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Dedicated to the 100th birthday of Academician N.N. Moiseev

Primal Newton Method for the Linear Cone Programming Problem

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Abstract—A linear cone programming problem containing among the constraints a second-order cone is considered. For solving this problem, a primal Newton method which is constructed with the help of the optimality conditions is proposed. Local convergence of this method is proven.

Keywords: linear cone programming problem, second-order cone, primal Newton method.

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INTRODUCTION

Problems of mathematical programming and optimal control were an important area of N.N. Moiseev's research. He made a special contribution to the development of efficient numerical methods for finding their solutions (see [1–3]). Recently, new formulations of optimization problems appeared, which required new numerical methods. The linear cone programming problems are among such formulations. They are more general in comparison with linear programming problems; the requirement of nonnegativity of variables is replaced in them by the belonging of these variables to some convex cone. Of particular interest are problems with a second-order cone (the Lorentz cone) and semidefinite programming problems, the cone in which is the cone of positive-semidefinite matrices (see [4]). Since recently, all these problems attract increased attention. This is related both to theoretical issues and to the possibility of reducing many other optimization problems, including combinatorial optimization problems, to cone programming and semidefinite programming problems (see [5]).

Numerical methods for solving cone programming problems are usually constructed by adjusting to these problems appropriate linear programming methods. The most popular of them are the primal-dual interior point methods (see [4, 6, 7]). There are also simplex-type methods (see [8–10]). In this paper, we consider the general cone programming problem in which a part of the variables belongs to the nonnegative orthant of the space and the other part, to the direct product of second-order cones. In this paper, for solving this problem, a primal Newton-type method is proposed. Earlier, in [11], this method was considered in connection with the linear semidefinite programming problem.

The work consists of three sections. Section 1 gives the optimality conditions for cone programming problems. Section 2 gives the formulation of the general cone programming problem containing second-order cones. Finally, in Section 3, a numerical method is developed and its convergence is proven. Symbol I_s is used to denote a unit matrix of order s, and symbol 0_s denotes an s-dimensional zero vector. The diagonal matrix with a vector x on the diagonal is denoted by $\mathrm{Diag}(x)$.

1. OPTIMALITY CONDITIONS IN CONE PROGRAMMING

Suppose that, in $\mathbb{R}^n$, there is a convex closed cone K with a nonempty interior $K_0 = \mathrm{int}\, K$. We also assume that the cone K is directed in the sense that $K \cap (-K) = \{0_n\}$. Then, K defines a partial order: namely, $x_1 \succeq_K x_2$ if $x_1 - x_2 \in K$. The strict inequality $x_1 \succ_K x_2$ means that $x_1 - x_2 \in \mathrm{int}\, K$.

Suppose, in addition, that there are an $m \times n$-matrix $\mathcal{A}$ and nonzero vectors $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. The linear cone programming problem in the standard formulation is

$$\min \langle c, x\rangle, \quad \mathcal{A}x = b, \quad x \succeq_K 0_n. \tag{1}$$

The problem dual to (1) is

$$\max \langle b, u\rangle, \quad v = c - \mathcal{A}^\top u, \quad v \succeq_{K^*} 0_n, \tag{2}$$

where $K^* = \{y \in K : \langle x, y\rangle \geq 0\}$ is the dual cone to K. It is assumed that both problems (1) and (2) have solutions and the rows of the matrix $\mathcal{A}$ are linearly independent.

The necessary and sufficient optimality conditions for the pair of problems (1) and (2) are written in the form of the equalities (see [12])

$$\langle x, v\rangle = 0, \quad \mathcal{A}x = b, \quad v = c - \mathcal{A}^\top u, \tag{3}$$

in which $x \succeq_K 0_n$ and $v \succeq_{K^*} 0_n$, where the first equality can be replaced by n other equalities.

Further on, we assume that the cone K is self-dual, i.e., $K^* = K$ and consider two examples of problems with such cones (see [12]).

Example 1. $K = \mathbb{R}^n_+$. This cone is polyhedral; for any vector $x \in \mathbb{R}^n$, we have the expansion $x = \alpha^1 e_1 + \dots + \alpha^n e_n$, where e_i is the ith unit vector and $\alpha^i \in \mathbb{R}$, $1 \leq i \leq n$. A point x belongs to $\mathbb{R}^n_+$ if and only if $\alpha^i \geq 0$ for all $1 \leq i \leq n$. Problem (1) with the cone $\mathbb{R}^n_+$ is an ordinary linear programming problem.

