operator with nonsmooth periodic coefficients was analyzed in a series of papers by Badanin and Korotyaev. In [23] spectral zones and characteristics of the spectrum were investigated, and the asymptotics of the eigenvalues was obtained. The last result was later elaborated in [24]. In [25, 26], various spectral characteristics of the operator L_θ ($\theta \in \{0,1\}$) were analyzed, including the asymptotics of its eigenvalues. In [27], the differential operator of an arbitrary order the domain of which is determined by periodic or antiperiodic boundary conditions was considered. In that paper, asymptotic formulas for the eigenvalues were derived, and the conditions under which the eigenfunctions and associated functions form a Riesz basis in $L_2(0,1)$ were obtained.

Now, we present the main results of the current paper. The first theorem deals with the asymptotics of the eigenvalues of L_θ, $\theta \in \{0,1\}$. Compared with [25, Theorem 1], [26, Theorem 1], and [27, Theorems 1 and 2], the result below gives an improved formula for the second-order approximation and the formula for the remainder term.

Theorem 5. *The operator L_θ for $\theta \in \{0,1\}$ is operator with discrete spectrum, and there exists a number $m \in \mathbb{Z}_+$ such that its spectrum can be represented in form* (0.2). *Here $\sigma_{(m)}$ is a finite set with the number of points not exceeding $2m+1$, and the set σ_n has the form $\sigma_n = \{\tilde{\lambda}_n^+\} \cup \{\tilde{\lambda}_n^-\}$. The eigenvalues $\tilde{\lambda}_n^\pm$ ($n \geq m+1$) satisfy the asymptotic bound*

$$\Bigg|\tilde{\lambda}_n^\pm - \pi^4(2n+\theta)^4 - \pi^2(2n+\theta)^2 a_0 + 2(2n+\theta)^2 \sum_{\substack{s=1\\ s\neq n}} \frac{(2s+\theta)^2(a_{n-s}a_{s-n} + a_{n+s+\theta}a_{-n-s-\theta})}{(2s+\theta)^4 - (2n+\theta)^4}$$
$$\pm (2n+\theta)^2 \left(\pi^2 a_{-2n-\theta} - 2\sum_{\substack{s\in\mathbb{Z}\\ s\neq n, s\neq -n-\theta}} \frac{(2s+\theta)^2 a_{s-n}a_{-n-s-\theta}}{(2s+\theta)^4 - (2n+\theta)^4}\right)^{1/2} \tag{0.5}$$
$$\times \left(\pi^2 a_{2n+\theta} - 2\sum_{\substack{s\in\mathbb{Z}\\ s\neq n, s\neq -n-\theta}} \frac{(2s+\theta)^2 a_{n-s}a_{n+s+\theta}}{(2s+\theta)^4 - (2n+\theta)^4}\right)^{1/2}\Bigg| \leq m\eta_n, \quad n \geq m+1,$$

where η_n is a summable sequence.

Remark 2. Detailed bounds on the asymptotic terms will be given in the proof of this theorem.

As before, we consider some special cases.

Corollary 1. Let the elements a and b be real functions. Then

$$\left|\lambda_n^{\pm} - \pi^4(2n+\theta)^4 - \pi^2(2n+\theta)^2 a_0 + 2(2n+\theta)^2 \sum_{\substack{s=1\\ s\neq n}} \frac{(2s+\theta)^2(a_{n-s}a_{s-n} + a_{n+s+\theta}a_{-n-s-\theta})}{(2s+\theta)^4 - (2n+\theta)^4}\right.$$
$$\left. \pm (2n+\theta)^2 \left|\pi^2 a_{2n+\theta} - 2\sum_{\substack{s\in\mathbb{Z}\\ s\neq n, s\neq -n-\theta}} \frac{(2s+\theta)^2 a_{n-s}a_{n+s+\theta}}{(2s+\theta)^4 - (2n+\theta)^4}\right|\right| \leq n\tilde{\eta}_n, \quad n \geq m+1,$$

where $(\tilde{\eta}_n)$ is a summable sequence.

Theorem 6. *Let the elements a and b be functions of bounded variation. Then, the spectrum $\sigma(L_\theta)$ of the operator L_θ $(\theta \in \{0,1\})$ can be represented in form* (0.2), *and $\tilde{\lambda}_n^{\pm}$ $(n \geq m+1)$ satisfy the relation*

$$\tilde{\lambda}_n^{\pm} = \pi^4(2n+\theta)^4 + \pi^2(2n+\theta)^2 a_0 - 2(2n+\theta)^2 \sum_{\substack{s=1\\ s\neq n}} \frac{(2s+\theta)^2(a_{n-s}a_{s-n} + a_{n+s+\theta}a_{-n-s-\theta})}{(2s+\theta)^4 - (2n+\theta)^4}$$
$$\pm (2n+\theta)^2 \left(\pi^2 a_{-2n-\theta} - 2\sum_{\substack{s\in\mathbb{Z}\\ s\neq n, s\neq -n-\theta}} \frac{(2s+\theta)^2 a_{s-n}a_{-n-s-\theta}}{(2s+\theta)^4 - (2n+\theta)^4}\right)^{1/2}$$
$$\times \left(\pi^2 a_{2n+\theta} - 2\sum_{\substack{s\in\mathbb{Z}\\ s\neq n, s\neq -n-\theta}} \frac{(2s+\theta)^2 a_{n-s}a_{n+s+\theta}}{(2s+\theta)^4 - (2n+\theta)^4}\right)^{1/2} - b_0 + \mathcal{O}(n^{-1}), \quad n \geq m+1.$$

The result of Theorem 6 strengthens the corresponding result in [25, Theorem 2] and [26, Theorem 2].

It is known (see [26, Theorem 8]), that $-L_\theta$ $(\theta \in \{0,1\})$ is a sectorial operator, and it generates an analytic semigroup of operators. In that paper, an asymptotic representation of this semigroup was also obtained. However, this representation is extremely cumbersome. On the basis of the improved eigenvalue asymptotics in Theorem 5 and the form of the semigroup obtained in [28, Chapter 1, § 6], we obtain in Theorem 14 a better and more compact representation of the semigroup under examination.

The paper is organized as follows. In Section 1, we investigate the abstract operators the spectral characteristics of which are close to the spectral characteristics of the operator L_θ. In particular, we prove the basic theorem for the eigenvalue asymptotics. In Section 2, we preliminary perform the similarity transform of the operator L_θ to an operator with the spectral characteristics studied in Section 1. In Section 3, we prove the main results of the paper for $\theta \in (0,2)$, $\theta \neq 1$. Section 4 is devoted to the proof of results for the operator L_θ in the one-dimensional case subject to periodic and antiperiodic boundary conditions.

The results of this paper were partly announced in [29].

1. ABSTRACT OPERATORS CLOSE TO THE OPERATOR L_θ AND THEIR PROPERTIES

In this section, we study the spectral properties of an abstract operator the structure of which is similar to that of the operator L_θ. We construct an adaptive scheme of the method and apply it directly to the operator L_θ. However, we first formulate the basic principles.

Let $\mathcal{H}$ be a complex separable Hilbert space, and End $\mathcal{H}$ be the Banach algebra of linear bounded operators acting in $\mathcal{H}$ with the norm $\|\cdot\|$.

Definition 2. Two linear operators $A_j : D(A_j) \subset \mathcal{H} \to \mathcal{H}$ $(j = 1,2)$ are said to be *similar* if there exists a continuously invertible operator $U \in \operatorname{End}\mathcal{H}$ such that $A_1Ux = UA_2x$ for $x \in D(A_2)$ and $UD(A_2) = D(A_1)$. The operator U is called the *transform operator* of A_1 to A_2.

The interest in the study of such operators is due to the fact that some of their spectral properties are identical (see [17, Lemma 1]). In particular, similar operators have identical spectra.

Consider a closed linear operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$. We denote by $\sigma(A)$ and $\rho(A)$ the spectrum and the resolvent set of A, respectively. Denote by $\mathfrak{L}_A(\mathcal{H})$ the Banach space of operators acting in $\mathcal{H}$ and subordinate to the operator A. A linear operator $B : D(B) \subset \mathcal{H} \to \mathcal{H}$ belongs to the space $\mathfrak{L}_A(\mathcal{H})$ if $D(B) \supseteq D(A)$ and the quantity $\|B\|_A = \inf\{C > 0 : \|Bx\| \le C(\|x\| + \|Ax\|), x \in D(A)\}$ is finite. This quantity is considered as the norm in $\mathfrak{L}_A(\mathcal{H})$.

Now we examine the operator $A - B$. Usually the spectral properties of $A - B$ we are interested in are well studied for the operator A; however, A and $A - B$ are not similar. The method of similar operators suggests the following solution to this problem. Using similarity, the investigation of the spectral properties of the operator $A - B$ is reduced to the investigation of the spectral properties of an operator $A - B_0$, where B_0 has a simple structure and the operator $A - B_0$ is sufficiently simple for the investigation of its spectral properties. Then, by Definition 2, the operator $A - B$ has the same properties. To perform such a similarity transformation of operators, we need certain special techniques.

Definition 3 (see [15, 16]). Let $\mathfrak{U}$ be a linear subspace of the operators in $\mathfrak{L}_A(\mathcal{H})$, and let $J : \mathfrak{U} \to \mathfrak{U}$ and $\Gamma : \mathfrak{U} \to \operatorname{End}\mathcal{H}$ be transformers (i.e., linear operators in the space of linear operators). The triple $(\mathfrak{U}, J, \Gamma)$ is said to be an admissible triple for the operator A, and $\mathfrak{U}$ is called the space of admissible perturbations if the following conditions hold:

(1) $\mathfrak{U}$ is a Banach space with its own norm $\|\cdot\|_*$ continuously embedded in $\mathfrak{L}_A(\mathcal{H})$;

(2) J and Γ are continuous transformers, where J is a projection;

(3) $(\Gamma X)D(A) \subset D(A)$ and $A(\Gamma X) - (\Gamma X)A = X - JX$ for every $X \in \mathfrak{U}$, and $Y = \Gamma X$ is the unique solution to the equation $AY - YA = X - JX$ satisfying the condition $JY = 0$;

(4) $X(\Gamma Y)$ and $(\Gamma Y)X \in \mathfrak{U}$ for all $X, Y \in \mathfrak{U}$, and there exists a constant $\gamma > 0$ such that $\|\Gamma\| \le \gamma$ and $\max\{\|X(\Gamma Y)\|_*, \|(\Gamma X)Y\|_*\} \le \gamma \|X\|_* \|Y\|_*$;

(5) for every $X \in \mathfrak{U}$ and any $\varepsilon > 0$, there exists a $\lambda_\varepsilon \in \rho(A)$ such that $\left\|X(A - \lambda_\varepsilon I)^{-1}\right\| < \varepsilon$.

Let us give some explanations for the objects just defined. Usually, we choose a convenient Banach or Hilbert space as the space $\mathfrak{U}$. The transformer J is directly related to the form of the operator B_0. The introduction of the operator Γ is closely related to the construction of the transform operator U in Definition 2. This is explained in more detail after the formulation of the main similarity theorem. Note that the admissible triple is not unique. We select it on the basis of the convenience of its use, the presence of certain properties of the transformers in this triple, and the final results.

Theorem 7 (see [15, 16]). *Let $(\mathfrak{U}, J, \Gamma)$ be an admissible triple for the operator A and the operator B belong to $\mathfrak{U}$. If it holds that*

$$\|J\|_* \|B\|_* \|\Gamma\|_* < 1/4, \tag{1.1}$$

then the operator $A - B$ is similar to the operator $A - JX_*$, where the operator $X_* \in \mathfrak{U}$ is a solution of the nonlinear operator equation

$$X = B + B\Gamma X - (\Gamma X)JB - (\Gamma X)J(B\Gamma X) = \Phi(X). \tag{1.2}$$

Such a solution can be found using the simple iteration method by setting $X_0 = 0$, $X_1 = B$ and so on. Here $\Phi : \mathfrak{U} \to \mathfrak{U}$ is a contraction operator in the ball $\{X \in \mathfrak{U} : \|X - B\|_* \le 3\|B\|_*\}$. The similarity transform of the operator $A - B$ to the operator $A - JX_*$ is done by the invertible operator $I + \Gamma X_* \in \operatorname{End}\mathcal{H}$.

