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Methods of the Convex Cone Theory in the Feasibility Problem of Multicommodity Flow

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Abstract—The feasibility problem of multicommodity flow is reduced to finding out if a multidimensional vector determined by the network parameters belongs to a convex polyhedral cone determined by the set of paths in the network. It is shown that this representation of the feasibility problem makes it possible to formulate the feasibility criterion described in [1] in a different form. It is proved that this criterion is sufficient. The concepts of reference vectors and networks are defined, and they are used to describe a method for solving the feasibility problem for an arbitrary network represented by a complete graph.

Keywords: multicommodity flow, feasibility criterion, polyhedral cone, multivertex graph.

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INTRODUCTION

The feasibility problem of a multicommodity flow is a part of the general theory of flows in networks, and this problem was stated simultaneously with the appearance of this theory (e.g., see [2, 3]). It is a natural generalization of the maximal flow and minimal cut problem, which can be called *feasibility* problem of a single-commodity flow; the solution of this problem obtained using the augmenting flow algorithm is determined by the capacity of the minimal cut of the network.

In the general case, multiple source–sink pairs and intensities of flows are given in a network. The problem is to find out if it is possible to organize flows with the given intensities in this network in such a way that the edge capacities are not exceeded. An algorithmic method was also applied to this problem [3]; however, it was only proved that the solution of the two-commodity problem is reduced to checks on cuts. However, this is generally not true.

In [1], a generic feasibility criterion of a multicommodity flow in the form of nonnegativity of scalar products of a multidimensional vector determined by the network parameters and intensities of commodities with the set of metric vectors was formulated. In [1], metrics are defined as vectors satisfying a certain system of inequalities. No proof of the sufficiency of this criterion was given in [1] and in subsequent publications (e.g., [4]).

The main purpose of the current paper is to prove the sufficiency criterion for the feasibility of the multicommodity flow proposed in [1]. This is achieved by reformulating the criterion given in [1]. More precisely, it turned out that the multicommodity feasibility flow problem can be formulated in terms of the theory of convex polyhedral cones, and it is equivalent to the following problem: does a given vector belong to a given polyhedral cone? In this formulation, the convex polyhedral cone is a cone with the generators formed by all paths of all commodities and all positive unit vectors of the network edges; the components of the network vector are the capacities of the network edges and the intensities of commodities.

In [1, 4], considerable effort was devoted to finding out which networks are "purely cut networks" (the term *purely cut network* denotes a network in which all metric vectors are cuts). It is claimed in [1] that all networks with flow schema (the definition of this concept see in this paper) in the form of two stars, complete four-vertex graph, or five-vertex cycle possess this property. In [4], it is claimed that the networks possessing this property are confined to these three flow schemata. The representation of the problem in terms of the theory of convex polyhedral cones and introduction of the concept of reference networks provides a new look at the problem of purely cut networks.

1. FEASIBILITY CRITERIA FOR MULTICOMMODITY FLOWS

1.1. Formulation of the Problem in Terms of Convex Cones

The feasibility problem of a multicommodity flow in the network F is denoted by $F(N,C,T)$; it is formulated as follows. In the N-vertex graph (network), a symmetric (triangular) real positive $N \times N$ matrix $\mathbf{C}$ of edge capacities of F with zero diagonal elements and a symmetric (triangular) real nonnegative $N \times N$ matrix $\mathbf{T}$ of flow intensities with zero diagonal elements are given:

$$c_{ik} = c_{ki} \geq 0, \quad t_{ik} = t_{ki} \geq 0, \quad i,k = \overline{1,N}, \quad i \neq k, \quad c_{ii} = t_{ii} = 0, \quad i = \overline{1,N}. \tag{1.1}$$

The positive elements of C are called *network edges* with the capacities c_{ik}, and the positive elements of $\mathbf{T}$ are called *flow edges* or commodities with the intensities of those commodities t_{ik}. The set of network and flow edges is called the graph of the network F, and the subgraph of F formed by the edges of $\mathbf{T}$ is called the flow schema.

We assume that the connectivity of F is such that the following conditions are satisfied.

1. Let $(X,\bar{X})$ be a partition of the set of F vertices; F^1 and F^2 are the subgraphs of F formed by the vertices X and $\bar{X}$, respectively; and let there be among the flow edges at least one edge such that one of its vertices belongs to the first subgraph and the other one belongs to the second subgraph. In this case, the subgraphs F^1 and F^2 must be connected. The set of edges connecting the nodes of these subgraphs is conventionally called the cut.

2. According to Condition 1, every cut contains more than one edge (an equivalent formulation is that the removal of any network edge does not violate the connectivity of F).

We want to find out if it is possible to organize the flows of commodities with intensities t_{ik} such that the sum of all commodities through each network edged does not exceed the capacity of the edge.

Let M_1 be the number of network edges and M_2 be the number of flow edges. It was shown in [2] that, without the loss of generality, the sets of network and flow edges can be assumed to be disjoint. Below, we assume that this condition is fulfilled.

Consider the real space $\mathbb{R}^{M_3}$ of dimension $M_3 = M_1 + M_2$. The unit vectors corresponding to the elements of the matrix $\mathbf{C}$, i.e., to the network edges of F, are denoted by $e_k, k = \overline{1,M_1}$, the subspace spanned by these vectors is denoted by R^C, the unit vectors corresponding to the elements of the matrix T, i.e., the flow edges of F, are denoted by e_m, $m = \overline{(M_1+1),M_3}$, and the corresponding subspace is denoted by R^T. In this notation, the elements of the matrices C and T are assumed to be (positive) components of the vectors $C \in R^C \supset R^{M_3}$ and $T \in R^T \supset R^{M_3}$. We also define the network vector $f = (C - T) \in R^{M_3}$.

