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Local Bifurcations in the Cahn–Hilliard and Kuramoto–Sivashinsky Equations and in Their Generalizations

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Abstract—A periodic boundary value problem for a nonlinear evolution equation that takes the form of such well-known equations of mathematical physics as the Cahn–Hilliard, Kuramoto–Sivashinsky, and Kawahara equations for specific values of its coefficients is studied. Three bifurcation problems arising when the stability of the spatially homogeneous equilibrium states changes are studied. The analysis of these problems is based on the method of invariant manifolds, the normal form techniques for dynamic systems with an infinite-dimensional space of initial conditions, and asymptotic methods of analysis. Asymptotic formulas for the bifurcation solutions are found, and stability of these solutions is analyzed. For the Kuramoto–Sivashinsky and Kawahara equations, it is proved that a two-dimensional local attractor exists such that all solutions on it are unstable in Lyapunov's sense.

Keywords: nonlinear boundary value problem, stability, local bifurcations, normal form, asymptotic formulas

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INTRODUCTION

We consider the nonlinear partial differential equation

$$\begin{aligned} u_t + \gamma u_{xxxx} + \beta u_{xx} + \alpha u + \gamma_1 u_x + \gamma_2 u_{xxx} + \gamma_3 u_{xxxxx} \\ + a_2(u^2)_x + b_2(u^2)_{xx} + a_3(u^3)_x + b_3(u^3)_{xx} = 0, \end{aligned} \tag{0.1}$$

where $\beta, \gamma_1, \gamma_2, \gamma_3, a_2, b_2, a_3, b_3 \in R, \gamma,$ and α are nonnegative constants and $u = u(t, x)$.

For various combinations of coefficients, Eq. (0.1) occurs in many branches of mechanics and mathematical physics. For example, in the case $\gamma = \beta = \alpha = \gamma_3 = b_2 = b_3 = a_3 = 0$, we obtain the Korteweg–de Vries equation. If $\gamma > 0$, $\alpha \geq 0$, and $b_2 = a_3 = b_3 = 0$, then Eq. (0.1) is called the Kuramoto–Sivashinsky equation [1–3]. In [2], the corresponding equation was obtained in the analysis of the two-dimensional Navier–Stokes system of equations in Kolmogorov's modification after introducing the stream function and some additional assumptions about the parameters of the problem. In [4–8], the version of Eq. (0.1) with $\alpha = 0$, $\gamma > 0$ $(\gamma = 1)$, and $b_2 = a_3 = b_3 = 0$ was studied. This version is called the Kawahara (or Kawahara–Benney–Lin) equation. The Kawahara equation describes the evolution of long waves in hydrodynamics.

In applications to hydrodynamics, Eq. (0.1) was considered for $\gamma = 1$, $\alpha = 0$, and $\gamma_1 = \gamma_2 = \gamma_3 = 0$. When $a_2 = a_3 = b_2 = 0$ and $b_3 \neq 0$, this version is known as the Cahn–Hilliard equation [9].

In many practical cases, the equation derived in [10] for describing the formation of relief on the surface of plates under the action of a flow of ions and as a result of laser or electrochemical processing (see [11]) can be reduced to Eq. (0.1). A number of mathematical problems for the Bradley–Harper equations and equations derived from it were studied in [12–14].

In many studies mentioned above, Eq. (0.1) is considered subject to periodic boundary conditions (e.g., see [2, 3, 10, 11]). In this paper, we consider Eq. (0.1) subject to boundary conditions

$$u(t, x + 2\pi) = u(t, x), \tag{0.2}$$

and we assume that $\gamma = 1$ (if $\gamma > 0$, then we can make $\gamma = 1$ by the normalization with respect to time).

We supplement the boundary value problem (0.1), (0.2) with the initial condition

$$u(0,x) = f(x). \tag{0.3}$$

Let $f(x) \in H_2^s$ where $s = 5$ if $\gamma_3 \neq 0$ and $s = 4$ if $\gamma_3 = 0$. Here H_2^s denotes the Sobolev space [15] consisting of 2π periodic functions with square integrable derivatives up to the order s on an interval of the period length. For such functions $f(x)$, the results obtained in [16, 17] imply that the mixed problem (0.1), (0.2), (0.3) is locally well resolvable and its solutions in the phase space (the space of initial conditions H_2^s) generate a local semiflow [18]

$$f(x) \to f_t(x) = u(t,x).$$

These remarks provide reasons to assume that the boundary value problem can be analyzed using dynamical systems theory techniques with an infinite-dimensional phase state (e.g., see [18]).

1. LINEARIZED BOUNDARY VALUE PROBLEM

Let us analyze the stability of the trivial equilibrium state of the boundary value problem (0.1), (0.2); for this purpose, we consider the auxiliary linear boundary value problem

$$u_t = Au, \tag{1.1}$$

$$u(t, x + 2\pi) = u(t,x). \tag{1.2}$$

Here, the linear differential operator (LDO) A is defined by

$$Av = -v^{(IV)} - \beta v'' - \alpha v - \gamma_1 v' - \gamma_2 v''' - \gamma_3 v^{(V)},$$

where $v = v(x)$ is a smooth 2π periodic function. The LDO A has a countable set of eigenvalues

$$\lambda_n = \tau_n + i\sigma_n, \quad \tau_n = -n^4 + \beta n^2 - \alpha, \quad \sigma_n = -\gamma_1 n + \gamma_2 n^3 - \gamma_3 n^5, \quad n = 0, \pm 1, \pm 2, \ldots$$

The corresponding eigenfunctions $\exp(inx)$ form a complete orthogonal system of functions in the space $L_2(-\pi,\pi)$. Therefore, the solutions to the linear boundary value problem (1.1), (1.2) are asymptotically stable if $\tau_n < 0$ for all n (in the case under examination, $\lim\limits_{|n|\to\infty} \tau_n = -\infty$), and these solutions are unstable if $\tau_k > 0$ for a certain k. In turn, the zero solution to the nonlinear boundary value problem (0.1), (0.2) is asymptotically (exponentially) stable if $\tau_n < 0$ for all n, and it is unstable if there exists an integer k such that $\tau_k > 0$. For the boundary value problem (0.1), (0.2), there is a critical case if $\tau_n \leq 0$ and $\tau_k = 0$ at certain integer k.

Let us identify all possible critical cases in the stability problem of the zero solution of the boundary value problem (0.1), (0.2). First, let $\alpha > 0$. Then, the inequality $n^4 - \beta n^2 + \alpha > 0$ and the corresponding equality $n^4 - \beta n^2 + \alpha = 0$ at integer n imply that there are two critical cases in the case $\alpha \neq 0$.