A proof of this theorem can be found in [15, Theorem 1.5] and in [16, Theorem 19.2]. Condition (1.1) provides the existence condition for the solution of the nonlinear equation (1.2). The form of this equation is directly related to the transform operator $I + \Gamma X_*$. Conditions (3)–(5) in Definition 3 guarantee the invertibility of this operator and its invariance under the domain $D(A)$. Thus, all properties of similar operators in Definition 2 hold true.

Since J is a projection, Theorem 7 can be considered as a theorem on the similarity of the operator $A - B$ to the operator $A - JX_*$ of block diagonal form relative to the “basis” in which the operator A has diagonal form. Thus, the scheme just described makes it possible to significantly simplify the analysis of spectral properties of the original operator $A - B$.

Now we apply the scheme described above to an operator the spectral properties of which are similar to those of the operator L_θ. To the end of this section, we assume that $\theta \in (0, 2)$, $\theta \ne 1$. As the unperturbed operator, we consider the normal linear operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ with a discrete spectrum. Assume that the eigenvalues of this operator have multiplicity k and have the form

$$\lambda_{n,j} = \pi^4(2n+\theta)^4, \quad n \in \mathbb{Z}, \quad j = 1, \dots, k.$$

Thus, the spectrum $\sigma(A)$ of A can be written as $\sigma(A) = \bigcup_{n\in\mathbb{Z}} \sigma_n$, where $\sigma_n \cap \sigma_r = \varnothing$, $r \neq n$, $r, n \in \mathbb{Z}$, and σ_n $(n \in \mathbb{Z})$ are finite sets. Denote the corresponding eigenfunctions by $e_{n,j}$, $n \in \mathbb{Z}$, $j = 1, \dots, k$. Assume that they form an orthonormal basis in $\mathcal{H}$. Denote by P_n $(n \in \mathbb{Z})$ the Riesz projection constructed on the basis of the spectral set σ_n. For every $x \in \mathcal{H}$, this projection is determined by formula (0.1). Therefore, $AP_n = \lambda_{n,j}P_n$, $n \in \mathbb{Z}$, $j = 1, \dots, k$.

Remark 3. It has already been mentioned above that the operator A has exactly the same spectral properties as the operator L_θ^0 for $\theta \in (0,2)$, $\theta \neq 1$. For the eigenvalues, eigenfunctions, and projections, we will use the same notation as for the operator L_θ^0.

Denote by $\mathfrak{S}_2(\mathcal{H})$ the ideal of the Hilbert–Schmidt operators with the norm $\|\cdot\|_2$ (see [30, Chapter 3, §9]). Each operator $X \in \operatorname{End}\mathcal{H}$ is assigned the block matrix $X = (X_{sr})$ composed of the operators $X_{sr} = P_sXP_r$, $s, r \in \mathbb{Z}$. Since the projections P_s $(s \in \mathbb{Z})$ are orthoprojections, the norm in $\mathfrak{S}_2(\mathcal{H})$ is given by the formula $\|X\|_2 = \left(\sum_{s,r\in\mathbb{Z}} \|P_sXP_r\|_2^2\right)^{1/2}$.

For an arbitrary $\alpha \in (-1,0) \cup (0,1)$, consider the normal operator $A^\alpha : D(A^\alpha) \subset \mathcal{H} \to \mathcal{H}$ defined by

$$A^\alpha x = \sum_{n\in\mathbb{Z}} \lambda_{n,j}^\alpha P_n x, \quad j = 1, \dots, k,$$

with the domain

$$D(A^\alpha) = \left\{ x \in \mathcal{H} : \sum_{n\in\mathbb{Z}} |\lambda_{n,j}|^{2\alpha} \|P_n x\|^2 < \infty \right\}.$$

Now we have everything we need to construct an admissible triple. According to Definition 3, it consists of the space of admissible perturbations $\mathfrak{U}$ and two transformers.

The Banach space of admissible perturbations $\mathfrak{U}$ consists of the operators $X \in \mathfrak{L}_A(\mathcal{H})$ that can be represented as

$$X = X_0A^{1/2}, \quad X_0 \in \mathfrak{S}_2(\mathcal{H}).$$

The norm of the operator X in $\mathfrak{U}$ is defined as $\|X\|_* = \|X_0\|_2$.

Let us proceed to constructing the necessary transformers, which we will denote by J_m and Γ_m. To this end, determine more general transformers J_0 and Γ_0, which will play an auxiliary role. Let

$$J_0X = \sum_{s\in\mathbb{Z}} P_sXP_s, \quad X \in \mathfrak{U}. \tag{1.3}$$

For $X \in \mathfrak{U}$, define the operator $\Gamma_0 : \mathfrak{U} \to \mathfrak{S}_2(\mathcal{H})$ on the operator blocks $X_{sr} = P_sXP_r = P_sX_0A^{1/2}P_r = \lambda_{r,j}^{1/2}P_sX_0P_r$. For each X_{sr} $(s \neq r)$, define the transformer Γ_0 as $\Gamma_0X_{sr} = Y_{sr}$, where Y_{sr} is a solution to the equation $AY_{sr} - Y_{sr}A = X_{sr}$ $(s \neq r)$ and $Y_{ss} = 0$ for every $s \in \mathbb{Z}$. Note that the last equation can be rewritten as

$$A_sY_{sr} - Y_{sr}A_r = X_{sr}, \tag{1.4}$$

where A_s is the restriction of A to the subspace $\operatorname{Im} P_s$ for any $s \in \mathbb{Z}$. Since $\sigma(A_s) \cap \sigma(A_r) = \varnothing$ for $s, r \in \mathbb{Z}$, each equation in (1.4) has a solution and

$$\|Y_{sr}\|_* \le C\|X_{sr}\|_* / \operatorname{dist}(\sigma(A_s), \sigma(A_r)), \quad Y_{ss} = 0, \quad s, r \in \mathbb{Z}.$$

The correctness of the definition of the operators J_0X and Γ_0X and their boundedness are proved as described in [26, Lemma 1].

The extensions of the transformers J_0 and Γ_0 to the space $\mathfrak{L}_A(\mathcal{H})$, which will be below denoted by the same symbols, will be defined for any operator $X \in \mathfrak{L}_A(\mathcal{H})$ by

$$J_0X = (J_0XA^{-1/2})A^{1/2}, \quad \Gamma_0X = (\Gamma_0XA^{-1/2})A^{1/2}, \quad X \in \mathfrak{L}_A(\mathcal{H}). \tag{1.5}$$

For any $m \in \mathbb{Z}_+$, define the family of transformers J_m and Γ_m by

$$J_mX = J_0(X - P_{(m)}XP_{(m)}) + P_{(m)}XP_{(m)} = P_{(m)}XP_{(m)} + \sum_{|n|\geq m+1} P_nXP_n, \quad X \in \mathfrak{U}, \tag{1.6}$$

$$\Gamma_mX = \Gamma_0X - \Gamma_0(P_{(m)}XP_{(m)}) = \Gamma_0X - P_{(m)}(\Gamma_0X)P_{(m)}, \quad X \in \mathfrak{U}, \tag{1.7}$$

where $P_{(m)} = \sum_{|j|\leq m} P_j$. Since X belongs to the space $\mathfrak{U}$, the series in (1.3), (1.6), and (1.7) are convergent in the uniform operator topology.

Since the transformers J_m and Γ_m are constructed from the transformers J_0 and Γ_0, then extensions of J_m and Γ_m to the spaces $\mathfrak{L}_A(\mathcal{H})$ and $\mathfrak{U}$ are also constructed by formulas (1.5).

Remark 4. In essence, the transformers J_m and Γ_m are formed by "cutting" a finite-dimensional block from the corresponding transformers J_0 and Γ_0. Thus, J_m and Γ_m differ from J_0 and Γ_0 by operators of finite rank. The dependence on m in the transformers J_m and Γ_m is introduced for two purposes. Firstly, for controlling $\|\Gamma_mB\|_*$, which must be sufficiently small. This requirement will be clarified when the preliminary similarity transform is performed in the next section (see Lemma 5). The other purpose is to remove certain restrictions on the operator B. A closer examination of Γ_0 allows us to prove the bound $\|\Gamma_0\|_* \leq C$. By Theorem 7, similarity is possible only under condition (1.1). Therefore, for further constructions, we need $\|B\|_*$ to be small. Hence, by using the transformer Γ_m and choosing a sufficiently large m, we can remove this restriction.

Let us discuss some properties of the transforms just constructed. Definitions (1.6) and (1.7) immediately imply that all operators $X, Y \in \mathfrak{U}$ for $|n| \geq m+1$ satisfy the equalities

$$(J_mX)P_n = P_n(J_0X)P_n = P_nXP_n, \quad \Gamma_m(P_nXP_n) = 0, \quad P_n(J_mX)(\Gamma_mY)P_n = 0. \tag{1.8}$$

Thus, the construction of the triple $(\mathfrak{U}, J_m, \Gamma_m)$ is completed. To apply Theorem 7 this triple must be admissible. This result is formulated in the following lemma.

Lemma 1. *For every $m \in \mathbb{Z}_+$, the triple $(\mathfrak{U}, J_m, \Gamma_m)$ is admissible for the operator A and*

$$\|\Gamma_m\|_* \leq 1/(4\pi^2(2m - 1 + \theta)).$$

The proof is similar to the proof of [26, Lemma 4].

Up to the end of this section, we assume that the perturbation B belongs to the space $\mathfrak{U}$ constructed above. Then, on the basis of the abstract Theorem 7, we can formulate the main similarity theorem for the operator $A - B$.

Theorem 8. *Let the number $m \in \mathbb{Z}_+$ be such that*

$$\|B\|_* < \pi^2(2m - 1 + \theta). \tag{1.9}$$

Then, the operator $A - B$ is similar to the operator $A - J_mX_*$, where $X_* \in \mathfrak{U}$ is a solution of the nonlinear equation

$$X = B\Gamma_mX - (\Gamma_mX)(J_mB) - (\Gamma_mX)J_m(B\Gamma_mX) + B = \Phi(X), \tag{1.10}$$

which can be found by the simple iteration method by setting $X_0 = 0$, $X_1 = B$, ... The operator $\Phi : \mathfrak{U} \to \mathfrak{U}$ is a contraction operator in the ball $\{X \in \mathfrak{U} : \|X - B\|_* \leq 3\|B\|_*\}$. The similarity transform of the operator U is the operator $I + \Gamma_mX_*$.

The proof immediately follows from Lemma 1 and Theorem 7.

Using this theorem, we proceed to the analysis of the spectrum of $A - B$.

Theorem 9. *Let condition* (1.9) *holds. Then, the operator $A - B$ has a discrete spectrum that coincides with the spectrum of the operator*

$$A - J_m X_* = A - P_{(m)} X_* P_{(m)} - \sum_{|n| \geq m+1} P_n X_* P_n. \tag{1.11}$$

In addition, the equalities

$$\sigma(A - B) = \sigma(A_{(m)}) \bigcup \left(\bigcup_{|n| \geq m+1} \sigma(A_n) \right) = \sigma_{(m)} \bigcup \left(\bigcup_{|n| \geq m+1} \sigma_n \right) \tag{1.12}$$

hold, where $A_{(m)}$ is a restriction of the operator $A - J_m X_*$ to the invariant subspace $\mathcal{H}_{(m)} = \operatorname{Im} P_{(m)}$, and A_n is the restriction of $A - J_m X_*$ to the subspace $\mathcal{H}_n = \operatorname{Im} P_n$, $|n| \geq m + 1$.

Proof. Since A has a discrete spectrum and $J_m X_*$ is a bounded operator, the operator $A - J_m X_*$ also has a discrete spectrum. By Theorem 8, $A - B$ is similar to $A - J_m X_*$. This similarity implies that $A - B$ also has a discrete spectrum, and it holds that $\sigma(A - B) = \sigma(A - J_m X_*)$.