The paths in the network F will be represented by the vectors in the space $\mathbb{R}^{M_3}$; more precisely,

$$\ell^m \Rightarrow e_{i_1i_2} + e_{i_2i_3} + \ldots + e_{i_{n-1}i_n} - e_{i_1i_n} = e_{k_1} + e_{k_2} + \ldots + e_{k_{n-1}} - e_m, \tag{1.2}$$

where ℓ^m is the path of commodity m, $i_1,i_2,\ldots,i_n$ are the vertices of F included in the path ℓ^m, $e_{i_1i_2} = e_{k_1}, e_{i_2i_3} = e_{k_2},\ldots,e_{i_{n-1}i_n} = e_{k_{n-1}}$ are the unit vectors of the space R^C corresponding to the edges in ℓ^m, and $e_{i_1i_n} = e_m$ is the flow edge of ℓ^m, $e_m \in R^T$. The row of components of the path vector in the space R^{M_3} is

$$\ell^m \to \overbrace{0\,1\,0\,0\,1\ldots 1}^{M_1}\;\overbrace{\underbrace{0\;0\;-1}_{M_1+m}\;0\ldots 0}^{M_2}, \tag{1.3}$$

where the ones correspond to the edges of R^C included in the path ℓ^m and -1 are at the places of the network edges of the path in R^T. We will use the following notation:

L^m $(m = \overline{1,M_2})$ are the sets of vectors of type (1.2) representing all paths of each commodity m in F;

$\hat{R}^C \equiv \bigcup_{i=1}^{M_1} e_i$ is the set of network unit vectors, i.e., the unit vectors of the subspace R^C;

$\hat{R}^T \equiv \bigcup_{m=1}^{M_2} e_{M_1+m}$ is the set of unit vectors of all commodities, i.e, the unit vectors of the subspace R^T;

$\hat{R}^{M_3} = \hat{R}^C \cup \hat{R}^T$ for the networks represented by complete graphs, $\hat{R}^{M_3} \equiv \hat{R}^M$, $M = N(N-1)/2$;

$L^T \equiv \bigcup_{m=1}^{M_2} L^m$ is the set of all paths of all commodities.

The flow of commodity m ($m = \overline{1, M_2}$) of intensity t^m in the network $F(N,C,T)$ is the vector $\Pi^m \in \mathbb{R}^{M_3}$ defined by

$$\Pi^m = \sum_i \alpha_i \ell_i^m, \quad \alpha_i \geq 0, \quad \sum_i \alpha_i = \Pi_{M_1+m}^m = -t^m, \tag{1.4}$$

where $\Pi_{M_1+m}^m$ is the component of this vector corresponding to the unit vector e_{M_1+m}. The other components of this vector are

$$\Pi_k^m = \sum_i \alpha_i \delta_{i,k}^m, \quad k = \overline{1, M_1}, \quad m = \overline{1, M_2},$$

$$\begin{aligned} \delta_{i,k}^m = 1 \quad &\text{if the path} \quad \ell_i^m \quad \text{includes the edge} \quad e_k, \\ \delta_{i,k}^m = 0 \quad &\text{if the path} \quad \ell_i^m \quad \text{does not include the edge} \quad e_k, \end{aligned} \tag{1.5}$$

$$\Pi_{M_1+i}^m = 0, \quad i = \overline{1, M_2}, \quad i \neq m.$$

The multicommodity flow with the intensities of the commodity flows $t^m(m = \overline{1, M_2})$ is realizable in the network $F(N,C,T)$ if there exist flows $\Pi^m(m = \overline{1, M_2})$ such that

$$\Pi_{M_1+m}^m = t^m, \quad m = \overline{1, M_2} \quad \text{and} \quad \sum_{m=1}^{M_2} \Pi_k^m \leq c_k, \quad k = \overline{1, M_1}. \tag{1.6}$$

Furthermore, let $M = N(N-1)/2 \geq M_3$ be the number of edges in the complete N-vertex network and R^M be the real space of dimension M. Let $\gamma \in R^M$ and

$$\gamma_{ij} + \gamma_{jk} - \gamma_{ik} \geq 0, \quad \gamma_{ij} - \gamma_{jk} + \gamma_{ik} \geq 0, \quad -\gamma_{ij} + \gamma_{jk} + \gamma_{ik} \geq 0, \quad i, j, k \leq N, \quad i < j < k. \tag{1.7}$$

In [1], such vectors are called *metrics*. Inequalities (1.7) specify a convex polyhedral cone in the space R^M (e.g., see [5]); this is the cone of metrics, which is denoted by Γ.

Extend the vectors C and T to the space R^M by assuming the additional components equal to zero. The extended vectors will be denoted by $\tilde{C}$ and $\tilde{T}$, respectively. The feasibility criterion of the multicommodity flow formulated in [1] (it is called *FM* inequality in that paper) is as follows (we give three equivalent formulations):

$$\langle \gamma^C, C \rangle \geq \langle \gamma^T, T \rangle, \quad \langle \gamma, \tilde{C} \rangle \geq \langle \gamma, \tilde{T} \rangle, \quad \langle \gamma, f \rangle \geq 0, \quad \langle \gamma, \tilde{f} \rangle \geq 0 \quad \forall \gamma \in \Gamma, \tag{1.8}$$

where γ^C is the trace of the vector γ in the space R^C and γ^T is the trace of the vector γ in the space R^T, $\tilde{f} = \tilde{C} - \tilde{T}$, and the angle brackets denote the scalar product of vectors.