The first critical case. There exists a natural m such that $\tau_m = \tau_{-m} = 0$ for $n = \pm m$ and $\tau_n < 0$ for $n \neq \pm m$. This case occurs if $\beta = \beta_1 = m^2 + (m+\delta)^2$, $\alpha = \alpha_1 = m^2(m+\delta)^2$, and $\delta \in (-1,1)$, which follows from Viète's formulas. For such a choice of α and β, the linear boundary value problem (1.1), (1.2) has two linearly independent t periodic solutions

$$q_m = \exp(imx + i\sigma_m t), \quad q_{-m} = \bar{q}_m, \quad \sigma_m = \gamma_2 m^3 - \gamma_1 m - \gamma_3 m^5, \quad \sigma_{-m} = -\sigma_m.$$

The second critical case. There exists a natural m such that $\tau_n = 0$ occurs when $n = \pm m$ and $n = \pm(m+1)$. For other $n \neq \pm m$ and $n \neq \pm(m+1)$, it holds that $\tau_n < 0$. This critical case occurs when $\alpha = \alpha_2 = m^2(m+1)^2$ and $\beta = \beta_2 = m^2 + (m+1)^2$. Then, the boundary value problem (1.1), (1.2) has the t periodic solutions

$$q_m = \exp(imx + i\sigma_m t), \quad q_{-m} = \bar{q}_m, \quad q_{m+1} = \exp(i(m+1)x + i\sigma_{m+1}t), \quad q_{-(m+1)} = \bar{q}_{m+1},$$
$$\sigma_m = \gamma_2 m^3 - \gamma_1 m - \gamma_3 m^5, \quad \sigma_{m+1} = \gamma_2(m+1)^3 - \gamma_1(m+1) - \gamma_3(m+1)^5.$$

A special critical case in the analysis of stability occurs when $\alpha = 0$. Then, the LDO A has a zero eigenvalue ($\lambda_0 = 0$), and the corresponding eigenfunction is $e_0(x) = 1$. For $\beta = \beta_3 = 1$ ($\alpha = \alpha_3 = 0$), the LDO A has the eigenvalues $\lambda_{\pm 1} = \pm i\sigma$, where $\sigma = \gamma_2 - \gamma_1 - \gamma_3$. If $n \neq 0, \pm 1$, then $\tau_n < 0$.

In the following sections of the paper, we analyze the bifurcation problems arising in the cases that are close to the three critical cases listed above. This allows us to find the solutions that branch from the equilibrium state $u = 0$.

2. BIFURCATION PROBLEM IN THE CLOSE TO CRITICAL CASE WITH A SINGLE PAIR OF PURELY IMAGINARY EIGENVALUES

In Eq. (0.1), set $\beta = \beta_1 + \nu_1\varepsilon$ and $\alpha = \alpha_1 - \nu_2\varepsilon$, where $\nu_1, \nu_2 \in R$, $\varepsilon \in (0, \varepsilon_0)$, and $0 < \varepsilon_0 \ll 1$. Rewrite the boundary value problem (0.1), (0.2) in the form

$$u_t = A_1 u + \varepsilon B_1 u + F(u), \tag{2.1}$$

$$u(t, x + 2\pi) = u(t, x). \tag{2.2}$$

Here

$$A_1 u = -u_{xxxx} - \beta_1 u_{xx} - \alpha_1 u - \gamma_1 u_x - \gamma_2 u_{xxx} - \gamma_3 u_{xxxxx}, \quad B_1 u = \nu_2 u - \nu_1 u_{xx},$$

$$F(u) = F_2(u) + F_3(u), \quad F_2(u) = -a_2(u^2)_x - b_2(u^2)_{xx}, \quad F_3(u) = -a_3(u^3)_x - b_3(u^3)_{xx}.$$

The assumptions made in Section 1 imply that the LDO $A(\varepsilon) = A_1 + \varepsilon B_1$ has the eigenvalues

$$\lambda_\pm(\varepsilon) = \tau(\varepsilon) \pm i\sigma(\varepsilon), \quad \sigma(\varepsilon) = \sigma_m, \quad \tau(\varepsilon) = \varepsilon\tau'_m, \quad \tau'_m = \nu_1 m^2 + \nu_2.$$

This pair of eigenvalues is associated with the eigenfunctions $\exp(\pm imx)$. For the other eigenvalues of $A(\varepsilon)$, we have the inequality $\operatorname{Re}\lambda_n(\varepsilon) \leq -\gamma_0 < 0$, where γ_0 is a sufficiently small positive constant independent of ε if $|\varepsilon|$ is sufficiently small.

These properties allow us to conclude that the boundary value problem (2.1), (2.2) satisfies the Andronov–Hopf bifurcation theorem (e.g., see [18]). According to this theorem, the analysis of the dynamics of solutions with sufficiently small in the norm of the phase space can be reduced to the analysis of dynamics of an auxiliary two-dimensional system on a two-dimensional central manifold $M_2(\varepsilon)$ [18–20]. The other solutions with sufficiently small initial conditions $u(0, x) = f(x)$ approach $M_2(\varepsilon)$ at an exponential rate. Such an auxiliary system is called normal form (NF). In complex form, the normal form can be written as a single differential equation for an auxiliary complex-valued function $z = z(t)$

$$\dot{z} = \varepsilon[\tau'_m + (l_m + ig_m)|z|^2]z \tag{2.3}$$

if the first Poincaré–Lyapunov constant $l_m \neq 0$ and $\varepsilon \neq 0$. This is the "principal" part of the NF, and the omitted terms have an order $o(\varepsilon)$.

In applications to partial differential equations, the algorithm used to construct the NF (i.e., the algorithm for calculating the NF coefficients) is important. Below in this section, we describe such an algorithm, which can be interpreted as a modification of the well-known Krylov–Bogolyubov method.

We will seek the solutions of the boundary value problem (2.1), (2.2) lying in the central manifold $M_2(\varepsilon)$ in the form

$$u(t, x, \varepsilon) = \varepsilon^{1/2} u_1(t, x, z, \bar{z}) + \varepsilon u_2(t, x, z, \bar{z}) + \varepsilon^{3/2} u_3(t, x, z, \bar{z}) + O(\varepsilon^2). \tag{2.4}$$

Here $z = z(t)$ is the solution to the differential equation (2.3) and

$$u_1(t, x, z, \bar{z}) = zq_m + \overline{zq_m}, \quad q_m = \exp(imx + i\sigma_m t).$$

The functions u_2 and u_3 smoothly depend on their arguments. At fixed t, z, and $\bar{z}$, these functions considered as the functions of x lie in H_2^s and have period $2\pi/\sigma_m$ in the variable t. Finally,

$$M_k(u_j) = \frac{\sigma_m}{(2\pi)^2} \int_0^{2\pi/\sigma_m} \int_0^{2\pi} u_j q_k \, dx dt = 0, \quad j = 2, 3, \quad k = \pm m, \quad q_{-m} = \bar{q}_m.$$

The class of such functions will be denoted by V.

We can substitute sum (2.4) into the boundary value problem (2.1), (2.2) and equate the coefficients of the terms ε and $\varepsilon^{3/2}$ to obtain inhomogeneous boundary value problems for determining the functions u_2 and u_3. For u_2, we obtain the boundary value problem

$$u_{2t} - A_1u_2 = \Phi_2(t, x, z, \bar{z}), \tag{2.5}$$

$$u_2(t, x + 2\pi) = u_2(t, x), \tag{2.6}$$

where $\Phi_2(t, x, z, \bar{z}) = -a_2(u_1^2)_x - b_2(u_1^2)_{xx}$. The boundary value problem (2.5), (2.6) has a unique solution in the class of functions V, which can be written as

$$u_2(t, x, z, \bar{z}) = \eta_m(zq_m)^2 + \bar{\eta}_m(\bar{z}\bar{q}_m)^2,$$

where

$$\eta_m = \frac{2(2b_2m - ia_2)}{3m(c_m + 2id_mm)}, \quad c_m = 3m^2 - 2m\delta - \delta^2, \quad d_m = 5\gamma_3m^2 - \gamma_2.$$

For u_3, we have the similar inhomogeneous boundary value problem

$$u_{3t} - A_1u_3 = \Phi_3(t, x, z, \bar{z}), \tag{2.7}$$

$$u_3(t, x + 2\pi) = u_3(t, x),$$

$$\Phi_3(t, x, z, \bar{z}) - a_3(u_1^3)_x - b_3(u_1^3)_{xx} - 2a_2(u_1u_2)_x - 2b_2(u_1u_2)_{xx} - B_1u_1 \tag{2.8}$$
$$- (\tau'_m + (l_m + ig_m)|z|^2)zq_m - (\tau'_m + (l_m - ig_m)|z|^2)\bar{z}\bar{q}_m.$$

When writing the right-hand side of Eq. (2.7), we should take into account that the derivative of $z = z(t)$ with respect to t is calculated along the solutions to Eq. (2.3). The solvability conditions of problem (2.7), (2.8) in the class of functions $V(M_m(\Phi_3) = 0)$ imply that

$$l_m = \frac{4[c_m(2b_2^2m^2 - a_2^2) - 6a_2b_2d_mm^2]}{3(c_m^2 + 4m^2d_m^2)} + 3b_3m^2,$$

$$g_m = -3a_3m - \frac{8md_m(2b_2^2m^2 - a_2^2) + 12ma_2b_2c_m}{3(c_m^2 + 4m^2d_m^2)},$$

which once more confirms the supercriticality of $\tau'_m = \nu_1m^2 + \nu_2$.