Formula (1.6) immediately implies (1.11). Moreover, Theorem 8 and [17, Lemma 1] imply that the operator $A - J_m X_*$ (1.11) commutes with all projections $P_{(m)}$ and P_n, $|n| \geq m + 1$. Therefore, the subspaces $\mathcal{H}_{(m)} = \operatorname{Im} P_{(m)}$ and $\mathcal{H}_n = \operatorname{Im} P_n$ for $|n| \geq m + 1$ are invariant for the operator $A - J_m X_*$. If $\mu_0 \in \sigma(A - J_m X_*)$, then there exists an eigenvector $x_0 \in D(A)$ such that $(A - J_m X_*) x_0 = \mu_0 x_0$. Thus, the form of the operator $J_m X_*$ implies that

$$A_{(m)} P_{(m)} x_0 = \mu_0 P_{(m)} x_0, \quad A_n P_n x_0 = \mu_0 P_n x_0, \quad |n| \geq m + 1, \tag{1.13}$$

where $A_{(m)}$ is a restriction of $A - J_m X_*$ to $\mathcal{H}_{(m)}$ and A_n is a restriction of $A - J_m X_*$ to $\mathcal{H}_n$ for $|n| \geq m + 1$. Since $I = P_{(m)} + \sum_{|n| \geq m+1} P_n$, i.e., the system of projections forms a resolution of identity, it follows from (1.13) that at least one of the vectors $P_n x_0$ ($|n| \geq m + 1$) and $P_{(m)} x_0$ is nonzero. Therefore, μ_0 is an eigenvalue of the corresponding operator from the family of operators A_n ($|n| \geq m + 1$), $A_{(m)}$. Therefore, we have the embedding

$$\sigma(A_{(m)}) \cup \left(\bigcup_{|n| \geq m+1} \sigma(A_n) \right) \supset \sigma(A - J_m X_*) = \sigma(A - B).$$

The reverse embedding is obvious. Therefore, we have proved equality (1.12), which completes the proof of Theorem 9.

Thus, Theorems 8 and 9 reduce the analysis of the spectral characteristics of the operator $A - B$ to the analysis of the spectral characteristics of the operator $A - J_m X_*$. The next theorem deals with the algorithm for finding asymptotic formulas for the arithmetic mean of the eigenvalues of $A - B$.

Theorem 10. *Let condition* (1.9) *be satisfied and the spectrum of the operator $A - B$ can be represented in form* (1.12). *Then, the sets σ_n ($|n| \geq m + 1$) contain no more than k points. The arithmetic mean of the eigenvalues of each of these sets coincides with the arithmetic mean of the eigenvalues of the matrix $\mathcal{A}_n$, which can be represented as*

$$\mathcal{A}_n = \pi^4 (2n + \theta)^4 E - \mathcal{B}_n + \pi^2 (2n + \theta)^2 \mathcal{C}_n, \tag{1.14}$$

where E is the identity matrix and $\mathcal{B}_n$ is the matrix of size $k \times k$ composed of the elements $(Be_{n,j}, e_{n,j})$, $j = 1, \ldots, k$. Furthermore, for the number $n_1 = \max\{m + 1, 3\|B\|_* / (4\pi^2) + (1 + \theta)/2\}$, the norm of the matrices $\mathcal{C}_n$ satisfies the bounds

$$\|\mathcal{C}_n\|_* \leq \frac{1}{2\pi^2 |2|n| - 1 - \theta|} \|P_n B - P_n B P_n\|_* \|B P_n - P_n B P_n\|_*, \quad |n| \geq n_1. \tag{1.15}$$

Proof. In the proof, we will often use equalities (1.8). Apply the projection P_n to Eq. (1.10) with $X = X_*$ on the left and on the right. Then

$$A - P_n X_* P_n = A - P_n B P_n - P_n (B \Gamma_m X_*) P_n. \tag{1.16}$$

Represent the operator $P_n(B\Gamma_m X_*)P_n$ $(|n| \geq m+1)$ in the form

$$P_n(B\Gamma_m X_*)P_n = P_n(B - J_m B)(\Gamma_m X_*)P_n = ((P_n B - P_n B P_n)A^{-1/2})(A^{1/2}(\Gamma_m X_*)P_n).$$

Multiply both sides of the equality by the operator $A^{-1/2}$ on the right. Then

$$\|P_n(B\Gamma_m X_*)P_n\|_* \leq \|P_n B - P_n B P_n\|_* \left\|A^{1/2}(\Gamma_m X_*)P_n\right\|_*, \quad |n| \geq m+1. \tag{1.17}$$

Let us estimate the second factor on the right-hand side of (1.17). Using (1.8), we obtain

$$\left\|A^{1/2}(\Gamma_m X_*)P_n\right\|_* = \left\|A^{1/2}\Gamma_m(X_* - P_n X_* P_n)P_n\right\|_*$$
$$\leq \max_{s\neq n}\frac{\lambda_{s,j}^{1/2}}{\operatorname{dist}(\sigma_s,\sigma_n)}\left\|X_* P_n - P_n X_* P_n\right\|_* = d_n\left\|X_* P_n - P_n X_* P_n\right\|_*, \quad |n| \geq m+1,$$

where

$$d_n = \max_{s\neq n}\frac{\lambda_{s,j}^{1/2}}{\operatorname{dist}(\sigma_s,\sigma_n)} \leq \frac{1}{4\pi^2\,|2|n| - 1 - \theta|}, \quad j = 1, \ldots, k.$$

Let us now estimate $\|X_* P_n - P_n X_* P_n\|_*$. Again, we apply to Eq. (1.10) the projection P_n. Then

$$X_* P_n - P_n X_* P_n = BP_n - P_n B P_n + B\Gamma_m(X_* - P_n X_* P_n)P_n$$
$$- \Gamma_m(X_* - P_n X_* P_n)P_n B P_n - P_n B\Gamma_m(X_* - P_n X_* P_n)P_n.$$

By estimating both sides of the last equality, we obtain

$$\|X_* P_n - P_n X_* P_n\|_* \leq \|BP_n - P_n B P_n\|_* + d_n\|B\|_*\|X_* P_n - P_n X_* P_n\|_*$$
$$+ d_n\|B\|_*\|X_* P_n - P_n X_* P_n\|_* + d_n\|B\|_*\|X_* P_n - P_n X_* P_n\|_*$$
$$= \|BP_n - P_n B P_n\|_* + 3d_n\|B\|_*\|X_* P_n - P_n X_* P_n\|_*.$$

Therefore, if $3d_n\|B\|_* \leq 1/2$, then we have $\|X_* P_n - P_n X_* P_n\|_* \leq 2\|BP_n - P_n B P_n\|_*$. Hence,

$$\left\|A^{1/2}(\Gamma_m X_*)P_n\right\|_* = \left\|A^{1/2}\Gamma_m(X_* P_n - P_n X_* P_n)\right\|_* \leq 2d_n\|BP_n - P_n B P_n\|_*. \tag{1.18}$$

Inequalities (1.17) and (1.18) imply the bound

$$\|P_n(B\Gamma_m X)P_n\|_* \leq \frac{1}{2\pi^2\,|2|n| - 1 - \theta|}\|P_n B - P_n B P_n\|_*\|BP_n - P_n B P_n\|_*, \quad |n| \geq n_1, \tag{1.19}$$

where $n_1 = \max\{m+1, 3\|B\|_*/(4\pi^2) + (1+\theta)/2\}$. Note that the first value in n_1 is responsible for the representation of spectrum (1.12), and the second one for the fulfillment of the inequality $3d_n\|B\|_* \leq 1/2$.

Consider the restrictions of the operators in (1.16) to the subspaces $\operatorname{Im} P_n$ for $|n| \geq n_1$. The matrices corresponding to these operators are denoted by $\mathcal{A}_n$, $\mathcal{B}_n$, and $\mathcal{C}_n$, respectively. Then, taking into account the fact that the operator $B\Gamma_m X_*$ belongs to the space $\mathfrak{U}$, formula (1.16) implies representation (1.14). Moreover, for the matrix $\mathcal{C}_n$, which corresponds to the operator $P_n(B\Gamma_m X_*)P_n$, inequality (1.15) holds due to (1.19).

2. PRELIMINARY SIMILARITY TRANSFORM

Now we return to the analysis of the operator $L_\theta = L_\theta^0 - B$, where $\theta \in (0,2)$ and $\theta \neq 1$. Everywhere below, we use $L_2^k[0,1]$ as the space $\mathcal{H}$. To investigate the operator L_θ, we use the scheme described in Section 1. As the operator A, we will use the unperturbed operator L_θ^0. Next, we use the admissible triple constructed above. For the scheme described above to be applicable, it is necessary (see Theorem 8) that the operator B belongs to the space $\mathfrak{U}$. However, the straightforward calculation shows that this is not the case— the operator B only belongs to the space $\mathfrak{L}_{L_\theta^0}(\mathcal{H})$. For this reason, we perform a preliminary similarity transformation (see [17, Assumption]). This transformation allows us to reduce the analysis of the operator L_θ to the analysis of the operator $\tilde{L}_\theta^0 - \tilde{B}$, where the operator $\tilde{B}$ belongs to the space $\mathfrak{U}$, and the operator $\tilde{L}_\theta^0$ is constructed given the operator L_θ^0. The exact form of this operator will be described below. Then, we will be able to apply the scheme constructed in Section 1 to the operator $\tilde{L}_\theta^0 - \tilde{B}$.

Technically, we will check five properties (see [17, Assumption] and Lemma 5 in this paper) that are in part similar to the properties of the admissible triple. For this reason, the construction of the transform operator and the specific form of the operator $\tilde{B}$ actually deal with the same issues of the solvability of nonlinear equations as before.

Thus, we will use the triple constructed in Section 1 as the basis. Since B is in the space $\mathfrak{L}_{L_\theta^0}(\mathcal{H})$, the operators $J_m B$ and $\Gamma_m B$ are well defined by (1.6) and (1.7) using extensions (1.5). Recall that the spectrum of the operator L_θ^0 is discrete and its eigenvalues are

$$\lambda_{n,j} = \pi^4(2n+\theta)^4, \quad n \in \mathbb{Z}, \quad j = 1, \dots, k.$$

The corresponding eigenfunctions are

$$e_{n,j}(t) = e^{i\pi(2n+\theta)t} f_j(t), \quad n \in \mathbb{Z}, \quad j = 1, \dots, k, \quad t \in [0,1],$$

where the vectors f_j $(j = 1, \dots, k)$ form an orthonormal basis in $\mathbb{C}^k$. Therefore, the eigenspace $E_n = \operatorname{Span}\{e_{n,1}, \dots, e_{n,k}\}$ is k-dimensional. As before, we denote by P_n the Riesz projection, which for every $x \in \mathcal{H}$ is defined by (0.1).

We begin designing the preliminary similarity transform from examining the properties of the operator B. We represent it in the form

$$B = B_1 + B_2, \quad \text{where} \quad (B_1 y)(t) = \mathfrak{A}(t)y(t) \quad \text{and} \quad (B_2 y)(t) = \mathfrak{B}(t)y(t).$$

Since the elements a_{pj} and b_{pj} $(p, j = 1, \dots, k)$ of the matrices $\mathfrak{A}$ and $\mathfrak{B}$ are in $L_2[0,1]$, we have the expansions

$$a_{pj}(t) = \sum_{s\in\mathbb{Z}} a_{s,pj} e^{i2\pi st}, \quad b_{pj}(t) = \sum_{s\in\mathbb{Z}} b_{s,pj} e^{i2\pi st}, \quad t \in [0,1],$$

where $a_{s,pj}$ and $b_{s,pj}$ are the Fourier coefficients of the functions a and b, respectively. In addition, due to Parseval's identity, we have

$$\|a\|_{L_2}^2 = \sum_{p,j=1}^{k} \int_0^1 |a_{pj}(t)|^2 dt = \sum_{p,j=1}^{k} \sum_{s\in\mathbb{Z}} |a_{s,pj}|^2.$$

Using these expansions, we estimate the elements $b^1_{l_p,r_j}$ and $b^2_{l_p,r_j}$ of the blocks of the matrix representations of B_1 and B_2, respectively. We have the equalities

$$b^1_{l_p,r_j} = (B_1 e_{r,j}, e_{l,p}) = -\pi^2(2r+\theta)^2 \int_0^1 a_{pj}(t) e^{i\pi(2r+\theta)t} e^{-i\pi(2l+\theta)t} f_p \overline{f}_j dt$$
$$= -\pi^2(2r+\theta)^2 \sum_{s\in\mathbb{Z}} a_{s,pj} \int_0^1 e^{i2\pi(s+r-l)t} f_p \overline{f}_j dt = -\pi^2(2r+\theta)^2 a_{l-r,pj}, \quad p, j = 1, \dots, k.$$

Similarly, we obtain the relations for $b^2_{l_p,r_j}$. Hence, we obtain the bounds

$$\left|b^1_{l_p,r_j}\right| \le \pi^2(2r+\theta)^2 \left|a_{l-r,pj}\right|, \quad \left|b^2_{l_p,r_j}\right| \le \left|b_{l-r,pj}\right|, \quad p, j = 1, \dots, k, \quad r, l \in \mathbb{Z}. \tag{2.1}$$

Using these inequalities, we proceed to the analysis of the operators involved in the preliminary similarity transform.