Let us define new objects and reformulate this criterion.

Since the cone Γ is uniquely defined by the finite set of its extreme vectors (this set will be denoted by $\hat{\Gamma}$), the criterion formulated in [1] can be written in the form

$$\langle \tilde{f}, \gamma \rangle \geq 0 \quad \forall \gamma \in \hat{\Gamma}. \tag{1.9}$$

Denote the adjoint cone of Γ by G. The extreme vectors of this cone (the set of such vectors will be denoted by $\hat{G}$) are:

$$\hat{G} \equiv \Delta_{ijk,1} \cup \Delta_{ijk,2} \cup \Delta_{ijk,3}, \quad i, j, k \leq N, \quad i < j < k, \tag{1.10}$$

where

$$\Delta_{ijk,1} = e_{ij} + e_{jk} - e_{ik}, \quad \Delta_{ijk,2} = e_{ij} - e_{jk} + e_{ik}, \quad \Delta_{ijk,3} = -e_{ij} + e_{jk} + e_{ik}, \quad i, j, k \leq N, \quad i < j < k.$$

(The proof of the fact that this set is not redundant is omitted in this paper.)

Criterion (1.9) is equivalent to the following criterion:

$$\tilde{f} \in G. \tag{1.11}$$

Consider the convex polyhedral cone F determined by the set of its extreme vectors $\hat{F}$:

$$\hat{F} \equiv \hat{R}^C \cup L^T. \tag{1.12}$$

This cone is defined in the space R^{M_3}. Denote by Φ the adjoint cone of F so that $\langle \phi, f \rangle \geq 0$ $\forall \phi \in \Phi$ and $\forall f \in F$. The set of extreme vectors of Φ will be denoted by $\hat{\Phi}$. This cone is also defined in R^{M_3}.

1.2. Proof of Criterion [1]

Theorem 1. *The feasibility problem for a multicommodity flow has a solution or the multicommodity flow T is realizable in the network with the edge capacities C if and only if any of the following three conditions is satisfied*:

$$f = (C - T) \in F,$$

$$\langle (C - T), \phi \rangle \geq 0 \quad \forall \phi \in \Phi, \tag{1.13}$$

$$\langle (C - T), \phi \rangle \geq 0 \quad \forall \phi \in \hat{\Phi}.$$

Proof. *Necessity.* If a multicommodity flow is feasible, then the vector $f = (C - T)$ admits expansion (1.4)–(1.6), i.e. the representation by a nonnegative linear combination of vectors from the set $\hat{F}$; therefore, $f \in F$.

Sufficiency. Let the vector $f \in \mathbb{R}^{M_3}$ and $f = (C - T) \in F$, where the matrices **C** and **T** satisfy conditions (1.1) and determine the components of *f*. This implies that *f* can be represented by

$$f = \sum_{m=1}^{M_2} \sum_i \alpha_i^m \ell_i^m + \sum_{k=1}^{M_1} \beta_k e_k, \quad \alpha_i^m \geq 0, \quad \beta_k \geq 0, \quad \text{i.e.,} \quad f_i = \sum_{m=1}^{M_2} \Pi^m + \sum_{k=1}^{M_1} \beta_k e_k, \quad \beta_k \geq 0, \tag{1.14}$$

where

$$\Pi^m = \sum_i \alpha_i^m \ell_i^m, \quad m = \overline{1, M_2}.$$

Hence, we see that the flows Π^m satisfy conditions (1.6); therefore, they provide a solution to the multicommodity flow problem $F(N, C, T)$.

Let $\ell \in L^T$. Consider the path ℓ as an element in the space R^M, and represent it by the expansion

$$\begin{aligned} \ell^m &= e_{i_1 i_2} + e_{i_2 i_3} + \ldots + e_{i_{n-1} i_n} - e_{i_1 i_n} = (e_{i_1 i_2} + e_{i_2 i_3} - e_{i_1 i_3}) \\ &+ (e_{i_1 i_3} + e_{i_3 i_4} - e_{i_1 i_4}) + \ldots + (e_{i_1 i_{(n-1)}} + e_{i_{(n-1)} i_n} - e_{i_1 i_n}) = \Delta_{i_1 i_2 i_3, 1} + \Delta_{i_1 i_3 i_4, 1} + \ldots + \Delta_{i_1 i_{(n-1)} i_n, 1}, \end{aligned} \tag{1.15}$$

where $e_{i_1 i_n}$ is another notation for the unit vector $e_m \in R^T$.

All elements of expansion (1.15) are extreme vectors of the cone *G;* therefore, $\ell \in G \; \forall \ell \in L^T$.

Moreover, if $e_{i_1 i_2} \in R^C$ and i is the index of the vertex such that $i \neq i_1$ and $i \neq i_2$, then it is clear that $2e_{i_1 i_2} = \Delta_{i_1 i_2 i, 1} + \Delta_{i_1 i_2 i, 2}$. Therefore, $e_i \in G \; \forall e_i \in R^C$; hence, all extreme vectors of the cone F (1.12) belong to this cone, and, consequently,

$$F \subseteq G \quad \text{and} \quad \Phi \supseteq \Gamma. \tag{1.16}$$

The definition of the cones F, Φ, G, and Γ implies the following result.