Lemma 2.1. *The NF* (2.3) *has a periodic solution* P_1

$$z_m(t) = \rho_m\exp(i\varepsilon\omega_mt), \quad \rho_m = \sqrt{-\frac{\tau'_m}{l_m}}, \quad \omega_m = -g_m\frac{\tau'_m}{l_m},$$

which exists if $\tau'_ml_m < 0$. *The solution* P_1 *is stable if* $l_m < 0$ $(\tau'_m > 0)$ *and unstable if* $l_m > 0$ $(\tau'_m < 0)$.

Lemma 2.1 is easy to verify in the standard way. For example, the existence of the exact solution is verified by the straightforward substitution.

Lemma 2.1 and formula (2.4) for the solutions on the invariant (central) manifold $M_2(\varepsilon)$ imply the following result.

Theorem 2.1. *There exists an* $\varepsilon_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0)$, *the periodic solution* P_1 *of the NF* (2.3) *is associated with the family of periodic solutions* $P(\varphi_m)$ *of the boundary value problem* (2.1), (2.2)

$$u_m(t, x, \varepsilon) = \varepsilon^{1/2}\rho_m[\exp(i\varphi_m) + \exp(-i\varphi_m)]$$
$$+ \varepsilon\rho_m^2[\eta_m\exp(2i\varphi_m) + \bar{\eta}_m\exp(-2i\varphi_m)] + O(\varepsilon^{3/2}), \tag{2.9}$$

where $\varphi_m = \varphi_m(t, x) = mx + (\sigma_m + \varepsilon\omega_m)t + \varphi_0$ *and* φ_0 *is an arbitrary real constant. Each solution* (2.9) *is stable if* $l_m < 0$ *and unstable if* $l_m > 0$.

In other terms, the family of periodic solutions in the phase space H_2^s generates the cycle C_m, which is orbitally asymptotically stable if $l_m < 0$ (and unstable if $l_m > 0$).

Note that the solutions $u_m(t,x,\varepsilon)$ have the structure of the travelling wave $u_m(t,x,\varepsilon) = u_m(\Theta_m,\varepsilon)$, where $\Theta_m = mx + (\sigma_m + \varepsilon\omega_m + o(\varepsilon))t$.

From the physical point of view, the case when the cycle C_m is stable is more meaningful because the corresponding solutions are physically realizable. This situation occurs when the version of the Kuramoto–Sivashinsky equation with $b_2 = a_3 = b_3 = \gamma_2 = \gamma_3 = 0$ is considered. In this case,

$$l_m = -\frac{4a_2^2}{3(3m^2 - 2m\delta - \delta^2)} < 0 \tag{2.10}$$

independently of the natural m and $\delta \in (-1,1)$.

If we consider the generalized Cahn–Hilliard equation [9] ($a_2 = b_2 = a_3 = \gamma_1 = \gamma_2 = \gamma_3 = 0$), then $\sigma_m = \omega_m = 0$. In this case, we obtain a family of equilibrium states S_m rather than a family of periodic solutions. This family of equilibrium states generates a one-dimensional invariant set, which is asymptotically stable if $b_3 < 0$ ($l_m < 0$) and unstable if $b_3 > 0$ ($l_m > 0$).

Here we examined the version with $\alpha > 0$. The conventional version of the Cahn–Hilliard equations assumes that $\alpha = 0$. This version should be analyzed separately.

3. THE CASE OF TWO PAIRS OF PURELY IMAGINARY EIGENVALUES THAT IS CLOSE TO THE CRITICAL CASE

In Eq. (0.1), set

$$\alpha = \alpha_2 - \nu_2\varepsilon, \quad \beta = \beta_2 + \nu_1\varepsilon, \quad \nu_1,\nu_2 \in \mathbb{R}, \quad \varepsilon \in (0,\varepsilon_0),$$

where $0 < \varepsilon_0 \ll 1$, and the constants $\alpha_2, \beta_2 > 0$ were chosen in Section 1. Consider the resulting boundary value problem

$$u_t = A_2u + \varepsilon B_2u + F(u), \tag{3.1}$$

$$u(t, x + 2\pi) = u(t,x). \tag{3.2}$$

Here $A_2u = -u_{xxxx} - \beta_2u_{xx} - \alpha_2u - \gamma_1u_x - \gamma_2u_{xxx} - \gamma_3u_{xxxxx}$, and $B_2 = B_1$ (see the definition of B_1 in Section 2). The nonlinear operator $F(u)$ was also defined in Section 2. The LDO $A(\varepsilon) = A_2 + \varepsilon B_2$ has the eigenvalues

$$\lambda_{\pm m}(\varepsilon) = \tau_m(\varepsilon) \pm i\sigma_m(\varepsilon), \quad \lambda_{\pm(m+1)}(\varepsilon) = \tau_{m+1}(\varepsilon) \pm i\sigma_{m+1}(\varepsilon),$$

$$\tau_m(\varepsilon) = \tau'_m\varepsilon, \quad \tau'_m = \nu_1m^2 + \nu_2,$$

$$\tau_{m+1}(\varepsilon) = \tau'_{m+1}\varepsilon, \quad \tau'_{m+1} = \nu_1(m+1)^2 + \nu_2,$$

$$\sigma_m(\varepsilon) = \sigma_m, \quad \sigma_m = \gamma_2m^3 - \gamma_1m - \gamma_3m^5,$$

$$\sigma_{m+1}(\varepsilon) = \sigma_{m+1}, \quad \sigma_{m+1} = \gamma_2(m+1)^3 - \gamma_1(m+1) - \gamma_3(m+1)^5.$$

These eigenvalues are associated with the eigenfunctions $\exp(\pm imx)$ and $\exp(\pm i(m+1)x)$, respectively. For the other n, it holds that $\operatorname{Re}\lambda_n \le -\gamma_0 < 0$, where $n \ne \pm m$, $n \ne \pm(m+1)$, and γ_0 is a positive constant, which is independent of ε.

In this case, the boundary value problem (3.1), (3.2) has a four-dimensional smooth invariant (central) manifold $M_4(\varepsilon)$ [18–20], on which the dynamics of solutions to the boundary value problem is determined by the solutions to the auxiliary system of four ordinary differential equations (or the system of two such equations written in complex form).

In this section, we first consider the situation of generic position. We assume that either $m \ne 1$ or $\gamma_2 \ne 5\gamma_3$ if $m = 1$. Otherwise, in addition to the "resonance" of modes, the resonance of eigenfrequencies 1 : 2 in the linearized at $\varepsilon = 0$ boundary value problem (3.1), (3.2) is realized. The assumption that $m = 1$ and $\gamma_2 = 5\gamma_3$ gives the NF of a different structure.