Lemma 2. *For every $q \in \mathbb{Z}_+$, the operators $\Gamma_q B$ belong to the space $\mathfrak{U}$, and the inequality $\|\Gamma_q B\|_* \le 1/2$ holds for sufficiently large q. Moreover, for $n \in \mathbb{Z}$, we have the bounds*

$$\|P_n(\Gamma_q B)\|_* \le 2\left\|P_n(\Gamma_0 B_1)(L_\theta^0)^{-1/2}\right\|_2 \le \frac{\|a\|_{L_2}}{2\pi^4|\theta-1|(1-|\theta-1|)(2n+\theta)^2\,|2|n|-3|}, \tag{2.2}$$

$$\|(\Gamma_q B)P_n\|_* \le 2\left\|(\Gamma_0 B_1)P_n(L_\theta^0)^{-1/2}\right\|_2 \le \frac{\|a\|_{L_2}}{2\pi^4|\theta-1|(1-|\theta-1|)(2n+\theta)^2\,|2|n|-3|}. \tag{2.3}$$

Proof. Since $B = B_1 + B_2$, we obtain due to the linearity of the transformer Γ_q that $\Gamma_q B = \Gamma_q B_1 + \Gamma_q B_2$. Let us prove that the operator $\Gamma_q B_1$ belongs to the space $\mathfrak{U}$. To this end, we first show that $\Gamma_0 B_1 \in \mathfrak{U}$. Let us use the first inequality in (2.1). Then we have

$$\sum_{l,r\in\mathbb{Z}}\sum_{p,j=1}^{k}\left|(\Gamma_0 B_1(L_\theta^0)^{-1/2}e_{r,j}, e_{l,p})\right|^2 = \sum_{\substack{l,r\in\mathbb{Z}\\ l\ne r}}\sum_{p,j=1}^{k}\frac{\left|b^1_{l_p,r_j}\right|^2}{|\lambda_{l,p}-\lambda_{r,j}|^2\lambda_{r,j}}$$

$$\le \frac{1}{\pi^8}\sum_{\substack{l,r\in\mathbb{Z}\\ l\ne r}}\sum_{p,j=1}^{k}\frac{(2r+\theta)^4|a_{l-r,pj}|^2}{((2l+\theta)^4-(2r+\theta)^4)^2(2r+\theta)^4}$$

$$\le \frac{1}{16\pi^8}\sum_{\substack{l,r\in\mathbb{Z}\\ l\ne r}}\sum_{p,j=1}^{k}\frac{|a_{l-r,pj}|^2}{(l-r)^2(l+r+\theta)^2(2l+\theta)^4} = \frac{1}{16\pi^8}\sum_{\substack{l\in\mathbb{Z}\\ l\ne 0}}\frac{1}{(2l+\theta)^4}\sum_{\substack{s\in\mathbb{Z}\\ s\ne 0}}\sum_{p,j=1}^{k}\frac{|a_{s,pj}|^2}{s^2(2l-s+\theta)^2} \tag{2.4}$$

$$\le \frac{\|a\|^2_{L_2}}{16\pi^8(\theta-1)^2(1-|\theta-1|)^2}\sum_{\substack{l\in\mathbb{Z}\\ l\ne 0}}\frac{1}{(2l+\theta)^4(2|l|-3)^2}$$

$$\le \frac{\|a\|^2_{L_2}}{16\pi^8(\theta-1)^2(1-|\theta-1|)^2}\sum_{\substack{l\in\mathbb{Z}\\ l\ne 0}}\frac{1}{(2l+\theta)^4}\left(\frac{1}{l^2}+1\right) < \infty.$$

Therefore, $\Gamma_0 B_1 \in \mathfrak{U}$. According to formula (1.7), the operator $\Gamma_q B_1$ differs from $\Gamma_0 B_1$ by a finite rank operator. Therefore, $\Gamma_q B_1 \in \mathfrak{U}$. Taking into account the second inequality in (2.1), we similarly prove that $\Gamma_q B_2 \in \mathfrak{U}$. Therefore, $\Gamma_q B$ belongs to the space of admissible perturbations. Furthermore,

$$\lim_{q\to\infty}\|\Gamma_q B\|_*^2 = \lim_{q\to\infty}\|\Gamma_0 B - P_{(q)}(\Gamma_0 B)P_{(q)}\|_*^2 = \lim_{q\to\infty}\sum_{\max\{|l|,|r|\}\ge q+1}\|P_l(\Gamma_q B)P_r\|_*^2 = 0.$$

Hence, we can always find a sufficiently large q for which $\|\Gamma_q B\|_* \le 1/2$.

Again, use (1.7) and the properties of the operator $\Gamma_q B$ to obtain

$$\|P_n(\Gamma_q B)\|_* \le \|P_n(\Gamma_q B_1)\|_* + \|P_n(\Gamma_q B_2)\|_* \le 2\|P_n(\Gamma_q B_1)\|_* \le 2\|P_n(\Gamma_0 B_1)\|_*.$$

Using this bound and setting $l = n$ in inequalities (2.4), we obtain (2.2). Inequality (2.3) is proved using the same technique by setting $r = n$ in these inequalities. Thus, the lemma is proved.

The straightforward verification proves the following lemma.

Lemma 3. *For every $q \in \mathbb{Z}_+$, $J_q B$ is a bounded operator.*

Lemma 4. *For every $q \in \mathbb{Z}_+$, the operators $B\Gamma_q B$ belong to the space $\mathfrak{U}$ and satisfy the bounds*

$$\|P_n(B\Gamma_q B)\|_* \le \frac{8\sqrt{2}\,\beta(n)}{\pi^2|\theta-1|(1-|\theta-1|)}, \quad \|(B\Gamma_q B)P_n\|_* \le \frac{8\|a\|_{L_2}\,\alpha(2n)}{\pi|\theta-1|(1-|\theta-1|)\sqrt{3}}, \tag{2.5}$$

$$\|P_n(B\Gamma_q B)P_n\|_* \le C/(|2n+\theta||\theta-1|(1-|\theta-1|)), \quad n\in\mathbb{Z}, \tag{2.6}$$

where $(\alpha(2n))$ and $(\beta(n))$ are square summable sequences.

Proof. The reasoning is the same as in the proof of Lemma 2. First, we prove the lemma assertion for the operator $B_1\Gamma_0 B_1$. Using the first inequality in (2.1), we obtain the inequality

$$\sum_{l,r\in\mathbb{Z}}\sum_{p,j=1}^{k}\left|(B_1\Gamma_0 B_1(L_\theta^0)^{-1/2}e_{r,j},e_{l,p})\right|^2=\frac{1}{\pi^4}\sum_{l,r\in\mathbb{Z}}\sum_{p,j=1}^{k}\left|\sum_{\substack{s\in\mathbb{Z}\\ s\neq r}}\sum_{i=1}^{k}\frac{(2r+\theta)^2(2s+\theta)^2 a_{l-s,pi}a_{s-r,ij}}{((2s+\theta)^4-(2r+\theta)^4)(2r+\theta)^2}\right|^2$$
$$\leq\frac{1}{\pi^4}\sum_{l,r\in\mathbb{Z}}\sum_{p,j=1}^{k}\left|\sum_{\substack{s\in\mathbb{Z}\\ s\neq r}}\sum_{i=1}^{k}\frac{a_{l-s,pi}a_{s-r,ij}}{(s-r)(s+r+\theta)}\right|^2. \tag{2.7}$$

The right-hand side in this inequality is finite. This is proved as in [26, Lemma 7]. Therefore, $B_1\Gamma_0 B_1$ is in the space $\mathfrak{U}$. Since B_2 is the multiplication operator by the function b, then the operators $B_2\Gamma_0 B_1$, $B_1\Gamma_0 B_2$, and $B_2\Gamma_0 B_2$ also belong to $\mathfrak{U}$. Thus, $B\Gamma_0 B\in\mathfrak{U}$. Since $B\Gamma_q B$ differs from $B\Gamma_0 B$ by a finite rank operator, $B\Gamma_q B$ also belongs to the space of admissible perturbations $\mathfrak{U}$.

To complete the proof of the lemma, it remains to derive bounds (2.5) and (2.6). In (2.7), set $l=n$ and apply the Hölder inequality to obtain

$$\|P_n(B_1\Gamma_0 B_1)\|_*^2\leq\frac{1}{\pi^4}\sum_{r\in\mathbb{Z}}\sum_{p,j=1}^{k}\left|\sum_{\substack{s\in\mathbb{Z}\\ s\neq r}}\sum_{i=1}^{k}\frac{a_{n-s,pi}a_{s-r,ij}}{(s-r)(s+r+\theta)}\right|^2=\frac{1}{\pi^4}\sum_{r\in\mathbb{Z}}\sum_{p,j=1}^{k}\left|\sum_{\substack{h\in\mathbb{Z}\\ h\neq 0}}\sum_{i=1}^{k}\frac{a_{n-r-h,pi}a_{h,ij}}{h(2r+h+\theta)}\right|^2$$
$$\leq\frac{1}{\pi^4}\sum_{r\in\mathbb{Z}}\sum_{p,j=1}^{k}\left(\sum_{\substack{h\in\mathbb{Z}\\ h\neq 0}}\sum_{i=1}^{k}\frac{|a_{n-r-h,pi}|^2}{h^2}\right)\left(\sum_{\substack{h\in\mathbb{Z}\\ h\neq 0}}\sum_{i=1}^{k}\frac{|a_{h,ij}|^2}{(2r+h+\theta)^2}\right)$$
$$\leq\frac{1}{\pi^4}\sum_{r\in\mathbb{Z}}\sum_{p,j=1}^{k}\left(\sum_{\substack{h\in\mathbb{Z}\\ h\neq 0}}\sum_{i=1}^{k}\frac{|a_{n-r-h,pi}|^2}{h^2}\right)\left(\sum_{\substack{s\in\mathbb{Z}\\ s\neq 2r}}\sum_{i=1}^{k}\frac{|a_{s-2r,ij}|^2}{(s+\theta)^2}\right)$$
$$\leq\frac{2}{\pi^4}\max_{s\in\mathbb{Z}}\frac{s^2}{(s+\theta)^2}\sum_{r\in\mathbb{Z}}\sum_{p,j=1}^{k}\left(\sum_{\substack{h\in\mathbb{Z}\\ h\neq 0}}\sum_{i=1}^{k}\frac{|a_{n-r-h,pi}|^2}{h^2}\right)\left(\sum_{\substack{s\in\mathbb{Z}\\ s\neq 0}}\sum_{i=1}^{k}\frac{|a_{s-2r,ij}|^2}{s^2}\right)$$
$$\leq\frac{8}{\pi^4(\theta-1)^2(2-\theta)^2}\sum_{r\in\mathbb{Z}}\sum_{p,j=1}^{k}\left(\sum_{\substack{h\in\mathbb{Z}\\ h\neq 0}}\sum_{i=1}^{k}\frac{|a_{n-r-h,pi}|^2}{h^2}\right)\left(\sum_{\substack{s\in\mathbb{Z}\\ s\neq 0}}\sum_{i=1}^{k}\frac{|a_{s-2r,ij}|^2}{s^2}\right)$$
$$\leq\frac{8}{\pi^4(\theta-1)^2(1-|\theta-1|)^2}\sum_{r\in\mathbb{Z}}\sum_{p,j=1}^{k}\left(\sum_{\substack{h\in\mathbb{Z}\\ h\neq 0}}\sum_{i=1}^{k}\frac{|a_{n-r-h,pi}|^2}{h^2}\right)\left(\sum_{\substack{s\in\mathbb{Z}\\ s\neq 0}}\sum_{i=1}^{k}\frac{|a_{s-2r,ij}|^2}{s^2}\right).$$