Lemma 1. *Let $F(N, C, T)$ be a network for which the feasibility problem for the multicommodity flow is formulated. Then,* $\hat{R}^C \subset \hat{\Phi}$, $-e \in \hat{\Phi} \; \forall e \in \hat{R}^T$, $\hat{R}^C \not\subset \hat{\Gamma}$, $-e \not\subset \hat{\Gamma} \; \forall e \in \hat{R}^T$.

Proof. Let $e \in \hat{R}^C$. Define the set F^e by $F^e = \left\{\hat{R}^C \backslash e \subset F^e,\ \ell \in F^e / \ell \in L^T \wedge \ell \notin e\right\}$. In other words, F^e contains all unit vectors of the space R^C except for the vector *e* and all paths from the set L^T that do not pass through the edge *e*. (Such a path always exists because, due to our assumptions, the removal of any network edge does not make the network *F* disconnected.) This set determines *e* as an element of the set $\hat{\Phi}$, $e \in \hat{\Phi}$. Now, let $e \in \hat{R}^T$. Define the set F^eas follows: $F^e = \left\{\hat{R}^C \subset F^e,\ \ell \in F^e / \ell \in L^T \backslash L^e\right\}$. This set determines $-e$ as an element of the set $\hat{\Phi}$, $-e \in \hat{\Phi}$. This completes the proof of the first part of Lemma 1.

The proof of the second part of this lemma is as follows. Let $e_{ij} \in \hat{R}^C$. The element $\Delta_{ijk,3} = -e_{ij} + e_{jk} + e_{ik} \in G$, and $\langle e_{ij}, \Delta_{ijk,3}\rangle = -1 < 0$. Therefore, $\hat{R}^C \not\subset \hat{\Gamma}$. In the same way, we prove that $-e \not\subset \hat{\Gamma}\ \forall e \in \hat{R}^T$.

If $a \in R^{M_3}$, then we denote by $\tilde{a}$ the extension of this vector to the space R^M assuming that the additional components are equal to zero.

Theorem 2. *Necessity of the criterion* (*see* [1]).

Proof. Let $f = (C - T) \in F$. According to (1.16), $F \subseteq G$; therefore, $\tilde{f} \in G$; i.e., due to (1.11), criterion [1] holds.

Let $F(N,T,C)$ be an *N*-vertex network with a given arbitrary flow schema **T**. The network $F(N-1,T',C')$ is said to be the reduced network $F(N,T,C)$ with respect to the edge $e_{ij} \in R^{M_3}$ if it is formed by merging two vertices A_i and A_j; if two network edges are merged, then the newly formed edge is a network one; and if two flow edges or a network edge is merged with a flow edge, then the newly formed edge is a flow one.

Lemma 2. *Let* $F(N,C,T)$ *be a multicommodity feasibility problem and* $\phi \in \hat{\Phi}$ *be not a unit vector nor a cut. Furthermore, let* $\hat{F}(\hat{N},\hat{T},\hat{C})$ $(\hat{N} \le N, \hat{M}_1 \le M_1, \hat{M}_2 \le M_2)$ *be the reduced network with respect to the zero components of the vector* ϕ *and* $\hat{\hat{\Phi}}$ *be the set of extreme vectors of the cone* $\hat{\Phi}$ *adjoint of the cone* $\hat{F}$. *Then, there exists* $\hat{\phi} \in \hat{\hat{\Phi}}$ *such that all its components are positive*.

Proof. If all components of the vector $\phi \in \hat{\Phi}$ are positive, then the lemma is proved. In the general case, the condition of the lemma implies that there exists the system of linear homogeneous equations

$$
\begin{array}{llll}
 & x_1 \ldots\ldots\ldots x_{M_1'} & x_{M_1'+1} \ldots x_{M_3'} & x_{M_3'+1} \ldots x_{M_3} \\
\ell_1' & 1\,1\,0\,0 \ldots 1 & 0 \ldots -1 \ldots 0 & 0 \ldots\ldots 0 \\
\ell_2' & 1\,0\,1\,1 \ldots 0 & -1 \ldots 0 \ldots 0 & 0 \ldots\ldots 0 \\
 & \ldots\ldots\ldots\ldots\ldots\ldots & & \\
\ell'_{M_3'-1} & 0\,1\,1\,0 \ldots 1 & 0 \ldots -1 & 0 \ldots\ldots 0 \\
 & 0 \ldots\ldots\ldots\ldots & \ldots 0 & 1 \ldots\ldots 0 \\
 & \ldots\ldots\ldots\ldots\ldots\ldots & & \\
 & 0 \ldots\ldots\ldots\ldots & \ldots 0 & 0 \ldots\ldots 1
\end{array}
; \qquad (1.17)
$$