We will seek the solutions belonging to $M_4(\varepsilon)$ in the form that is similar to sum (2.4):

$$u(t,x,\varepsilon) = \varepsilon^{1/2}u_1(t,x,z_1,\bar{z}_1,z_2,\bar{z}_2) + \varepsilon u_2(t,x,z_1,\bar{z}_1,z_2,\bar{z}_2) + \varepsilon^{3/2}u_3(t,x,z_1,\bar{z}_1,z_2,\bar{z}_2) + O(\varepsilon^2). \tag{3.3}$$

The functions $z_1(t)$ and $z_2(t)$ on the right-hand side of (3.3) are solutions to a system of differential equations that is called the NF. This system is written below. In addition,

$$u_1(t,x,z_1,\bar{z}_1,z_2,\bar{z}_2) = z_1 q_m + \bar{z}_1\bar{q}_m + z_2 q_{m+1} + \bar{z}_2\bar{q}_{m+1},$$
$$q_m = \exp(imx + i\sigma_m t), \quad q_{m+1} = \exp(i(m+1)x + i\sigma_{m+1}t).$$

Finally, the smooth functions $u_j(t,x,z_1,\bar{z}_1,z_2,\bar{z}_2)$ $(j = 2,3)$ considered as functions of x belong to H_2^s; they are trigonometric polynomials of t and satisfy the identities

$$\int_0^{2\pi} u_j \exp(\pm ikx)dx \equiv 0, \quad k = \pm m, \quad k = \pm(m+1).$$

In the nonresonance case (we assume that $\sigma_m : \sigma_{m+1} \neq 1;2;1/2;3;1/3$), the principal part of the NF is

$$\begin{aligned} z_1' &= \varepsilon z_1[\tau_m' + (l_{11} + ig_{11})|z_1|^2 + (l_{12} + ig_{12})|z_2|^2], \\ z_2' &= \varepsilon z_2[\tau_{m+1}' + (l_{21} + ig_{21})|z_1|^2 + (l_{22} + ig_{22})|z_2|^2]. \end{aligned} \tag{3.4}$$

Here $\tau_m' = \nu_1 m^2 + \nu_2$, $\tau_{m+1}' = \nu_1(m+1)^2 + \nu_2$. The other coefficients l_{jk}, $g_{jk} \in \mathbb{R}$ are determined when the algorithm for constructing the NF is implemented.

Substitute sum (3.3) into the boundary value problem (3.1), (3.2) and equate the coefficients of the identical powers of $\varepsilon^{1/2}$. As a result, we obtain inhomogeneous boundary value problems for determining the functions u_2 and u_3. When making up these problems, it must be taken into account that the derivatives of the functions $z_1(t)$ and $z_2(t)$ with respect to t are calculated along the solutions to the system of differential equations (3.4). As a result, we obtain the following two problems:

$$u_{2t} - A_2 u_2 = \Phi_2(t,x,z_1,\bar{z}_1,z_2,\bar{z}_2), \tag{3.5}$$

$$u_2(t,x+2\pi) = u_2(t,x), \tag{3.6}$$

$$u_{3t} - A_3 u_3 = \Phi_3(t,x,z_1,\bar{z}_1,z_2,\bar{z}_2), \tag{3.7}$$

$$u_3(t,x+2\pi) = u_3(t,x). \tag{3.8}$$

Here

$$\Phi_2(t,x,z_1,\bar{z}_1,z_2,\bar{z}_2) = -a_2(u_1^2)_x - b_2(u_1^2)_{xx},$$
$$\Phi_3(t,x,z_1,\bar{z}_1,z_2,\bar{z}_2) = -a_3(u_1^3)_x - b_3(u_1^3)_{xx} - 2a_2(u_1u_2)_x - 2b_2(u_1u_2)_{xx}$$
$$+ B_2 u_1 - z_1 q_m\left[\tau_m' + (l_{11} + ig_{11})\left|z_1^2\right| + (l_{12} + ig_{12})\left|z_2^2\right|\right]$$
$$- \bar{z}_1\bar{q}_m\left[\tau_m' + (l_{11} - ig_{11})\left|z_1^2\right| + (l_{12} - ig_{12})\left|z_2^2\right|\right]$$
$$- z_2 q_{m+1}\left[\tau_{m+1}' + (l_{21} + ig_{21})\left|z_1^2\right| + (l_{22} + ig_{22})\left|z_2^2\right|\right]$$
$$- \bar{z}_2\bar{q}_{m+1}\left[\tau_{m+1}' + (l_{21} - ig_{21})\left|z_1^2\right| + (l_{22} - ig_{22})\left|z_2^2\right|\right].$$

The solutions to the boundary value problems (3.5), (3.6) and (3.7), (3.8) should be sought in the form of trigonometric polynomials of the variables t and x that are orthogonal to the functions $\exp(\pm imx)$ and $\exp(\pm i(m+1)x)$ in the sense of the scalar product in the space $L_2(0,2\pi)$ at all t. The corresponding solution to the boundary value problem (3.5), (3.6) has the form

$$\begin{aligned} u_2(t,x,z_1,\bar{z}_1,z_2,\bar{z}_2) &= \eta_1 z_1^2 q_m^2 + \eta_2 z_2^2 q_{m+1}^2 + \eta_3 z_1 z_2 q_m q_{m+1} \\ &+ \eta_4 \bar{z}_1 z_2 \bar{q}_m q_{m+1} + \bar{\eta}_1 \bar{z}_1^2 \bar{q}_m^2 + \bar{\eta}_2 \bar{z}_2^2 \bar{q}_{m+1}^2 + \bar{\eta}_3 \bar{z}_1 \bar{z}_2 \bar{q}_m \bar{q}_{m+1} + \bar{\eta}_4 z_1 \bar{z}_2 q_m \bar{q}_{m+1}. \end{aligned} \tag{3.9}$$

By substituting sum (3.9) into the boundary value problem (3.5), (3.6), we find that

$$\eta_1 = \frac{2(2b_2m - a_2i)}{3m(c_1 + 2imd_1)}, \quad \eta_2 = \frac{2(2b_2(m+1) - a_2i)}{3(m+1)(c_2 + 2i(m+1)d_2)},$$

$$\eta_3 = \frac{2(2m+1)(b_2(2m+1) - ia_2)}{m(c_3 + id_3)}, \quad \eta_4 = \frac{2(b_2 - ia_2)}{m(c_4 + id_4)}, \quad c_1 = 3m^2 - 2m - 1,$$

$$c_2 = 3m^2 + 8m + 4, \quad c_3 = (m+1)(9m^2 + 9m + 2),$$

$$c_4 = (m^2 - 1)(m + 2), \quad d_1 = 5m^2\gamma_3 - \gamma_2, \quad d_2 = 5(m+1)^2\gamma_3 - \gamma_2,$$

$$d_3 = 5\gamma_3(6m^4 + 15m^3 + 14m^2 + 6m + 1) - 3\gamma_2(2m^2 + 3m + 1),$$

$$d_4 = 3\gamma_2(m+1) - 5\gamma_3(m^3 + 2m^2 + 2m + 1).$$

The coefficients of the NF (3.4) are found from the solvability conditions of the boundary value problem (3.7), (3.8) in the class of trigonometric polynomials. To this end, the coefficients on the right-hand side of Eq. (3.7) must vanish for q_m, $\overline{q}_m$, q_{m+1}, and $\overline{q}_{m+1}$. Hence, we find that