Consider the summable sequence $f\in l^1(\mathbb{Z})$, where $f(n)=1/n^2$ if $n\neq 0$ and $f(n)=0$ if $n=0$. Then

$$\sum_{r\in\mathbb{Z}}\sum_{p,j=1}^{k}\left(\sum_{\substack{h\in\mathbb{Z}\\ h\neq 0}}\sum_{i=1}^{k}\frac{a(n-r-h,pi)}{h^2}\right)\left(\sum_{\substack{s\in\mathbb{Z}\\ s\neq 0}}\sum_{i=1}^{k}\frac{a(s-2r,ij)}{s^2}\right)$$
$$=\sum_{r\in\mathbb{Z}}\sum_{p,j=1}^{k}(f*a)(n-r)(f*a)(-2r)=\beta^2(n), \tag{2.8}$$

where $a(n-r-h,pi) = |a_{n-r-h,pi}|^2$, $a(s-2r,ij) = |a_{s-2r,ij}|^2$, and $(\beta(n))$ is a square summable sequence. Taking into account (1.7) and the relations just obtained, we have

$$\|P_n(B\Gamma_q B)\|_* \le \|P_n(B\Gamma_0 B)\|_* \le \|P_n(B_1\Gamma_0 B_1)\|_* + \|P_n(B_1\Gamma_0 B_2)\|_* + \|P_n(B_2\Gamma_0 B_1)\|_* + \|P_n(B_2\Gamma_0 B_2)\|_* \le 4\|P_n(B_1\Gamma_0 B_1)\|_* \le \frac{8\sqrt{2}\beta(n)}{\pi^2|\theta-1|(1-|\theta-1|)}. \tag{2.9}$$

This proves the first inequality in (2.5).

Next, set $r = n$ in inequalities (2.7). Consider the square summable sequence

$$\alpha(n) = \left(\sum_{\substack{|s|<n\\ s\ne 0}}\sum_{p,j=1}^{k}\frac{\tilde{a}(s-n)+\tilde{a}(s+n)}{s^2} + \frac{\|a\|_{L_2}^2}{n^2} + \frac{|a_{-2n,pj}|^2}{\theta^2}\right)^{1/2}, \tag{2.10}$$

where $\tilde{a}(s) = \max\left\{|a_{s,pj}|^2, |a_{-s,pj}|^2\right\}$, $s\in\mathbb{Z}$, $p,j = 1,\ldots,k$.

Using the Hölder inequality, we obtain

$$\begin{aligned}\|(B_1\Gamma_0 B_1)P_n\|_*^2 &\le \frac{1}{\pi^4}\sum_{l\in\mathbb{Z}}\sum_{p,j=1}^{k}\left|\sum_{\substack{s\in\mathbb{Z}\\ s\ne n}}\sum_{i=1}^{k}\frac{a_{l-s,pi}a_{s-n,ij}}{(s-n)(s+n+\theta)}\right|^2 = \frac{1}{\pi^4}\sum_{l\in\mathbb{Z}}\sum_{p,j=1}^{k}\left|\sum_{\substack{h\in Z\\ h\ne 0}}\sum_{i=1}^{k}\frac{a_{l-h-n,pi}a_{h,ij}}{h(h+2n+\theta)}\right|^2\\
&\le \frac{1}{\pi^4}\sum_{l\in\mathbb{Z}}\sum_{p,j=1}^{k}\left(\sum_{\substack{h\in\mathbb{Z}\\ h\ne 0}}\sum_{i=1}^{k}\frac{|a_{l-h-n,pi}|^2}{h^2}\right)\left(\sum_{\substack{h\in\mathbb{Z}\\ h\ne 0}}\sum_{i=1}^{k}\frac{|a_{h,ij}|^2}{(h+2n+\theta)^2}\right)\\
&\le \frac{\|a\|_{L_2}^2}{3\pi^2}\left(\sum_{\substack{s\in\mathbb{Z}\\ s\ne 2n}}\sum_{i,j=1}^{k}\frac{|a_{s-2n,ij}|^2}{(s+\theta)^2}\right) \le \frac{\|a\|_{L_2}^2}{3\pi^2}\max_{s\in\mathbb{Z}}\frac{s^2}{(s+\theta)^2}\left(\sum_{\substack{s\in\mathbb{Z}\\ s\ne 0,2n}}\sum_{i,j=1}^{k}\frac{|a_{s-2n,ij}|^2}{s^2} + \frac{|a_{-2n,ij}|^2}{\theta^2}\right)\\
&\le \frac{4\|a\|_{L_2}^2}{3\pi^2(2-\theta)^2(\theta-1)^2}\left(\sum_{\substack{|s|<2n\\ s\ne 0}}\sum_{i,j=1}^{k}\frac{|a_{s-2n,ij}|^2}{s^2} + \frac{\|a\|_{L_2}^2}{4n^2} + \frac{|a_{-2n,ij}|^2}{\theta^2}\right) \le \frac{4\|a\|_{L_2}^2\alpha^2(2n)}{3\pi^2(1-|\theta-1|)^2(\theta-1)^2}.\end{aligned} \tag{2.11}$$

Reasoning in the same way as in (2.9), we obtain the second bound in (2.5) for the operator $(B\Gamma_q B)P_n$.

It remains to prove bound (2.6). Consider the matrix of the restriction of the operator $B_1\Gamma_0 B_1$ to the subspace $\operatorname{Im}P_n$ in the basis $e_{n,j}$ $(n\in\mathbb{Z},\ j = 1,\ldots,k)$. Its elements are

$$(B_1\Gamma_0 B_1 e_{n,j}, e_{n,p}) = (2n+\theta)^2\sum_{\substack{s\in\mathbb{Z}\\ s\ne n}}\sum_{i=1}^{k}\frac{(2s+\theta)^2 a_{n-s,pi}a_{s-n,ij}}{(2s+\theta)^4-(2n+\theta)^4},\qquad p,j = 1,\ldots,k. \tag{2.12}$$

Set $l = r = n$ in (2.7). Using the same reasoning as in the derivation of (2.11), we obtain

$$\begin{aligned}\|P_n(B_1\Gamma_0 B_1)P_n\|_*^2 &= \sum_{p,j=1}^{k}\left|(B_1\Gamma_0 B_1(L_\theta^0)^{-1/2}e_{n,j}, e_{n,p})\right|^2 \le \frac{1}{\pi^4}\sum_{p,j=1}^{k}\left|\sum_{\substack{s\in\mathbb{Z}\\ s\ne n}}\sum_{i=1}^{k}\frac{a_{n-s,pi}a_{s-n,ij}}{(s-n)(s+n+\theta)}\right|^2\\
&\le \frac{1}{\pi^4(2n+\theta)^2}\sum_{p,j=1}^{k}\left|\sum_{\substack{s\in\mathbb{Z}\\ s\ne n}}\sum_{i=1}^{k}\frac{a_{n-s,pi}a_{s-n,ij}}{s-n} - \sum_{\substack{s\in\mathbb{Z}\\ s\ne n}}\sum_{i=1}^{k}\frac{a_{n-s,pi}a_{s-n,ij}}{s+n+\theta}\right|^2\end{aligned}$$

$$\leq \frac{2}{\pi^4(2n+\theta)^2}\sum_{p,j=1}^{k}\left(\sum_{\substack{s\in\mathbb{Z}\\ s\neq n}}\sum_{i=1}^{k}\frac{|a_{s-n,ij}|^2}{(s-n)^2}\right)\left(\sum_{\substack{s\in\mathbb{Z}\\ s\neq n}}\sum_{i=1}^{k}|a_{n-s,pi}|^2\right)+\frac{8\|a\|_{L_2}^2\alpha^2(2n)}{\pi^4(2n+\theta)^2(2-\theta)^2(\theta-1)^2}$$

$$\leq \frac{2\|a\|_{L_2}^4}{\pi^4(2n+\theta)^2(\theta-1)^2}+\frac{8\|a\|_{L_2}^2\alpha^2(2n)}{\pi^4(2n+\theta)^2(\theta-1)^2(1-|\theta-1|)^2}\leq\frac{C}{(2n+\theta)^2(\theta-1)^2(1-|\theta-1|)^2}.$$

Using bound (2.9) for the operator $P_n(B\Gamma_qB)P_n$, we obtain the desired bound (2.6), which completes the proof.

Now, we formulate the final lemma under the conditions of which the preliminary similarity transformation will be possible.

Lemma 5. *There exists a number* $q\in\mathbb{Z}_+$ *such that the operators* B, J_qB, *and* Γ_qB *satisfy the following conditions*: (a) $\Gamma_qB\in\operatorname{End}\mathcal{H}$, $\|\Gamma_qB\|_* < 1$; (b) $(\Gamma_qB)D(L_\theta^0)\subset D(L_\theta^0)$; (c) $B\Gamma_qB$, $(\Gamma_qB)J_qB\in\mathfrak{U}$; (d) $L_\theta^0(\Gamma_qB)x-(\Gamma_qB)L_\theta^0x=(B-J_qB)x$, $x\in D(L_\theta^0)$; (e) *for every* $\varepsilon>0$, *there exists a* $\lambda_\varepsilon\in\rho(L_\theta^0)$ *such that* $\|B(L_\theta^0-\lambda_\varepsilon I)^{-1}\|_* < 1$.

Proof. By Lemma 2 the operator Γ_qB belongs to $\mathfrak{U}\subset\mathfrak{S}_2(\mathcal{H})\subset\operatorname{End}\mathcal{H}$, and there exists a $q\in\mathbb{Z}_+$ such that $\|\Gamma_qB\|_*\leq 1/2<1$. Therefore, property (a) holds.

Let us prove property (c). The first part of the lemma holds due to Lemma 4. By Lemma 3, J_qB is a bounded operator. Then, using Lemma 2, we conclude that the operator $(\Gamma_qB)J_qB$ belongs to $\mathfrak{U}$.

Properties (b), (d), and (e) are proved as in [26, Lemma 4]. This completes the proof of Lemma 5.

Based on this lemma and [17, Theorem 9], we can formulate the first similarity theorem.

Theorem 11. *Let there exist a number* $q\in\mathbb{Z}_+$ *such that* $\|\Gamma_qB\|_*\leq 1/2$. *Then, the operator* $L_\theta=L_\theta^0-B$ *is similar to the operator* $L_\theta^0-J_qB-\widetilde{B}$, *and it holds that*

$$(L_\theta^0-B)(I+\Gamma_qB)=(I+\Gamma_qB)(L_\theta^0-J_qB-\tilde{B}),$$

where the operator $\tilde{B}$ *belongs to the space* $\mathfrak{U}$ *and can be represented as*

$$\tilde{B}=(I+\Gamma_qB)^{-1}(B\Gamma_qB-(\Gamma_qB)J_qB). \tag{2.13}$$

In this theorem, the form of $\tilde{B}$ can be described in more detail. From formula (2.13), we conclude that

$$\begin{aligned}\tilde{B}&=\left(\sum_{l=0}^{\infty}(-1)^l(\Gamma_qB)^l\right)(B\Gamma_qB-(\Gamma_qB)J_qB)=B\Gamma_qB-(\Gamma_qB)J_qB\\ &-(\Gamma_qB)\left(\sum_{l=0}^{\infty}(-1)^l(\Gamma_qB)^l\right)(B\Gamma_qB-(\Gamma_qB)J_qB)=B\Gamma_qB-(\Gamma_qB)J_qB\\ &-(\Gamma_qB)(I+\Gamma_qB)^{-1}(B\Gamma_qB-(\Gamma_qB)J_qB).\end{aligned} \tag{2.14}$$

The last equality will also be used below.

Theorem 11 reduces the investigation of the operator L_θ to the investigation of $L_\theta^0-J_qB-\tilde{B}$, where $\tilde{B}$ is in the space $\mathfrak{U}$. Therefore, for this operator we may apply the main scheme described in Section 1. In particular, we are now able to use Theorem 10. Note that below we will use $\tilde{L}_\theta^0=L_\theta^0-J_qB$ as an unperturbed operator; this operator is not self-adjoint; however, it is normal (see [31, Chapter 1, § 6]).