the solution to this system is ϕ, M_3' is the number of nonzero components of ϕ, $M_3 - M_3' = M_3'' > 0$ is the number of zero components of ϕ (all of them are network components), and ℓ_1', $i = \overline{1, M_3' - 1}$ are the projections of the paths $\ell_i \in L^T$ on the space $R^{M_3'}$. Consider the space of zero components of ϕ. In the general case, it consists of a collection of connected subgraphs the edges of which belong to the space R^C. Let $e_{ij} \in R^{C'}$, i.e., $\phi_{ij} > 0$. If the vertices A_i and A_j belong to nonzero subgraphs (and at least one of them includes more than one vertex), $e_{mn} \in R^C$, and A_m (A_n) either coincides with A_i (A_j) or belongs to its zero subgraph, then $\phi_{ij} = \phi_{mn}$. Let us prove this fact. Assume the converse, e.g., let $\phi_{ij} > \phi_{mn}$. The edge e_{ij} is a part of the path $\ell \in L^\phi$, i.e., $\langle \ell, \phi\rangle = 0$. Consider the path ℓ' in which the edge e_{ij} is replaced by the edge e_{mn}

and the paths $A_i - A_m$ and $A_j - A_n$ pass through zero edges. For this path, $\langle \ell', \phi \rangle < 0$, which cannot be true. This contradiction completes the proof. Now let $\phi_{ij} = 0$ and $e_{mi} \in R^{C'}$, $e_{mj} \in R^T$. In this case, we also have $\phi_{mi} = \phi_{mj}$. If this were not the case, e.g., $\phi_{mi} < \phi_{mj}$, for the path ℓ of commodity m (i.e., for the flow edge e_{mj}) consisting of the edges e_{mi} and e_{ij}, it would hold that $\langle \ell, \phi \rangle < 0$, which is impossible. Now assume that $\phi_{mi} > \phi_{mj}$ and let $\ell \in L^T$ be an extremal path passing through the edge e_{mi}, i.e., $\langle \ell, \varphi \rangle = 0$. Furthermore, let $\ell' \in L^m$ be an extremal path of the commodity m, i.e., $\langle \ell', \phi \rangle = 0$. Form the path ℓ'' by replacing the edge e_{mi} in ℓ by the edge e_{ij} and the edge e_{mj} by the set of network edges of the path $\ell' - \ell'_C$. Therefore, $\left\langle \ell'_C, \phi \right\rangle = \phi_{mj}$. In turn, this implies that $\langle \ell'', \phi \rangle < \langle \ell, \phi \rangle < 0$, which contradicts the assumption $\phi \in \hat{\Phi}$. Finally, let $\phi_{ij} = 0$ and $e_{mi} \in R^T$, $e_{mj} \in R^T$. In this case, we also have $\phi_{mi} = \phi_{mj}$; indeed, otherwise, as in the preceding case, we could find a path ℓ' such that $\langle \ell', \phi \rangle < 0$. Thus, the network (cone) F and the extreme vector of the adjoint cone ϕ determine the network (cone) $\hat{F}$ that has the following form: all connected subgraphs consisting of the zero edges of the vector ϕ form a vertex and all edges connecting the vertices of one zero subgraph with the vertices of another zero subgraph form one edge. We have already proved that the components of all edges of the vector ϕ merged into one edge of the network $\hat{F}$ have the same positive values; therefore, the vector ϕ determines in this network a vector $\hat{\phi}$ with positive components. Let us prove that $\hat{\phi} \in \hat{\hat{\Phi}}$. To this end, consider, along with the network $\hat{F}$, the network F' which, by contrast with $\hat{F}$, contains all merged edges. The set of paths ℓ'_i $(i = \overline{1, M'_3 - 1})$ in F' for which $\left\langle \phi, \ell'_i \right\rangle = 0$ $(i = \overline{1, M'_3 - 1})$ forms the set of simple paths $\hat{\ell}_i$ $(i = \overline{1, M'_3 - 1})$ in $\hat{F}$ (which can be identical) satisfying the conditions $\left\langle \hat{\phi}, \hat{\ell}_i \right\rangle = 0$, $i = \overline{1, M'_3 - 1}$. We should prove that this set contains $\hat{M}_3 - 1$ linearly independent paths that determine the vector $\hat{\phi}$. Assume the converse. Then, there exists a vector $\hat{\phi}_1$ in $\hat{F}$ satisfying conditions (1.18). Extend this vector to the space $R^{M'_3}$ by assigning to the merged edges the same components as in the network $\hat{F}$ and denote this extended vector by ϕ_1. It satisfies the equalities $\left\langle \phi_1, \ell'_i \right\rangle = 0$ for $i = \overline{1, M'_3 - 1}$ and, therefore, the set of paths ℓ'_i $(i = \overline{1, M'_3 - 1})$ is not linearly independent, which contradicts the assumption.

Corollary 1. *Let $F^0(N^0, T^0, C^0)$ be an N^0-vertex network, $\phi \in \hat{\Phi}^0$, and this vector have no zero components. Furthermore, let $F_i(C_i), i = \overline{1, N^0}$ be arbitrary connected graphs. Let us form the network $F(N, T, C)$ $(N > N^0, M_1 > M_1^0, M_2 = M_2^0)$ as follows. To each vertex A_i^0 of the network F^0, we assign the graph C_i and, if there exists the edge e_{ij}^0, then we form the edges in F connecting vertices of the graphs C_i and C_j (their number is arbitrary). The flow schema of F is formed by the flow edges of the network F^0, and the set of its network edges is formed by the network edges of the network N^0, the edges of the graphs C_i, $i = \overline{1, N^0}$ and the edges e_{ij} connecting the vertices of the graphs C_i and C_j if the edge e_{ij}^0 exists in N^0. Then, in the cone $\hat{\Phi}$ adjoint of the cone (network) $F(N, T, C)$, there exists an extreme vector $\phi \in \hat{\Phi}$ that is neither a unit vector nor a cut.*

This corollary is proved by the fact that the reduction of the network F with respect to the network edges gives the network F^0.