$$l_{11} = 3b_3m^2 + \frac{4(c_1(2b_2^2m^2 - a_2^2) - 6a_2b_2d_1m^2)}{3(c_1^2 + 4m^2d_1^2)},$$

$$g_{11} = -3a_3m - \frac{4m(2d_1(2b_2^2m^2 - a_2^2) + 3b_2a_2c_1)}{3(c_1^2 + 4m^2d_1^2)},$$

$$l_{22} = 3b_3(m+1)^2 + \frac{4(c_2(2b_2^2(m+1)^2 - a_2^2) - 6a_2b_2d_2(m+1)^2)}{3(c_2^2 + 4(m+1)^2d_2^2)},$$

$$g_{22} = -3a_3(m+1) - \frac{4(m+1)(2d_2(2b_2^2(m+1)^2 - a_2^2) + 3b_2a_2c_2)}{3(c_2^2 + 4(m+1)^2d_2^2)},$$

$$l_{12} = 6b_3m^2 + \frac{4(2m+1)}{c_3^2 + d_3^2}[c_3(b_2^2m(2m+1) - a_2^2) - (3m+1)a_2b_2d_3] + 4\left[\frac{c_4(b_2^2m + a_2^2) - (m-1)a_2b_2d_4}{c_4^2 + d_4^2}\right],$$

$$g_{12} = -6a_3m - 4(2m+1)\left[\frac{d_3(b_2^2(2m+1)m - a_2^2) + (3m+1)c_3b_2a_2}{c_3^2 + d_3^2}\right]$$
$$+ 4\frac{d_4(b_2^2m + a_2^2) + (m-1)a_2b_2c_4}{c_4^2 + d_4^2},$$

$$l_{21} = 6(m+1)^2b_3 + 4\frac{m+1}{m}\left[\frac{c_4(b_2^2(m+1) - a_2^2) - (m+2)d_4a_2b_2}{c_4^2 + d_4^2}\right]$$
$$+ 4\frac{(m+1)(2m+1)}{m}\left[\frac{c_3(b_2^2(2m+1)(m+1) - a_2^2) - a_2b_2d_3(3m+2)}{c_3^2 + d_3^2}\right],$$

$$g_{21} = -6a_3(m+1) - 4\frac{m+1}{m}\left[\frac{d_4(b_2^2(m+1) - a_2^2) + (m+2)a_2b_2c_4}{c_4^2 + d_4^2}\right]$$
$$- 4\frac{(m+1)(2m+1)}{m}\left[\frac{d_3(b_2^2(2m+1)(m+1) - a_2^2) + (3m+2)c_3b_2a_2}{c_3^2 + d_3^2}\right].$$

In the system of differential equations (3.4), we set

$$z_1 = \rho_1\exp(i\varphi_1), \quad z_2 = \rho_2\exp(i\varphi_2);$$

then, we obtain the following system of ordinary differential equations (NF) for the real functions $\rho_1(t)$, $\rho_2(t)$, $\varphi_1(t)$, and $\varphi_2(t)$:

$$\begin{aligned}\dot\rho_1 &= \varepsilon[\tau'_m + l_{11}\rho_1^2 + l_{12}\rho_2^2]\rho_1,\\ \dot\rho_2 &= \varepsilon[\tau'_{m+1} + l_{21}\rho_1^2 + l_{22}\rho_2^2]\rho_2,\end{aligned} \tag{3.10}$$

$$\begin{aligned}\dot\varphi_1 &= \varepsilon[g_{11}\rho_1^2 + g_{12}\rho_2^2],\\ \dot\varphi_2 &= \varepsilon[g_{21}\rho_1^2 + g_{22}\rho_2^2].\end{aligned} \tag{3.11}$$

In system (3.10), (3.11), the key role is played by the closed subsystem (3.10) for the amplitude variables ρ_1 and ρ_2. It certainly has the zero equilibrium state $\rho_1 = \rho_2 = 0$ corresponding to the equilibrium state $u = 0$ of the boundary value problem (3.1), (3.2); however, it also can have nonzero equilibrium states.

Lemma 3.1. *System* (3.10) *has the following nonzero equilibrium states:*

$$S_1 : \rho_{11} = \sqrt{-\tau'_m/l_{11}}, \quad \rho_{21} = 0 \quad \text{if} \quad \tau'_m l_{11} < 0,$$

$$S_2 : \rho_{12} = 0, \quad \rho_{22} = \sqrt{-\tau'_{m+1}/l_{22}} \quad \text{if} \quad \tau'_{m+1} l_{22} < 0,$$

$$S_3 : \rho_{13} = \sqrt{\Delta_1/\Delta}, \quad \rho_{23} = \sqrt{\Delta_2/\Delta} \quad \text{if} \quad \Delta_1\Delta > 0 \text{ and } \Delta_2\Delta > 0.$$

In the last two formulas,

$$\Delta = l_{11}l_{22} - l_{21}l_{12}, \quad \Delta_1 = l_{12}\tau'_{m+1} - l_{22}\tau'_m, \quad \Delta_2 = l_{21}\tau'_m - l_{11}\tau'_{m+1}.$$

The equilibrium state S_1 is asymptotically stable if $l_{11} < 0$ and $\Delta_2 < 0$, and it is unstable if at least one of the numbers l_{11} or Δ_2 is positive. The equilibrium state S_2 is asymptotically stable if $l_{22} < 0$ and $\Delta_1 < 0$, and it is unstable if at least one of the numbers l_{22} or Δ_1 is positive.

Finally, S_3 is asymptotically stable if

$$\Delta > 0 \text{ and } l_{11}\Delta_1 + l_{22}\Delta_2 < 0.$$

If $\Delta < 0$ or $l_{11}\Delta_1 + l_{22}\Delta_2 > 0$, then S_3 is unstable.

The coordinates of S_3 are found by solving the algebraic system of equations

$$l_{11}\rho_1^2 + l_{12}\rho_2^2 = -\tau'_m, \quad l_{21}\rho_1^2 + l_{22}\rho_2^2 = -\tau'_{m+1}.$$

The stability conditions of these equilibrium states are checked in the standard fashion. For this purpose, the system of differential equations (3.10) should be linearized at the corresponding equilibrium state.

The results obtained in [21–24] imply the following theorem.

Theorem 3.1. *There exists a positive constant ε_0 such that, for all $\varepsilon \in (0, \varepsilon_0)$, the nonzero equilibrium state S_1 (S_2) is associated with the cycle $L_m(\varepsilon)$ ($L_{m+1}(\varepsilon)$) of the nonlinear boundary value problem* (3.1), (3.2). *The corresponding cycle $L_m(\varepsilon)$ ($L_{m+1}(\varepsilon)$) is orbitally asymptotically stable* (*unstable*) *if the corresponding equilibrium state is asymptotically stable* (*unstable*).

The cycle $L_m(\varepsilon)$ is generated by the family of periodic solutions

$$u_m(t, x, \varepsilon) = \varepsilon^{1/2}\rho_{11}(\exp(i\psi_m) + \exp(-i\psi_m)) + \varepsilon\rho_{11}^2[\eta_1 \exp(2i\psi_m) + \bar\eta_1 \exp(-2i\psi_m)] + o(\varepsilon),$$

where $\psi_m = mx + (\sigma_m + \omega_m\varepsilon)t + \psi_0$, $\psi_0 \in R$.