We now formulate the main similarity theorem, which immediately follows from Theorems 8 and 11. In essence, this theorem reduces the analysis of the operator L_θ to the analysis of a block diagonal operator.

Theorem 12. *Let there exist a number* $m\in\mathbb{Z}_+$ *such that* $m\geq q+1$ *and* $\|\tilde{B}\|_*<\pi^2(2m-1+\theta)$. *Then, the operator* L_θ *is similar to the operator* $\tilde{L}_\theta^0-J_mX_*$, *where* X_* *is a solution to the nonlinear equation*

$$X=\tilde{B}+\tilde{B}\Gamma_mX-(\Gamma_mX)(J_m\tilde{B})-(\Gamma_mX)J_m(\tilde{B}\Gamma_mX), \tag{2.15}$$

and the operator $\tilde{B}$ *is determined by formula* (2.14).

It has been mentioned above that we can use the scheme described in Section 1 for the further investigation; in particular, we can use Theorem 10. We now obtain the bounds used in this theorem. Let us apply the projections P_n to equality (2.14) on the left and on the right. As before, we will take into account (1.8). Then

$$\tilde{B}P_n - P_n\tilde{B}P_n = (B\Gamma_m B)P_n - (\Gamma_m B)P_nBP_n - P_n(B\Gamma_m B)P_n$$
$$+ (P_n(\Gamma_m B) - \Gamma_m B)(I + \Gamma_m B)^{-1}((B\Gamma_m B)P_n - (\Gamma_m B)P_nBP_n),$$
$$P_n\tilde{B} - P_n\tilde{B}P_n = P_n(B\Gamma_m B) - P_n(\Gamma_m B)J_mB - P_n(B\Gamma_m B)P_n$$
$$+ P_n(\Gamma_m B)(I + \Gamma_m B)^{-1}(-B\Gamma_m B - (\Gamma_m B)P_nBP_n + (\Gamma_m B)J_mB + (B\Gamma_m B)P_n).$$

Next, we apply the operator $(L_\theta^0)^{-1/2}$ to both sides of these equalities on the right and estimate them. We have the inequalities

$$\begin{aligned}\|\tilde{B}P_n - P_n\tilde{B}P_n\|_* \le \|(B\Gamma_m B)P_n\|_* + \|(\Gamma_m B)P_n\|_* \|P_nBP_n\|_* + \|P_n(B\Gamma_m B)P_n\|_* \\ + \frac{\|P_n(\Gamma_m B)\|_* + \|\Gamma_m B\|_*}{1 - \|\Gamma_m B\|_*}\left(\|(B\Gamma_m B)P_n\|_* + \|(\Gamma_m B)P_n\|_* \|P_nBP_n\|_*\right),\end{aligned} \tag{2.16}$$

$$\begin{aligned}\|P_n\tilde{B} - P_n\tilde{B}P_n\|_* \le \|P_n(B\Gamma_m B)\|_* + \|P_n(\Gamma_m B)J_mB\|_* + \|P_n(B\Gamma_m B)P_n\|_* \\ + \frac{\|P_n(\Gamma_m B)\|_*}{1 - \|\Gamma_m B\|_*}(\|B\Gamma_m B\|_* + \|(\Gamma_m B)P_nBP_n\|_* + \|(\Gamma_m B)J_mB\|_* + \|(B\Gamma_m B)P_n\|_*).\end{aligned} \tag{2.17}$$

Note that Theorem 11 implies that $\|\Gamma_m B\|_* \le 1/2$, and formula (1.6) implies the bound $\|J_mB\|_* \le \|B\|_*$. Taking these facts into account, we substitute bounds (2.2), (2.3), (2.5), and (2.6) into (2.16) and (2.17). Then

$$\|\tilde{B}P_n - P_n\tilde{B}P_n\|_* \le \frac{C\alpha(2n)}{|\theta - 1|(1 - |\theta - 1|)}, \quad \|P_n\tilde{B} - P_n\tilde{B}P_n\|_* \le \frac{C\beta(n)}{|\theta - 1|(1 - |\theta - 1|)}, \tag{2.18}$$

where α and β are determined by (2.8) and (2.10).

3. PROOFS OF THE MAIN RESULTS FOR $\theta \in (0, 2)$, $\theta \ne 1$

In this section, we prove the main results of the paper announced in the Introduction. The proofs are based on Theorem 12.

Proof of Theorem 1. We assume that the matrix $\mathfrak{A}_0$ is similar to a diagonal matrix. Thus, without loss of generality, we assume that $\mathfrak{A}_0$ is diagonal with the eigenvalues $\mu_1, \mu_2, \ldots, \mu_k$. As the basis, it is convenient to use the normalized eigenvectors $f_1, f_2, \ldots, f_k$.

By Theorem 12, the operator L_θ is similar to the operator $\tilde{L}_\theta^0 - J_mX_*$, where X_* is a solution to Eq. (2.15). Then, Theorem 9 implies that the spectrum of L_θ can be represented as

$$\sigma(L_\theta) = \sigma(\tilde{L}_\theta^0 - J_mX_*) = \sigma(A_{(m)}) \cup \left(\bigcup\nolimits_{|n|\ge m+1} \sigma(A_n)\right) = \sigma_{(m)} \cup \left(\bigcup\nolimits_{|n|\ge m+1} \sigma_n\right),$$

where $A_{(m)}$ is the restriction of $\tilde{L}_\theta^0 - J_mX_*$ to the space $\operatorname{Im} P_{(m)}$, $P_{(m)} = \sum_{|j|\le m} P_j$, and A_n is the restriction of $\tilde{L}_\theta^0 - J_mX_*$ to the space $\operatorname{Im} P_n$, $|n| \ge m + 1$. Since $\operatorname{Im} P_{(m)}$ is a finite dimensional space, $\sigma(A_{(m)})$ is a finite set. Therefore, we have representation (0.2).

Let us write the operator $\tilde{L}_\theta^0 - J_mX_*$ in the form

$$\begin{aligned}\tilde{L}_\theta^0 - J_mX_* = L_\theta^0 - J_mB - J_m(X_* - \tilde{B} + \tilde{B}) \\ = L_\theta^0 - J_mB - J_m\tilde{B} - J_m(X_* - \tilde{B}) = L_\theta^0 - JB - J\tilde{B} + J(X_* - \tilde{B}) + T.\end{aligned} \tag{3.1}$$

The operator $J(X_* - \tilde{B})$ belongs to the space $\mathfrak{U}$ because X_* and $\tilde{B}$ belong to $\mathfrak{U}$. The operator $T = JB - J_mB + J\tilde{B} - J_m\tilde{B} + J(X_* - \tilde{B}) - J_m(X_* - \tilde{B})$ has a finite rank and, therefore, belongs to $\mathfrak{U}$.

Using this representation, we can calculate the asymptotics of the arithmetic mean of the eigenvalues of the operator $\tilde{L}_\theta^0 - J_m X_*$.

In the finite-dimensional space, the spectral trace coincides with the matrix trace; therefore, the arithmetic mean of the eigenvalues of the operator $\tilde{L}_\theta^0 = L_\theta^0 - JB$ is $\pi^4(2n+\theta)^4 + (\pi^2(2n+\theta)^2/k)\sum_{j=1}^k \mu_j$. Therefore, bounds (1.15), (2.6), (2.18), and the equality $P_n(\tilde{B}\Gamma_m X_*)P_n = P_n(X_* - \tilde{B})P_n$ (see (1.16)) imply (0.3), which completes the proof of the theorem.

Proof of Theorem 2. Assume that the eigenvalues μ_j $(j = 1, \ldots, k)$ of the matrix $\mathfrak{A}_0$ are simple. In this case, we may speak about the asymptotics of the eigenvalues of L_θ. By repeating the proof of Theorem 1, we obtain the assertion of Theorem 2.

The form of the remainder term obtained in Theorems 1 and 2 is an improvement of the result obtained in [11, Theorems 1 and 2].

Proof of Theorem 3. Let the matrix coefficients $\mathfrak{A}$ and $\mathfrak{B}$ have size 1×1. The elements of these matrices will be denoted by a and b, where $a, b \in L_2[0,1]$. In this case, the projections P_n $(|n| \geq m+1)$ have rank one; therefore, the operators $P_n X_* P_n$ can be represented as $P_n X_* P_n = (X_* e_{n,1}, e_{n,1})P_n$ $(|n| \geq m+1)$. By Theorem 12, the operator L_θ is similar to $\tilde{L}_\theta^0 - J_m X_*$, where X_* is a solution to Eq. (2.15). In this case, the spectrum $\sigma(L_\theta)$ of L_θ can be written as

$$\sigma(L_\theta) = \sigma(\tilde{L}_\theta^0 - J_m X_*) = \sigma_{(m)} \cup \left(\bigcup_{|n|\geq m+1} \{\sigma(\tilde{L}_\theta^0) - (X_* e_{n,1}, e_{n,1})\} \right),$$

where $\sigma_{(m)}$ is the spectrum of the restriction of $\tilde{L}_\theta^0 - J_m X_*$ to the subspace $\operatorname{Im} P_{(m)}$ and $\sigma(\tilde{L}_\theta^0)$ is the spectrum of $\tilde{L}_\theta^0 = L_\theta^0 - J_m B$. Let us calculate the eigenvalues $\tilde{\lambda}_{n,1}$ $(|n| \geq m+1)$ of the operator $\tilde{L}_\theta^0$. We have

$$\begin{aligned}\breve{\lambda}_{n,1} &= \pi^4(2n+\theta)^4 - (Be_{n,1}, e_{n,1}) = \pi^4(2n+\theta)^4 - (B_1 e_{n,1}, e_{n,1}) - (B_2 e_{n,1}, e_{n,1}) \\ &= \pi^4(2n+\theta)^4 + \pi^2(2n+\theta)^2 \int_0^1 a(t)dt - \int_0^1 b(t)dt = \pi^4(2n+\theta)^4 + \pi^2(2n+\theta)^2 a_0 - b_0.\end{aligned}$$

Next, using (2.14) and Theorem 10, we represent $(X_* e_{n,1}, e_{n,1})$ as

$$(X_* e_{n,1}, e_{n,1}) = (\tilde{B}e_{n,1}, e_{n,1}) + ((X_* - \tilde{B})e_{n,1}, e_{n,1}) = (B\Gamma_m B e_{n,1}, e_{n,1}) + \pi^2(2n+\theta)^2 \eta(n).$$

Since the major contribution to the eigenvalue asymptotics is made by the operator B_1, it is sufficient to consider the last equality for B_1 (the asymptotic terms for the other summands will be included in the remainder term). Taking into account (2.12), we obtain

$$(B_1 \Gamma_m B_1 e_{n,1}, e_{n,1}) = (2n+\theta)^2 \sum_{\substack{s\in\mathbb{Z}\\ s\neq n}} \frac{(2s+\theta)^2 a_{n-s} a_{s-n}}{(2s+\theta)^4 - (2n+\theta)^4}, \quad |n| \geq m+1. \tag{3.2}$$

We now estimate the quantity η. Let us use Theorem 10. Formula (1.15) in Theorem 10 and inequalities (2.18) imply that

$$\begin{aligned}|\eta(n)| &\leq \frac{1}{2\pi^2 \big|2|n|-1-\theta\big|} \|P_n\tilde{B} - P_n\tilde{B}P_n\|_* \|\tilde{B}P_n - P_n\tilde{B}P_n\|_* \\ &\leq \frac{C\alpha(2n)\beta(n)}{|n|(\theta-1)^2(1-|\theta-1|)^2} = \frac{C\gamma_n}{|n|(\theta-1)^2(1-|\theta-1|)^2}, \quad |n| \geq n_1,\end{aligned}$$

where $n_1 = \max\{3\|B\|_*/(4\pi^2) + (1+\theta)/2\}$ and (γ_n) is a summable sequence (because it is the product of two square summable sequences).

Inequality (2.6) implies that (3.2) is a second-order approximation in the eigenvalue asymptotics, and it is not included in the remainder term. Thus, we have bound (0.4), which proves the theorem.