Corollary 2. *Let the network $F(N-1, T, C)$ be the reduction of the network $F_1(N, T_1, C_1)$ with respect to the flow edge e, $\phi \in \hat{\Phi}$, and this vector be neither a unit vector nor a cut. Then, there exists $\phi_1 \in \hat{\Phi}_1$ such that it is neither a unit vector nor a cut and $\phi_1(e) = 0$.*

Consider the network F_1' in which e is a network edge. The reduction of such a network with respect to this edge gives the network F. Therefore, by Corollary 1, the assertion of Corollary 2 holds true.

Theorem 3. *Let F be the cone corresponding to the multicommodity problem $F(N, C, T)$ and $\phi \in \hat{\Phi}$ be not a unit vector nor a cut. Then, there exists an expansion $\tilde{\phi}$ of ϕ to the space R^M such that $\tilde{\phi} \in \hat{\Gamma}$.*

Proof. 1. Definition of the vector $\tilde{\phi}$. Let $e_{ij} \notin R^M$. Since the network F is connected, there exists at least one path connecting the vertices A_i and A_j. Set $\tilde{\phi}_{ij} = \min\langle \ell, \phi\rangle$ over all such paths in F. Due to Lemma 2, if this unit vector is a chord of a path $\ell \in L^{\phi}$, then $\tilde{\phi}_{ij} = \phi_{ii_1} + \phi_{i_1 i_2} + \ldots + \phi_{i_k j}$, where $e_{ii_1}, e_{i_1 i_2}, \ldots, e_{i_k j}$ are the segments (edges) in ℓ. If the unit vector e_{ij} belongs to one of the connected subgraphs with zero edges, then $\tilde{\phi}_{ij} = 0$.

2. Let us now prove that $\tilde{\phi} \in \Gamma$. To this end, we should prove that $\langle \tilde{\phi}, \Delta\rangle \geq 0\ \forall \Delta \in \hat{G}$. This is equivalent to the following proposition: for every unit vector $e_{ij} \in R^M$ and every vertex A_k ($k \neq i$, $k \neq j$), it holds that $\langle (e_{ik} + e_{jk} - e_{ij}), \tilde{\phi}\rangle = \langle \Delta_{ijk,1}, \tilde{\phi}\rangle \geq 0$ or $\tilde{\phi}_{ik} + \tilde{\phi}_{jk} \geq \tilde{\phi}_{ij}$. (For the triangles $\Delta_{ijk,2}$ and $\Delta_{ijk,3}$, another initial edge should be selected.) By the definition of the vector $\tilde{\phi}$, for each unit vector $e_{ij} \in R^M$, there exists a path $\ell \in R^C$ connecting the vertices A_i and A_j such that $\langle \phi, \ell\rangle = \tilde{\phi}_{ij}$. If $e_{ij} \in R^C$, then this path coincides with e_{ij}. Let ℓ_1 be such a path for the unit vector e_{ik}, ℓ_2 for the unit vector e_{jk}, and ℓ_3 be a simple path in the space R^C connecting the vertices A_i and A_j selected from the union of the paths ℓ_1 and ℓ_2. We have

$$\tilde{\phi}_{ik} + \tilde{\phi}_{jk} = \langle (e_{ik} + e_{jk}), \tilde{\phi}\rangle \geq \langle \ell_3, \phi\rangle \geq \tilde{\phi}_{ij}.$$

3. Next, we prove that $\tilde{\phi} \in \hat{\Gamma}$. Form the set of triangles $G^{\tilde{\phi}}$ such that $\langle \Delta, \tilde{\phi}\rangle = 0$, $\Delta \in G^{\tilde{\phi}}$, and dim $G^{\tilde{\phi}} = M - 1$. Let $\ell = e_{1,2} + e_{2,3} + \ldots + e_{(m-1),m} - e_{1,m} \in L^{\phi}$, $m > 3$, i.e., the path ℓ is not a triangle and $\langle \ell, \varphi\rangle = 0$. Consider the triangles the sides of which are chords of this path:

$$\Delta_{ijk,1}, \quad i < j < k \leq m. \tag{1.18}$$

By definition, each of these triangles belongs to the set $G^{\tilde{\phi}}$. We set the following sequence of introducing new unit vectors into the space R^{M_3}. First, all two-segment chords one-by-one, then three-segment chords, etc. Each such step increases the dimension of the space by one and increases by one the number of equations in system (1.17), that is,

$$x_{i,i+j} + x_{i+j,i+j+1} - x_{i,i+j+1} = 0. \tag{1.19}$$

The solution to this system at each step coincides with $\tilde{\phi}$. The vector $\ell \notin G^{\tilde{\phi}}$; however, since all triangles (1.18) together with ℓ form a linearly independent system of vectors, the equation with the vector ℓ may be replaced by an equation of type (1.19) with the triangle, e.g., $\Delta_{1(m-1)m,1} \in G^{\tilde{\phi}}$, without changing the solution $\tilde{\phi}$ of the system.

Corollary 3. *The values of the components of the vector* $\phi \in \hat{\Phi}$ *are independent of the numerical values of the elements of the matrices* **C** *and* **T** *and depend only on whether they are positive or equal to zero.*

Corollary 4. *If* $\phi \in \hat{\Phi}$ *and all components of this vector are positive, then all components of* $\tilde{\phi}$ *are also positive. Moreover, if* $\Delta_{ijk,1} \in G^{\tilde{\phi}}$, *then* $\Delta_{ijk,2}, \Delta_{ijk,3} \notin G^{\tilde{\phi}}$.