For the periodic solutions generating the cycle $L_{m+1}(\varepsilon)$, we have the asymptotic formulas

$$\begin{aligned}u_{m+1}(t, x, \varepsilon) &= \varepsilon^{1/2}\rho_{22}(\exp(i\psi_{m+1}) + \exp(-i\psi_{m+1}))\\ &+ \varepsilon\rho_{22}^2[\eta_2 \exp(2i\psi_{m+1}) + \bar\eta_2 \exp(-2i\psi_{m+1})] + o(\varepsilon),\end{aligned}$$

where $\psi_{m+1} = (m+1)x + (\sigma_{m+1} + \omega_{m+1}\varepsilon)t + \psi_0, \psi_0 \in R$.

At the same ε, the equilibrium state S_3 is associated with the two-dimensional invariant torus $T_2(\varepsilon)$, which is asymptotically stable if S_3 is asymptotically stable. The torus $T_2(\varepsilon)$ is a saddle one if the equilibrium state S_3 is unstable.

The torus is filled with solutions each of which satisfies the asymptotic formula

$$u_T(t,x,\varepsilon) = \varepsilon^{1/2}[\rho_{13}[\exp(i\psi_3) + \exp(-i\psi_3)] + \rho_{23}[\exp(i\psi_4) + \exp(-i\psi_4)]] + \varepsilon[\rho_{13}^2\eta_1 \exp(2i\psi_3)$$
$$+ \rho_{23}^2\eta_2 \exp(2i\psi_4) + \rho_{13}\rho_{23}\eta_3 \exp(i\psi_3 + i\psi_4) + \rho_{13}\rho_{23}\eta_4 \exp(i\psi_4 - i\psi_3) + \text{c.c.}] + o(\varepsilon).$$

Here $\psi_3 = mx + (\sigma_m + \varepsilon\omega_m + o(\varepsilon))t + \psi_{30}$, $\psi_4 = (m+1)x + (\sigma_{m+1} + \varepsilon\omega_{m+1} + o(\varepsilon))t + \psi_{40}$, ψ_{30}, $\psi_{40} \in R$, $\omega_m = g_{11}\rho_{13}^2 + g_{12}\rho_{23}^2$, *and* $\omega_{m+1} = g_{21}\rho_{13}^2 + g_{22}\rho_{23}^2$. *The letters c.c. in the second pair of square brackets denote the terms that are complex conjugate to the explicitly written terms. The corrections to the frequencies* σ_m and σ_{m+1} *are found by analyzing the system of differential equations* (3.11).

Theorem 3.1 is formulated for the generic position. If $\sigma_m = \sigma_{m+1} = \omega_m = \omega_{m+1} = 0$, then the family of solutions u_T is independent of t and the two-dimensional invariant set $T_2(\varepsilon)$ is filled with the family of inhomogeneous equilibrium states. The equilibrium states S_1 and S_2 are associated with one-dimensional invariant manifolds filled with inhomogeneous equilibrium states. This fact can be confirmed if we consider the Cahn–Hilliard equation (i.e., Eq. (0.1) with $\gamma_1 = \gamma_2 = \gamma_3 = a_2 = b_2 = a_3 = 0$, and $\alpha > 0$) if $b_3 < 0$.

If Eq. (0.1) is considered with $\gamma_2 = \gamma_3 = b_2 = a_3 = b_3 = 0$, $\gamma_1 \neq 0$, $a_2 \neq 0$, and $\alpha > 0$, then the situation is different. At such a choice of the coefficients, we obtain a version of the Kuramoto–Sivashinsky equation (see [2]). Let, in addition, $\alpha_2 = 36$, $\beta_2 = 13$ ($m = 2$), and $\nu_1 > 4\nu_2 > 0$. The check of the conditions of Theorem 3.1 shows that there are two cycles $L_2(\varepsilon)$, $L_3(\varepsilon)$, and the torus $T_2(\varepsilon)$. The torus is stable and the cycles are saddles. Therefore, from the physical point of view, two-frequency oscillations are realized, which fill the two-dimensional invariant torus (in the situation of generic position).

Now, let $m = 1$ and $\gamma_2 = 5\gamma_3$. In this case and at $\varepsilon = 0$, there is the resonance of "eigenfrequencies" 1 : 2, i.e., $\sigma_2 = 2\sigma_1$ in the bifurcation problem. Below, we restrict ourselves to a special case of the bifurcation problem. In its more general version, $\gamma_2 - 5\gamma_3 = \gamma_4\varepsilon$, where $\gamma_4 \in R$ and ε is a small parameter.

In this case, the normal form techniques is applied in a different form. The solutions belonging to $M_4(\varepsilon)$ should be sought in the form (see [25, 26])

$$u(t,x,\varepsilon) = \varepsilon u_1(x, z_1\bar{z}_1, z_2, \bar{z}_2) + \varepsilon^2 u_2(x, z_1, \bar{z}_1, z_2, \bar{z}_2) + o(\varepsilon^2), \tag{3.12}$$

where

$$u_1(t,x,z_1,\bar{z}_1,z_2,\bar{z}_2) = z_1q_1 + \bar{z}_1\bar{q}_1 + z_2q_2 + \bar{z}_2\bar{q}_2, \quad z_j = z_j(t), \quad j = 1,2,$$
$$q_1 = \exp(i\sigma t + ix), \quad q_2 = \exp(2i\sigma t + 2ix), \quad \sigma_1 = \sigma = \gamma_2 - \gamma_1 - \gamma_3.$$

Note that $q_2 = q_1^2$, $q_1\bar{q}_1 = 1$, and $q_2\bar{q}_2 = 1$. Finally, the complex-valued functions $z_1(t)$ and $z_2(t)$ satisfy the system of ordinary differential equations (NF)

$$\dot{z}_1 = \varepsilon\psi_1(z_1,\bar{z}_1,z_2,\bar{z}_2) + o(\varepsilon), \quad \dot{z}_2 = \varepsilon\psi_2(z_1,\bar{z}_1,z_2,\bar{z}_2) + o(\varepsilon). \tag{3.13}$$

The right-hand sides of the NF (3.13) will be determined later. By substituting sum (3.12) into the boundary value problem (3.1), (3.2) and collecting the terms proportional to ε^2, we obtain an inhomogeneous boundary value problem for determining the functions $u_2 = u_2(t,x,z_1,\bar{z}_1,z_2,\bar{z}_2)$. The boundary value problem is

$$u_{2t} = A_2u_2 + \varepsilon B_2u_1 - a_2(u_1^2)_x - b_2(u_1^2)_{xx} - \psi_1q_1 - \bar{\psi}_1\bar{q}_1 - \psi_2q_2 - \bar{\psi}_2\bar{q}_2, \tag{3.14}$$

$$u_2(t, x+2\pi, z_1,\bar{z}_1,z_2,\bar{z}_2) = u_2(t,x,z_1,\bar{z}_1,z_2,\bar{z}_2). \tag{3.15}$$

The conditions of its solvability in the class of $2\pi/\sigma$ periodic functions in the variable t imply that

$$\psi_1 = \tau_1'z_1 + (l_3 + ig_3)\bar{z}_1z_2, \quad \psi_2 = \tau_2'z_2 + (l_4 + ig_4)z_1^2.$$

In the case under examination,

$$\tau_1' = \nu_1 + \nu_2, \quad \tau_2' = 4\nu_1 + \nu_2, \quad l_3 + ig_3 = 2(b_2 - ia_2), \quad l_4 + ig_4 = 2(2b_2 - ia_2).$$

Below in this and in the next section, we consider only the equations with $b_2 = 0$. If, additionally, $a_3 = 0$, and $b_3 = 0$, then we obtain a version of the Kawahara equation; and in the case $\gamma_2 = \gamma_3 = 0$, $a_3 = 0$, and $b_2 = 0$, we obtain a version of the Kuramoto–Sivashinsky equation. Under such additional assumptions, the truncated normal form ($a_2 \neq 0$) takes the form

$$\dot{z}_1 = \varepsilon[\tau_1' z_1 - 2ia_2\bar{z}_1 z_2], \quad \dot{z}_2 = \varepsilon[\tau_2' z_2 - 2ia_2 z_1^2].$$