Proof of Theorem 4. Suppose that, under the conditions of the preceding theorem, a and b are functions of bounded variation. Then, the Fourier coefficients of a and b satisfy the inequalities (see [32, Chapter 2, Theorem 4.12])

$$|a_n| \le C/(|n|+1), \quad |b_n| \le C/(|n|+1), \quad n \in \mathbb{Z}.$$

Formulas (2.8) and (2.10) imply that $\alpha(2n) \le C/(|n|+1)$ and $\beta(n) \le C/(|n|+1)$ for $n \in \mathbb{Z}$. Using Theorem 3, we obtain the desired result, which completes the proof.

It is clear that the assertion of Theorem 4 holds also in the case when the functions a and b are smooth (with any degree of smoothness).

4. ONE-DIMENSIONAL PERIODIC CASE

In this section, we investigate the case $\theta \in \{0,1\}$. Since the investigation of the multi-dimensional case is very difficult, we consider only the one-dimensional case. We will use the abstract scheme described in Section 1 with some modifications. First, we describe the situation under examination. To simplify the presentation, we will use the same notation for the eigenvalues, eigenfunctions, and projections as above.

Up to the end of this paper, $\mathfrak{A}$ and $\mathfrak{B}$ are matrices of size 1×1 with the elements a and b from the space $L_2[0,1]$. Consider the differential operator $L_\theta : D(L_\theta) \subset L_2[0,1] \to L_2[0,1]$ defined by the differential expression

$$l(y) = y^{IV} - a(t)y'' - b(t)y.$$

The domain $D(L_\theta) = \{y \in W_2^4([0,1],\mathbb{C})\}$ of this operator is determined either by periodic boundary conditions (at $\theta = 0$) or by antiperiodic boundary conditions (at $\theta = 1$). As before, the operator L_θ can be represented as $L_\theta = L_\theta^0 - B$. The spectrum of L_θ^0 is discrete, and its eigenvalues are double (except for the simple eigenvalue $\lambda_0 = 0$) and have the form $\lambda_n = \pi^4(2n+\theta)^4$, $n \in \mathbb{Z}_+ = \mathbb{N}\cup\{0\}$. The corresponding eigenfunctions are $e_n(t) = e^{i\pi(2n+\theta)t}$, $n \in \mathbb{Z}$, $t \in [0,1]$, which form an orthonormal basis in $L_2[0,1]$. Denote by $\mathbb{P}_n$ the Riesz projection, which is defined for any vector $x \in L_2[0,1]$ by

$$\mathbb{P}_n x = P_{-n-\theta}x + P_n x = (x, e_{-n-\theta})e_{-n-\theta} + (x, e_n)e_n, \quad n \in \mathbb{Z}_+.$$

The basic scheme of the investigation in the one-dimensional case coincides with the scheme in Sections 1 and 2. Some general constructions were made in [26]. Here we briefly comment on the study in [26].

As the admissible triple, we use the triple $(\mathfrak{U}, J_m, \Gamma_m)$ constructed in Section 1. In this case, the transformers J_0 and Γ_0 are

$$J_0 X = \sum_{n=0}^{\infty} \mathbb{P}_n X \mathbb{P}_n, \quad \Gamma_0 X = \sum_{\substack{s,r\in\mathbb{Z} \\ \lambda_s\ne\lambda_r}} \frac{P_s X P_r}{\lambda_s - \lambda_r}.$$

As before, for a certain $m \in \mathbb{Z}_+$, the transformers J_m and Γ_m are determined by formulas (1.6) and (1.7). Let us formulate the following lemma (see [26, Lemma 4]).

Lemma 6. *The triple* $(\mathfrak{U}, J_m, \Gamma_m)$ *for the operator* L_θ $(\theta \in \{0,1\})$ *is admissible.*

Next, we should perform the preliminary similarity transform for the operator L_θ; i.e., we should obtain an analog of Theorem 11. This was done in [26, Theorem 6]. After that, Theorem 7 can be used. Thus, we have the following result.

Theorem 13. *There exists a number* $m \in \mathbb{Z}_+$ *for which* $\|\Gamma_m B\|_* \le 1/2$ *and the operator* L_θ *is similar to the operator* $L_\theta^0 - J_m B - J_m X_*$, *where* X_* *is a solution of the nonlinear equation* (2.15). *In this equation, the operator* $\tilde{B}$ *belongs to the space* $\mathfrak{U}$ *and is determined by formula* (2.14). *Moreover, for some square summable sequences* $(\tilde{\alpha}(2n+\theta))$ *and* $(\tilde{\beta}(n))$, *the following bounds hold*:

$$\|\tilde{B}\mathbb{P}_n - \mathbb{P}_n\tilde{B}\mathbb{P}_n\|_* \le C\tilde{\alpha}(2n+\theta), \quad \|\mathbb{P}_n\tilde{B} - \mathbb{P}_n\tilde{B}\mathbb{P}_n\|_* \le C\tilde{\beta}(n). \tag{4.1}$$

Proof. The first part of this theorem was proved in [26, Theorem 7]. Bounds (4.1) are proved by analogy with inequalities (2.18).

Remark 5. The sequences in the preceding theorem have a specific form. The sequence $(\tilde{\beta}(n))$ is in fact a special case of formula (2.8) for $p = j = 1$, and the sequence $\tilde{\alpha}(n)$ is determined by

$$\tilde{\alpha}(n) = \left(\sum_{\substack{|s|<n \\ s\neq 0}} \frac{\hat{a}(s-n) + \hat{a}(s+n)}{s^2} + \frac{\|a\|_{L_2}^2}{n^2} \right)^{1/2},$$

where $\hat{a}(s) = \max\left\{|a_s|^2, |a_{-s}|^2\right\}$, $s \in \mathbb{Z}$.

Now we have all necessary constructs and bounds, and we proceed to the proof of the main theorem of this section.

Proof of Theorem 5. By Theorem 13, the operator L_θ $(\theta \in \{0,1\})$ is similar to the operator $L_\theta^0 - J_m B - J_m X_*$ with a certain $m \in \mathbb{Z}_+$. Since $L_\theta^0 - J_m B$ is a normal operator with a discrete spectrum and $J_m X_*$ is a bounded operator, $L_\theta^0 - J_m B - J_m X_*$ also has a discrete spectrum. Therefore, the similar operator L_θ also has a discrete spectrum. Furthermore, Theorem 9 implies that

$$\sigma(L_\theta) = \sigma(L_\theta^0 - J_m B - J_m X_*) = \sigma(A_{(m)}) \cup \left(\bigcup_{n\geq m+1} \sigma(A_n) \right),$$

where A_n is the restriction of $L_\theta^0 - J_m B - J_m X_*$ to the subspace $\operatorname{Im}\mathbb{P}_n$ and $A_{(m)}$ is the restriction of $L_\theta^0 - J_m B - J_m X_*$ to the subspace $\operatorname{Im}\mathbb{P}_{(m)}$. Here $\mathbb{P}_{(m)} = \sum_{j\leq m} \mathbb{P}_j$.

To write the asymptotics of the eigenvalues of L_θ, we should describe the sets $\sigma(A_n)$ for $n \geq m+1$. By analogy with the reasoning used in the proof of Theorem 1, we conclude that the operator $L_\theta^0 - J_m B - J_m X_*$ can be represented in form (3.1). Consider the restrictions of the operators on the right-hand side of representation (3.1) to the space $\operatorname{Im}\mathbb{P}_n$, $n \geq m+1$. Then, the operators A_n can be represented as

$$A_n = (L_\theta^0)_n - B_n - C_n - D_n, \quad n \geq m+1,$$

where B_n, C_n, and D_n are the restrictions of the operators $\mathbb{P}_n B \mathbb{P}_n$, $\mathbb{P}_n \tilde{B} \mathbb{P}_n$, and $\mathbb{P}_n(X_* - \tilde{B})\mathbb{P}_n$ to the space $\operatorname{Im}\mathbb{P}_n$, respectively. The matrices of these operators satisfy the equality

$$\mathcal{A}_n = \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_n \end{pmatrix} - \mathcal{B}_n - \mathcal{C}_n + \pi^2(2n+\theta)^2 \mathcal{D}_n, \tag{4.2}$$

where

$$\mathcal{B}_n = -\pi^2(2n+\theta)^2 \begin{pmatrix} a_0 & a_{-2n-\theta} \\ a_{2n+\theta} & a_0 \end{pmatrix}, \quad \mathcal{C}_n = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

and $\mathcal{D}_n$ is the matrix of the operator D_n. By c_{ij} $(i, j = 1, 2)$ we denote the elements of the matrix $\mathcal{C}_n$ $(n \geq m+1)$, and for $s \neq n$ and $s \neq -n-\theta$ we have

$$c_{11} = (B_1 \Gamma_m B_1 e_{-n-\theta}, e_{-n-\theta}) = (2n+\theta)^2 \sum_{s\in\mathbb{Z}} \frac{(2s+\theta)^2 (a_{n-s} a_{s-n} + a_{n+s+\theta} a_{-n-s-\theta})}{(2s+\theta)^4 - (2n+\theta)^4}, \tag{4.3}$$

$$c_{12} = (B_1 \Gamma_m B_1 e_n, e_{-n-\theta}) = 2(2n+\theta)^2 \sum_{s\in\mathbb{Z}} \frac{(2s+\theta)^2 a_{s-n} a_{-n-s-\theta}}{(2s+\theta)^4 - (2n+\theta)^4}, \tag{4.4}$$

$$c_{21} = (B_1 \Gamma_m B_1 e_{-n-\theta}, e_n) = 2(2n+\theta)^2 \sum_{s\in\mathbb{Z}} \frac{(2s+\theta)^2 a_{n-s} a_{n+s+\theta}}{(2s+\theta)^4 - (2n+\theta)^4}, \tag{4.5}$$

$$c_{22} = (B_1 \Gamma_m B_1 e_n, e_n) = (2n+\theta)^2 \sum_{s\in\mathbb{Z}} \frac{(2s+\theta)^2 (a_{n-s} a_{s-n} + a_{n+s+\theta} a_{-n-s-\theta})}{(2s+\theta)^4 - (2n+\theta)^4}. \tag{4.6}$$

It is clear that $c_{11} = c_{22}$.

Below, we will need the following relations. For the sequences of complex numbers $\mathfrak{a}_n$ and $\mathfrak{b}_n$ $(n \geq 1)$ with $\mathfrak{a}_n\mathfrak{b}_n \neq 0$, define the matrices

$$U_n^{-1} = \begin{pmatrix} 1 & 1 \\ -\sqrt{\mathfrak{b}_n/\mathfrak{a}_n} & \sqrt{\mathfrak{b}_n/\mathfrak{a}_n} \end{pmatrix}, \quad U_n = \begin{pmatrix} 1/2 & -\sqrt{\mathfrak{a}_n}/(2\sqrt{\mathfrak{b}_n}) \\ 1/2 & \sqrt{\mathfrak{a}_n}/(2\sqrt{\mathfrak{b}_n}) \end{pmatrix}.$$

Note that

$$U_n\begin{pmatrix} 0 & \mathfrak{a}_n \\ \mathfrak{b}_n & 0 \end{pmatrix}U_n^{-1} = \begin{pmatrix} -\sqrt{\mathfrak{a}_n\mathfrak{b}_n} & 0 \\ 0 & \sqrt{\mathfrak{a}_n\mathfrak{b}_n} \end{pmatrix}, \quad n \geq 1.$$

Now we apply these relations in the case under examination. Set $\mathfrak{a}_n = \pi^2(2n+\theta)^2 a_{-2n-\theta} - c_{12}$ and $\mathfrak{b}_n = \pi^2(2n+\theta)^2 a_{2n+\theta} - c_{21}$. Multiply both sides of (4.2) by U_n on the left and by U_n^{-1} on the right to obtain

$$\begin{aligned} \mathcal{A}_n = \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_n \end{pmatrix} + \begin{pmatrix} \pi^2(2n+\theta)^2 a_0 - c_{11} & 0 \\ 0 & \pi^2(2n+\theta)^2 a_0 - c_{11} \end{pmatrix} \\ + \begin{pmatrix} -\sqrt{\mathfrak{a}_n\mathfrak{b}_n} & 0 \\ 0 & \sqrt{\mathfrak{a}_n\mathfrak{b}_n} \end{pmatrix} + \pi^2(2n+\theta)^2 U_n\mathcal{D}_n U_n^{-1}. \end{aligned} \tag{4.7}$$

Let us now estimate the last term in (4.7). We will use Theorem 10 for the case $\theta \in \{0,1\}$. In this case, straightforward computations show that $d_n = 1/(4\pi^2(2n-1+\theta))$. We will also use the equality

$$P_n(\tilde{B}\Gamma_m X_*)P_n = P_n(X_* - \tilde{B})P_n, \tag{4.8}$$

which immediately follows from (1.16) for the operator $\tilde{B}$. Applying formulas (1.15), (4.1), and (4.8), we obtain for $n \geq n_0$

$$\begin{aligned} \left\|U_n\mathcal{D}_nU_n^{-1}\right\|_* \leq \|U_n\|_* \left\|U_n^{-1}\right\|_* \|\mathcal{D}_n\|_* \leq C\left\|\mathbb{P}_n(X_* - \tilde{B})\mathbb{P}_n\right\|_* \\ \leq \frac{C}{4\pi^2(2n-1+\theta)}\left\|\mathbb{P}_n\tilde{B} - \mathbb{P}_n\tilde{B}P_n\right\|_* \left\|\tilde{B}\mathbb{P}_n - \mathbb{P}_n\tilde{B}\mathbb{P}_n\right\|_* \leq \frac{C\tilde{\alpha}(2n+\theta)\tilde{\beta}(n)}{n} = \frac{C\eta_n}{n}, \end{aligned} \tag{4.9}$$

where $n_0 = \max\{m+1, (3\|B\|_* + 2\pi^2(1-\theta))/(4\pi^2)\}$ and (η_n) is the summable sequence (because it is the product of two square summable sequences).