Corollary 5. *Let* $\gamma \in \hat{\Gamma}$ *and* $\gamma_{ij} = 0$. *Then* $\Delta_{ijk,1} \in G^{\gamma}$ *and* $\Delta_{ijk,2} \in G^{\gamma}$.

Theorem 4. *Sufficiency of the criterion* (see [1]). *Let* $f = C - T \in R^{M_3}$ *and* $\langle f, \gamma\rangle = \langle \tilde{f}, \gamma\rangle = \langle f, \gamma'\rangle \geq 0$ $\forall \gamma \in \hat{\Gamma}$. *Then* $f \in F$.

Proof. By Theorem 3, for each $\phi \in \hat{\Phi}$, $\phi \neq e_i$, $i = \overline{1, M_1}$, $\phi \neq -e_j$, $j = \overline{(M_1 + 1), M_3}$, there exists a $\gamma \in \hat{\Gamma}$, $\gamma' = \phi$. Therefore, it holds that $\langle f, \phi\rangle \geq 0\ \forall \phi \in \hat{\Phi}$, $\phi \neq e_i$, $i = \overline{1, M_1}$, $\phi \neq -e_j$, $j = \overline{(M_1 + 1), M_3}$. However, for the components of the extreme vectors of the cone $\mathbf{\Phi}$, the conditions $\langle f, \phi\rangle \geq 0$ are fulfilled by definition. Therefore, the last condition in (1.13) is satisfied and $f \in F$.

2. REFERENCE NETWORKS AND VECTORS

Denote by $\hat{G}^N$ the set of vectors (10) (the extreme vectors of the cone G^N) and by $\hat{\Gamma}^N$ the set of extreme vectors of the cone Γ^N, which the adjoint cone of G^N.

The vectors $\gamma \in \hat{\Gamma}^N$ possess the following properties.

1. All components of the vectors $\gamma \in \hat{\Gamma}^N$ are nonnegative. This follows from the fact that all unit vectors of R^M belong to the cone G^N.

2. All the vectors representing the cuts of the complete graph G^N are extreme vectors of the cone Γ^N; i.e., they belong to the set $\hat{\Gamma}^N$.

3. Let $\gamma \in \hat{\Gamma}^N$ be not a cut and $\gamma_i > 0$ for $i = \overline{1, M}$. Denote by G^γ the set of triangles determining the vector γ, i.e., the triangles satisfying the condition $\langle \Delta, \gamma \rangle = 0$. The condition that all components of γ are positive implies that, if $\Delta_{ijk,1} \in G^\gamma$, then $\Delta_{ijk,2}$, $\Delta_{ijk,3} \notin G^\gamma$. Consider all triangles in the set G^γ containing the edge e_{ik}. Three cases are possible: (1) The set G^γ includes only the triangles $\Delta_{ijk,1}$, i.e., the triangles in which e_{ik} appears with the negative sign. (2) The set G^γ includes both triangles $\Delta_{ijk,1}$ and triangles $\Delta_{mik,2}$, i.e., the triangles in which e_{ik} appears with the positive or negative sign. (3) The set G^γ includes only the triangles $\Delta_{mik,1}$ (or $\Delta_{mik,3}$), i.e., the triangles in which e_{ik} appears with the positive sign. For each vector γ, there exists the set of edges of the first kind γ^T and the set of edges of the second and third type γ^C. Consider the network $F^\gamma = F(N, \gamma^C, \gamma^T)$ in which the edges of the first kind form a flow schema and the edges of the second and third type are network edges. By definition, for this network we have $\phi \in \hat{\Phi}$, $\tilde{\phi} \equiv \phi = \gamma$.

The set of all vectors $\gamma \in \hat{\Gamma}^N$ with $\gamma_i > 0$ will be denoted by $\Im^N$. Each such vector uniquely determines the network with the flow schema γ^T and the set of network edges γ^C; in addition, the numerical values of the components of this vector depend only on the number *N*. For this reason, such vectors and networks will be called reference vectors and reference networks. The set of all reference networks (it is denoted by $\Im$) is the union of the sets of reference vectors and networks for all *N*, beginning from $N = 5$ because these sets are empty for $N = 3$ and $N = 4$:

$$\Im = \Im^5 \cup \Im^6 \cup \ldots \cup \Im^N \cup \ldots$$

4. **Theorem 5.** *Let $F(N, C, T)$ be a complete N-vertex network with an arbitrary flow schema* **T**. $\phi \in \hat{\Phi}$ *if and only if there exists a $\gamma \in \Im$ such that either $C \equiv \gamma^C$ and $T \equiv \gamma^T$ (and then $\phi = \gamma$) or the network F can be reduced to the network $F'(N', C', T')$, $N' < N$ in such a way that $C \equiv \gamma^C$ and $T' \equiv \gamma^T$ (and then $\phi = \gamma$ after adding zero components in the places of reduced edges).*

Proof. *Necessity.* The necessity follows from the definition of the vectors ϕ and γ and Corollaries 1 and 2 to Lemma 2.

Sufficiency. The sufficiency follows from Theorem 3.