The changes $z_1 = v_1/(2a_2)$, $z_2 = -v_2/(2a_2)$ reduce this system of differential equations to the system

$$\dot{v}_1 = \varepsilon[\tau_1' v_1 + i\bar{v}_1 v_2], \quad \dot{v}_2 = \varepsilon[\tau_2' v_2 + iv_1^2]. \tag{3.16}$$

In (3.16), we set

$$v_1 = \rho_1 \exp(i\varphi_1), \quad v_2 = \rho_2 \exp(i\varphi_2), \quad \rho_1, \rho_2 \geq 0, \quad \varphi_1, \varphi_2 \in \mathbb{R}.$$

As a result, it can be rewritten in the real form

$$\dot{\rho}_1 = \varepsilon[\tau_1'\rho_1 - \rho_1\rho_2 \sin\psi], \quad \dot{\rho}_2 = \varepsilon[\tau_2'\rho_2 + \rho_1^2 \sin\psi],$$
$$\dot{\varphi}_1 = \varepsilon\rho_2 \sin\psi, \quad \dot{\varphi}_2 = \varepsilon(\rho_1^2/\rho_2)\cos\psi, \quad \psi = \varphi_2 - 2\varphi_1.$$

In turn, we can separate the following closed subsystem of three equations from the last system of differential equations:

$$\dot{\rho}_1 = \varepsilon[\tau_1'\rho_1 - \rho_1\rho_2 \sin\psi], \quad \dot{\rho}_2 = \varepsilon[\tau_2'\rho_2 + \rho_1^2 \sin\psi],$$
$$\dot{\psi} = \varepsilon\left[\frac{\rho_1^2}{\rho_2} - 2\rho_2\right]\cos\psi;$$

it has the coarse equilibrium states

$$S_1 : \psi = \frac{\pi}{2}, \quad \rho_1 = \sqrt{-\tau_1'\tau_2'}, \quad \rho_2 = \tau_1',$$
$$S_2 : \psi = \frac{3\pi}{2}, \quad \rho_1 = \sqrt{-\tau_1'\tau_2'}, \quad \rho_2 = -\tau_1'.$$

The equilibrium state S_1 exists if $\tau_1' > 0$, $\tau_2' < 0$, and S_2 exists if $\tau_1' < 0$, $\tau_2' > 0$. The first of these equilibrium states is asymptotically stable, and the other one is unstable.

Let in Eq. (0.1) $b_2 = 0$, $\gamma_2 = 5\gamma_3$, $\alpha = \alpha_2 - \nu_2\varepsilon$, $\beta = \beta_2 - \nu_1\varepsilon$, $\alpha_2 = 4$, $\beta_2 = 5$, and $a_2 \neq 0$. The reasoning above and the results obtained in [25, 26] imply the following theorem.

Theorem 3.2. *There exists an* $\varepsilon_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0)$, *the boundary value problem* (0.1), (0.2) *has a limit cycle if* $\tau_1'\tau_2' < 0$. *This cycle is stable if* $\tau_2' < 0$ *and* $2\tau_1' + \tau_2' < 0$. *The solutions forming this cycle satisfy the asymptotic formula*

$$u(t, x, \varepsilon) = \frac{\varepsilon}{a_2}\sqrt{-\tau_1'\tau_2'}\cos(\sigma t + x + \varphi_0) + \frac{\varepsilon}{a_2}\tau_1' \sin(2\sigma t + 2x + 2\varphi_0) + O(\varepsilon^2), \quad \varphi_0 \in R.$$

Recall that $\tau_1' = \nu_1 + \nu_2$ and $\tau_2' = 4\nu_1 + \nu_2$.

4. A SPECIAL VERSION OF THE BIFURCATION PROBLEM

In this section, we consider the boundary value problem (0.1), (0.2) for $\alpha = 0$. We additionally assume that $a_3 = b_3 = b_2 = 0$. Under these assumptions, we obtain the Kawahara (Kawahara–Benney–Lin equation) [4, 5]. If additionally $\gamma_2 = \gamma_3 = 0$, then we have the Kuramoto–Sivashinsky equation.

Let us also set $\beta = 1 + \varepsilon$, $\varepsilon \in (0, \varepsilon_0)$, and $0 < \varepsilon_0 \ll 1$. Then, we obtain a special version of the critical case in which the stability spectrum includes $\lambda_0 = 0$ and $\lambda_{\pm 1} = \pm i\sigma_1$, $\sigma_1 = -\gamma_1 + \gamma_2 - \gamma_3$ (see Section 1). Rewrite the boundary value problem (0.1), (0.2) in the form similar to that used in Sections 2 and 3:

$$u_t = A_3 u + \varepsilon B_3 u - a_2(u)_x^2, \tag{4.1}$$

$$u(t, x + 2\pi) = u(t, x), \tag{4.2}$$

where $A_3 u = -u_{xxxx} - u_{xx} - \gamma_1 u_x - \gamma_2 u_{xxx} - \gamma_3 u_{xxxxx}$, $B_3 u = -u_{xx}$.

Any solution to the boundary value problem (4.1), (4.2) can be represented by the sum

$$u(t,x) = u_0(t) + v(t,x), \quad u_0(t) = M_0(u) \equiv \frac{1}{2\pi}\int_0^{2\pi} u(t,x)dx,$$

$$v(t,x) = \sum_{n\neq 0} u_n(t)\exp(inx), \quad u_n(t) = \frac{1}{2\pi}\int_0^{2\pi} u(t,x)\exp(-inx)dx.$$

It is clear that $M_0(v) = 0$. Note that the right-hand side of Eq. (4.1) has a zero spatial mean. Therefore, the boundary value problem (4.1), (4.2) may be replaced by the equation

$$\dot{u}_0(t) = 0, \quad u_0(t) = c,$$

where c is an arbitrary real constant, and the boundary value problem for $v(t,x)$

$$v_t = A(c)v + \varepsilon B_3 v - a_2(v^2)_x, \tag{4.3}$$

$$v(t, x + 2\pi) = v(t,x), \quad M_0(v) = 0, \tag{4.4}$$

where $A(c)v = A_3 v - 2a_2 c v_x$. Therefore, the LDO $A(c)$ has the eigenvalues

$$\lambda_{\pm 1}(\varepsilon, c) = \varepsilon \pm i\sigma(c), \quad \sigma(c) = \gamma_2 - \gamma_1 - \gamma_3 - 2a_2 c, \quad \sigma(0) = \sigma_1,$$

and the other $\lambda_n(\varepsilon, c)$ satisfy the inequality $\operatorname{Re}\lambda_n(\varepsilon, c) \le -\gamma_0 < 0$. Moreover γ_0 is independent of c. The phase space of the nonlinear boundary value problem (4.3), (4.4) is the space $H^s_{2,0} \subset H^s_2$, which consisting of the functions $f(x) \in H^s_2(u(0,x) = f(x))$ satisfying the equality $M_0(f) = 0$.