Next, reasoning as in the proof of formula (2.11) in Lemma 4, we obtain $|c_{11}| = |(B_1\Gamma_m B_1 e_{-n-\theta}, e_{-n-\theta})| \leq Cn\|a\|_{L_2}\tilde{\alpha}(2n+\theta)$, $n \geq m+1$. Recall that the sequence $(\tilde{\alpha}(2n+\theta))$ is square summable. The elements c_{12} and c_{21} are estimated similarly. Since the Fourier coefficients a_s $(s \in \mathbb{Z})$ are also square summable, $\mathfrak{a}_n$ and $\mathfrak{b}_n$ have the same order as the elements c_{11}. Therefore, bound (4.9) implies that $c_{11} \pm \sqrt{\mathfrak{a}_n\mathfrak{b}_n}$ can be separated as an individual asymptotic term.

Finally, we conclude from formulas (4.2), (4.7), and (4.9) that

$$\left|\tilde{\lambda}_n^{\pm} - \lambda_n - \pi^2(2n+\theta)^2 a_0 + c_{11} \pm \sqrt{\mathfrak{a}_n\mathfrak{b}_n}\right| \leq Cn\eta_n,$$

where $\mathfrak{a}_n = \pi^2(2n+\theta)^2 a_{-2n-\theta} - c_{12}$ and $\mathfrak{b}_n = \pi^2(2n+\theta)^2 a_{2n+\theta} - c_{21}$. This proves formula (0.5) and completes the proof of Theorem 5.

Corollary 1 immediately follows from Theorem 5.

Theorem 6 is proved similarly to the proof of Theorem 4.

Now, consider the last result devoted to the asymptotic representation of the semigroup of operators with the generator $-L_\theta$, $\theta \in \{0,1\}$. All the required concepts of the theory of semigroups can be found in [33].

Theorem 14. *The operator* $-L_\theta$ $(\theta \in \{0,1\})$ *is sectorial, and it generates an analytic semigropup of operators* $T : \mathbb{R}_+ \to \operatorname{End} L_2[0,1]$. *This semigroup is similar to the semigroup* $\tilde{T}(t) = \tilde{T}_{(m)}(t) \oplus \tilde{T}^{(m)}(t)$ $(t \in \mathbb{R}_+)$

that acts in $L_2[0,1] = \mathcal{H}_{(m)} \oplus \mathcal{H}^{(m)}$, *where* $\mathcal{H}_{(m)} = \operatorname{Im}\mathbb{P}_{(m)}$ *and* $\mathcal{H}^{(m)} = \operatorname{Im}(I - \mathbb{P}_{(m)})$. *Moreover, the semigroup* $\tilde{T}^{(m)} : \mathbb{R}_+ \to \operatorname{End}\mathcal{H}^{(m)}$ *admits the asymptotic representation*

$$\tilde{T}^{(m)}(t)x = \sum_{n\geq m+1} e^{-t(\pi^4(2n+\theta)^4+\pi^2(2n+\theta)^2a_0-c_{11}+(d_1+d_4)n/2)}\left\{\cosh\rho t\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right.$$
$$\left. - \frac{\sinh\rho t}{\rho}\begin{pmatrix}\sqrt{\mathfrak{a}_n\mathfrak{b}_n} + (d_4 - d_1)n/2 & nd_2\\ nd_3 & -\sqrt{\mathfrak{a}_n\mathfrak{b}_n} + (d_1 - d_4)n/2\end{pmatrix}\right\}\mathbb{P}_n x, \quad x \in L_2[0,1], \tag{4.10}$$

where $\mathfrak{a}_n = \pi^2(2n+\theta)^2 a_{-2n-\theta} - c_{12}$, $\mathfrak{b}_n = \pi^2(2n+\theta)^2 a_{2n+\theta} - c_{21}$, *and* $\rho = ((\sqrt{\mathfrak{a}_n\mathfrak{b}_n} + (d_4 - d_1)n/2)^2 + n^2 d_2 d_3)^{1/2}$. *In addition,* c_{ij} $(i, j = 1,2)$ *are determined by formulas* (4.3)–(4.6) *and* d_i $(i = 1,2,3,4)$ *are summable sequences.*

Proof. The first part of this theorem was proved in [26, Theorem 8]. Here we prove the asymptotic representation (4.10). For this purpose, we will use the following formula for the semigroup generated by the 2×2 matrix (see [28, Chapter 1, § 6]):

$$e^{t\begin{pmatrix}a & b\\ c & d\end{pmatrix}} = e^{t(a+d)/2}\left\{\cosh\rho t\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{\sinh\rho t}{\rho}\begin{pmatrix}(a-d)/2 & b\\ c & (d-a)/2\end{pmatrix}\right\},$$

where $\rho = \sqrt{(a-d)^2/4 + bc}$.

Applying this representation to formula (4.7), we obtain the asymptotic representation (4.10). The proof of the theorem is thus completed.

Remark 6. Theorem 14 is also valid in the case $\theta \in (0,2)$, $\theta \neq 1$, and $k = 1$. In this situation, the representation of the semigroup $\tilde{T}^{(m)} : \mathbb{R}_+ \to \operatorname{End}\mathcal{H}^{(m)}$ is

$$T^{(m)}(t)x = \sum_{|n|\geq m+1} e^{-t\tilde{\lambda}_{n,1}} P_n x,$$

where $x \in L_2[0,1]$ and the eigenvalues $\tilde{\lambda}_{n,1}$ satisfy bound (0.4).

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REFERENCES

1. F. R. Gantmakher, *The Theory of Matrices* (Nauka, Moscow, 1988; Chelsea, New York, 1959).
2. G. D. Birkhoff and R. E. Langer, “The boundary-value problems and developments associated with a system of ordinary linear differential equations of the first order,” Proc. Amer. Acad. **58**, 51–128 (1923).
3. M. A. Naimark, *Linear Differential Operators* (Nauka, Moscow, 1969) [in Russian].
4. N. Dunford, “A survey of the theory of spectral operators,” Bull. Amer. Math. Soc. **64** (5), 217–274 (1958).
5. A. A. Shkalikov, “On the basis problem of the eigenfunctions of an ordinary differential operator,” Russ. Math. Surveys. **34 (5)**, 249–250 (1979).
6. A. A. Shkalikov, “Boundary value problems for ordinary differential equations with a spectral parameter in the boundary conditions,” Trudy Petrovskii Seminar, No. 9, 190–229 (1983).
7. A. A. Shkalikov, *On the Riesz Basis Property of the Eigenfunctions and Adjoint Eigenfunctions of Ordinary Differential Equations in the Space of Vector Functions* (Mosc. Gos. Univ., Moscow, 1984) [in Russian].
8. L. M. Luzhina, “Regular spectral problems in spaces of vector functions,” Vestn. Mosk. Gos. Univ., Ser. 1 Mat. Mekh., No. 1, 31–35 (1988).

9. O. A. Veliev, "Non-self-adjoint Sturm–Liouville operators with matrix potentials," Math. Notes. **81**, 440–448 (2007).
10. N. B. Uskova, "On spectral properties of Sturm–Liouville operator with matrix potential," Ufa Math. J. **7** (3), 84–94 (2015).
11. O. A. Veliev, "Uniform convergence of the spectral expansion for a differential operator with periodic matrix coefficients," Bound. Value Probl. **2008**, 628973 (2008).
12. O. A. Veliev, "On the differential operators with periodic matrix coefficients," Abstr. Appl. Anal. Article ID 934905 (2009).
13. O. A. Veliev, "On the basis property of the root functions of differential operators with matrix coefficients," Cent. Eur. J. Math. **9**, 657–672 (2011).
14. F. Seref and O. A. Veliev, "On non-self-adjoint Sturm–Liouville operators in the space of vector functions," Math. Notes. **95**, 180–190 (2014).
15. A. G. Baskakov, "Methods of abstract harmonic analysis in the perturbation theory of linear operators," Sib. Mat. Zh. **24** (1), 21–39 (1983).
16. A. G. Baskakov, *Harmonic Analysis of Linear Operators* (Voronezh. Gos. Univ., Voronezh, 1987) [in Russian].
17. A. G. Baskakov and D. M. Polyakov, "The method of similar operators in the spectral analysis of the Hill operator with nonsmooth potential," Sb. Math. **208** (1), 1–43 (2017).
18. L. Collatz, *Eigenwertaufgaben mit Technischen Anwendungen* (Leipzig, 1963).
19. V. A. Yakubovich and V. M. Starzhinskii, *Linear Differential Equations with Periodic Coeffcients and Their Applications* (Nauka, Moscow, 1972).
20. E. Korotyaev and I. Lobanov, "Schrödinger operators on zigzag nanotubes," Ann. Henri Poincare **8**, 1151–1176 (2007).
21. S. G. Mikhlin, *Variational Methods in Mathematical Physics* (Nauka, Moscow, 1964; Pergamon, Oxford, 1964).
22. A. V. Badanin and B. P. Belinskii,"Osciliations of a liquid in a bounded cavity with a plate on the boundary," USSR Comput. Math. Math. Phys. **33**, 829–835 (1993).
23. A. V. Badanin and E. L. Korotyaev, "Spectral estimates for a periodic fourth-order operator," St. Petersburg Math. J. **22** (5), 703–736 (2011).
24. A. Badanin and E. Korotyaev, "Asymptotics for fourth order operators on the circle," J. Math. Anal. Appl. **417**, 804–818 (2014).
25. D. M. Polyakov, "On spectral properties of fourth-order differential operator with periodic and semiperiodic boundary conditions," Russian Math. (Iz. VUZ) **59**(5), 64–68 (2015).
26. D. M. Polyakov, "Spectral analysis of a fourth-order differential operator with periodic and antiperiodic boundary conditions," St. Petersburg Math. J. **27** (5), 789–811 (2016).
27. O. A. Veliev, "On the nonself-adjoint ordinary differential operators with periodic boundary conditions," Israel J. Math. **176**, 195–207 (2010).
28. A. A. Kirillov, *Elements of the Theory of Representations* (Nauka, Moscow, 1978; Springer, Berlin, 1976).
29. D. M. Polyakov, "On the spectral characteristics of non-self-adjoint fourth-order operators with matrix coefficients," Math. Notes **105**, 630–635 (2019).
30. I. Ts. Gokhberg and M. G. Krein, *Introduction to the Theory of Linear Non-Self-Adjoint Operators in Hilbert Space* (Nauka, Moscow, 1965; American Mathematical Society, Providence, R.I., 1969).
31. T. Kato, *Perturbation Theory for Linear Operators* (Springer, Berlin, 1966).
32. A. Zygmund, *Trigonometric Series* (Cambridge Univ. Press, Cambridge, 1959).
33. K.-J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equation* (Springer, New-York, 2000).

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