5. Let $F(N, C, T)$ be a multicommodity problem and $\phi \in \hat{\Phi}$. Furthermore, let $F'(N', C', T')$ $(N' \le N)$ be the network *F* reduced with respect to the zero components of ϕ. Therefore, $\phi' \in \hat{\Phi}'$ and $\phi'_i > 0$ for $i = \overline{1, N'}$. By Theorem 3, $\phi' \in \Im^{N'}$. Let c_i^k and t_i^m be the network and flow edges of $F(N, C, T)$ that were merged as the reduction of this network to the edge ϕ'_i was made. In this notation

$$\langle f, \varphi \rangle = \sum_i \phi'_i \left(\sum_k c_i^k - \sum_m t_i^m \right). \tag{1.20}$$

This formula is a generalization of the main result of the theory of flows in networks—the Ford—Fulkerson max flow min cut theorem for the single-commodity problem. The cut is a simple extreme vector $\phi \in \hat{\Phi}$. The reduction of every network $F(N, C, T)$ with respect to the zero components of the vector representing the cut gives a network consisting of a single edge; therefore, formula (1.20) is an independent proof of the max flow min cut theorem.

6. Consider the following question: which networks are purely cut networks, i.e., which networks have only vectors representing cuts as the extreme vectors of adjoint cones?

For the networks represented by complete graphs, the answer is given by Theorem 5. More precisely, this condition is

$$\Im(F(N,C,T)) \bigcap \Im \equiv \varnothing, \tag{1.21}$$

where $\Im(F(N,C,T))$ is the set of networks obtained by reducing the network F and the network F itself.

7. Consider some examples.

Example 1. The cone G^4 is defined in the space R^6, and it has only four extreme vectors (1.10). Therefore, $\Im^4 = \varnothing$.

Example 2. The cone G^5 is defined in the space R^{10}, and it has ten extreme vectors (1.10). The set $\Im^5$ is not empty and has a single type of reference vector. $\gamma^5 \in \Im^5$, $\gamma^5 = (\gamma_{12}^5 = \gamma_{23}^5 = \gamma_{13}^5 = \gamma_{45}^5 = 2$, $\gamma_{14}^5 = \gamma_{15}^5 = \gamma_{24}^5 = \gamma_{25}^5 = \gamma_{34}^5 = \gamma_{35}^5 = 1)$. The flow schema of this reference vector consists of a triangle and an edge not connected to this triangle. All paths belonging to the set L^γ, i.e., the paths satisfying the condition $\langle \ell, \gamma \rangle = 0$, consist of two segments. The vector with the flow schema in the form of the five-vertex cycle is not included in the set $\Im^5$.

Example 3. One of the elements of the set $\Im^6$—$\gamma^{1,6} \in \Im^6$—has the following components: $\gamma_{14}^{1,6} = \gamma_{15}^{1,6} = \gamma_{16}^{1,6} = \gamma_{24}^{1,6} = \gamma_{25}^{1,6} = \gamma_{26}^{1,6} = \gamma_{34}^{1,6} = \gamma_{36}^{1,6} = 1$, $\gamma_{12}^{1,6} = \gamma_{13}^{1,6} = \gamma_{23}^{1,6} = \gamma_{45}^{1,6} = \gamma_{46}^{1,6} = \gamma_{56}^{1,6} = 2$, $\gamma_{35}^{1,6} = 3$. The flow schema includes three edges—$\gamma_{12}^{1,6} = \gamma_{46}^{1,6} = 2$ and $\gamma_{35}^{1,6} = 3$. For these edges, the set L^γ does not include triangles in which these edges appear with the negative sign. The set L^γ includes not only two-segment paths, but also three-segment paths. In these paths, the chords are the edges $\gamma_{13}^{1,6} = \gamma_{23}^{1,6} = \gamma_{45}^{1,6} = \gamma_{56}^{1,6} = 2$. Note that the incomplete network $F(6, C, T)$, in which the flow schema consists of three disconnected edges and the set of network edges consists of eight elements, i.e., it does not contain chord–edges of three-segment paths, has the vector $\phi \in \hat{\Phi}$ such that $\tilde{\phi} = \gamma^{1,6}$.

Supposedly, the vectors with flow schemata in the form of the complete four-vertex graph and an edge not connected to it and in the form of two disconnected triangles are elements of the set $\Im^6$. The vector having the flow schema in the form of the six-vertex cycle is not included in the set $\Im^6$.

CONCLUSIONS

1. The representation of networks in terms of the theory of convex polyhedral cones shows that the feasibility problem of a multicommodity flow is reduced to checking if a given network vector belongs to a given cone defined by the set of all paths in the network and the set of positive unit vectors of the network edges. One way of checking this property is to check if the scalar products of the network vector with the extreme vectors of the adjoint cone are nonnegative. This method was proposed in [1], even though no proof was given and no references to the theory of convex cones were made. In the current paper, all necessary proofs were given. A feature (and difficulty) of the theoretical comprehension of the problem is, in the author's opinion, the fact that the criterion in [1] is universal, and it is independent of the characteristics of a particular network, even though it is known that the direct and adjoint cones are in one-to-one correspondence. In this paper, it was shown that this difficulty can be resolved by introducing the concept of reference networks and reference vectors and by establishing the relationship between these objects and the extreme vectors of the adjoint cone of each particular network.

2. The results obtained in this paper are purely theoretical; they do not affect the available computational methods for solving the feasibility problem and its modifications (e.g., see [6–8]) such as linear programming and integer linear programming (e.g., see [3, 9]) or the heuristic method [10].

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