The bifurcations of the nonlinear boundary value problem (4.3), (4.4) can be analyzed using the techniques described in Section 2 of this paper. Recall that the analysis of the neighborhood of the boundary value problem (4.3), (4.4) can be reduced to the analysis of the NF:

$$\dot{z} = \varepsilon[1 + (l + ig)|z|^2]z. \tag{4.5}$$

In this case,

$$l = -\frac{4a_2^2}{3(4 + d^2)} < 0, \quad g = \frac{2da_2^2}{3(4 + d^2)}, \quad d = 5\gamma_3 - \gamma_2.$$

The differential equation (4.5) has a stable periodic solution

$$z(t) = \rho_0 \exp(i\varepsilon\omega_0 t), \quad \rho_0 = \sqrt{-1/l}, \quad \omega_0 = -g/l,$$

which is associated with the family of periodic solutions having the travelling wave structure

$$v_c(t,x,\varepsilon) = 2\rho_0\varepsilon^{1/2}\cos(\psi_c) + O(\varepsilon), \quad \psi_c = x + \sigma(c)t + \alpha,$$

where α is an arbitrary constant. In this section, we have written out only the principal term of the asymptotics in real form. It is clear that formula for $v_c(t,x,\varepsilon)$ can be refined using the calculations described in Section 2. The periodic solutions $v_c(t,x,\varepsilon)$ form a cycle $L_c(\varepsilon)$ in the phase space of the solutions of the auxiliary boundary value problem (4.3), (4.4). All its solutions with sufficiently small (in the norm) initial values lying in a neighborhood $L_c(\varepsilon)$ approach this cycle at an exponential rate with an exponent $\kappa(\gamma_0) < 0$ that is independent of c.

Now, return to the boundary value problem (4.1), (4.2). It has the family of periodic solutions

$$u_c(t,x,\varepsilon) = c + 2\rho_0\varepsilon^{1/2}\cos(\psi_c) + O(\varepsilon), \tag{4.6}$$

which depends on two arbitrary parameters c and c_0. Moreover, this family of solutions forms a two-dimensional invariant set $\mathrm{Cil}(\varepsilon)$ for the solutions of the boundary value problem (4.1), (4.2) (the local attractor of its solutions). This invariant set may be called a cylinder because geometrically it is the Cartesian product of the closed trajectory $L_c(\varepsilon)$ and a straight line.

Note that all solutions to (4.6) are periodic functions of t with the period depending on the choice of c. Indeed,

$$T_c \simeq 2\pi/\sigma(c), \quad \sigma(c) = \gamma_2 - \gamma_1 - \gamma_3 - 2a_2 c \quad (a_2 \neq 0).$$

The proof of the fact that all solutions of the two-parameter family of periodic solutions (4.6) are unstable in Lyapunov's sense in the norm of the phase space of solutions is fairly standard. Select two solutions of this family

$$u_1 = u(t,x,c_1,\alpha_1) = c_1 + 2\rho_0\varepsilon^{1/2}\cos(x+\sigma(c_1)t+\alpha_1) + o(\varepsilon^{1/2}),$$

$$u_2 = u(t,x,c_2,\alpha_2) = c_2 + 2\rho_0\varepsilon^{1/2}\cos(x+\sigma(c_2)t+\alpha_2) + o(\varepsilon^{1/2}).$$

Denote the "principal" parts of these formulas by w_1 and w_2, i.e.,

$$w_j = w_j(t,x,c_j,\alpha_j) = c_j + 2\rho_0\varepsilon^{1/2}\cos(x+\sigma(c_j)t+\alpha_j), \quad j = 1,2.$$

Then, trigonometric calculations give

$$|\Delta w_x| = |w_{1x} - w_{2x}| = 4\rho_0\varepsilon^{1/2}|\sin\Theta_1(t)||\cos\Theta_2(t,x)|,$$

where

$$\Theta_1(t) = 0.5[(\sigma(c_1)-\sigma(c_2))t + \Delta_1], \quad \Theta_2(t,x) = x + 0.5[(\sigma(c_1)+\sigma(c_2))t + \Delta_2],$$
$$\Delta_1 = \alpha_1 - \alpha_2, \quad \Delta_2 = \alpha_1 + \alpha_2.$$

The elementary computations show that

$$\|\Delta w_x\|^2_{L_2(0,2\pi)} = \int_0^{2\pi}(\Delta w_x)^2 dx = 16\pi\rho_0^2\varepsilon\sin^2(\Theta_1(t)).$$

The above reasoning implies three remarks:

(1) $\|\Delta w_x\|_{L_2(0,2\pi)} = 4\pi^{1/2}\rho_0\varepsilon^{1/2}$ if $t = t_k$, where t_k is the solution to the equation

$$0.5[(\sigma(c_1)-\sigma(c_2))t_k] + \Delta_1 = \frac{\pi}{2} + \pi k,$$

where $k \in Z$, i.e., $t_k = \dfrac{\pi + 2\pi k - \Delta_1}{2a_2(c_2 - c_1)}$. A proper choice of k ensures the validity of the inequality $t_k > 0$. In addition, $\lim\limits_{|k|\to\infty}|t_k| = \infty$.

(2) Therefore, we have

$$\|\Delta w\|_{H_2^s} \ge \|\Delta w_x\|_{L_2(0,2\pi)} \ge 4\pi^{1/2}\rho_0\varepsilon^{1/2}$$

if $s = 4$ or 5 and $t = t_k$.

(3) Finally, for $t = 0$ we obtain the inequality $\|\Delta w\|_{H_2^s} \le \delta$ if $|c_1 - c_2| \le \delta_1$, $|\alpha_1 - \alpha_2| \le \delta_1$. In this case, $\delta = \delta(\delta_1) \to 0$ if $\delta_1 \to 0$.

Similar remarks hold for the solutions u_1 and u_2 and not only for their "principal" parts if ε_0 is a sufficiently small positive constant and $\varepsilon \in (0,\varepsilon_0)$.

A similar result was obtained in [27], where a different boundary value problem was studied.

CONCLUSIONS

The boundary value problem for Eq. (0.1) was studied. For various sets of coefficient values, this equation includes such well-known equations as the Kuramoto–Sivashinsky, Cahn–Hilliard, and Kawahara equation. For the boundary value problem (0.1), (0.2), we studied the local bifurcations in the cases when the stability of homogeneous equilibrium states changes at small variations of the parameters of the problem. Note that the local bifurcation problem for the boundary value problem (0.1), (0.2) was earlier studied for certain specific cases and often under additional conditions imposed on the class of solutions. In most cases, the local bifurcations for the simple Kuramoto–Sivashinsky equation subject to periodic boundary conditions and additional assumptions about solutions of the boundary value problem were considered. For example, these were assumptions of evenness (oddity) with respect to the spatial variable x (see [28, 29]). The boundary value problem was reduced to a finite-dimensional system of ordinary differential equations using the Galerkin method with a small number (e.g., four) of basis functions. Often, the detailed analysis of bifurcations used the computer analysis of the corresponding bifurcation problems. Note that many studies use the seminal paper [30], in which a theorem on the existence of a global attractor for the mixed problem

$$u_t + \nu u_{xxxx} + u_{xx} + \frac{1}{2}(u_x)^2 = 0,$$
$$u(0, x) = u_0(x), \quad x \in R,$$
$$u(t, x + L) = u(t, x), \quad u(t, -x) = u(t, x)$$

or

$$u_t + \nu u_{xxxx} + u_{xx} + (u^2)_x = 0,$$
$$u(0, x) = u_0(x), \quad x \in R,$$
$$u(t, x + L) = u(t, x), \quad u(t, -x) = -u(t, x),$$

was proved; these are the periodic boundary value problems for the basic version of the Kuramoto–Sivashinsky equation with additional conditions about the evenness or oddity of solutions. In a number of studies in situations similar to studies [28, 29], the analysis of the basic version of the Kuramoto–Sivashinsky equation was based on the Lyapunov–Schmidt technique (e.g., see [31]).

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