Define in $\mathbb{R}^n$ the Hadamard product $x \circ y$ of vectors $x, y \in \mathbb{R}^n$:

$$x \circ y = \left[x^1 y^1, \dots, x^n y^n\right]^\top, \tag{4}$$

and the square $x^2 = x \circ x$. Then, we can consider a quadratic transformation $x = \xi(y) = \frac{1}{2} y^2$. Obviously, the cone $\mathbb{R}^n_+$ is the image of the entire space $\mathbb{R}^n$ under the mapping $\xi(y)$. The first equality in (3) for $x \geq 0_n$ and $v \geq 0_n$ is satisfied if and only if $x \circ v = 0_n$. Taking the matrix $G(x) = \mathrm{Diag}(x)$, we obtain

$$x \circ v = G(x)v = G(v)x = G(x)G(v)\bar{e}, \tag{5}$$

where $\bar{e}$ is the n-dimensional vector with all components equal to one.

Example 2. $K = K^n_2$, where K^n_2 is a second-order cone (the Lorentz cone) in the space $\mathbb{R}^n$, defined as

$$K = \left\{x = [x^0, \bar{x}] \in \mathbb{R} \times \mathbb{R}^{n-1} : x^0 \geq \|\bar{x}\|\right\}. \tag{6}$$

The norm in (6) is the Euclidean norm. Any vector $x \in \mathbb{R}^n$ can be represented as

$$x = \alpha_p e_p + \alpha_q e_q, \tag{7}$$

where $\alpha_p = x^0 + \|\bar{x}\|$, $\alpha_q = x^0 - \|\bar{x}\|$,

$$e_p = \frac{1}{2}\begin{bmatrix} 1 \\ \dfrac{\bar{x}}{\|\bar{x}\|} \end{bmatrix}, \quad e_q = \frac{1}{2}\begin{bmatrix} 1 \\ -\dfrac{\bar{x}}{\|\bar{x}\|} \end{bmatrix}. \tag{8}$$

The vectors e_p and e_q belong to the boundary of the cone K^n_2. A vector x from $\mathbb{R}^n$ belongs to the cone K^n_2 if and only if $\alpha_p \geq 0$ and $\alpha_q \geq 0$.

Now consider instead of (4) the product of the vectors $x, y \in \mathbb{R}^n$:

$$x \circ y = \begin{bmatrix} x^\top y \\ x^0 \bar{y} + y^0 \bar{x} \end{bmatrix}. \tag{9}$$

Let $x^2 = x \circ x$, where the product of vectors is now determined according to (9). It is easy to check that the cone K_2^n is the image of the space $\mathbb{R}^n$ under the quadratic transformation $x = \xi(y) = \frac{1}{2}y^2$. We again find that the first equality in (3) for $x \succeq_{K_2^n} 0_n$ and $v \succeq_{K_2^n} 0_n$ is satisfied if and only if $x \circ v = 0_n$. Formula (5) is preserved if, for the matrix $G(x)$, we take the matrix $\mathrm{Arr}(x)$ having the form

$$G(x) = \mathrm{Arr}(x) = \begin{bmatrix} x^0 & \bar{x}^\top \\ \bar{x} & x^0 I_{n-1} \end{bmatrix}.$$

2. GENERAL CONE PROGRAMMING PROBLEM

Let $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2$, where $\mathcal{K}_1 = \mathbb{R}_+^{n_1}$ and $\mathcal{K}_2 = K_2^{n_2} \times \dots \times K_2^{n_r}$, $r > 2$. In addition, suppose that the vectors c and x are divided into subvectors $c = [c_1; \dots; c_r]$ and $x = [x_1; \dots; x_r]$, where $c_i, x_i \in \mathbb{R}^{n_i}$ and the matrix $\mathcal{A}$ is divided into submatrices:

$$\mathcal{A} = [A_1, \dots, A_r], \quad A_i \in \mathbb{R}^{m \times n_i}, \quad 1 \le i \le r.$$

A semicolon in the enumeration of the components of vectors c and x indicates that one of the components is placed under another.

Consider a cone programming problem (1) in the form

$$\begin{gathered} \min \sum_{i=1}^r \langle c_i, x_i \rangle, \\ \sum_{i=1}^r A_i x_i = b, \quad x_1 \ge 0_{n_1}, \quad x_i \succeq_{K_{n_i}^2} 0_{n_i}, \quad 2 \le i \le r. \end{gathered} \tag{10}$$

The dual to it is the problem

$$\begin{gathered} \max \langle b, u \rangle, \\ v_i = c_i - A_i^\top u, \quad v_1 \ge 0_{n_1}, \quad v_i \succeq_{K_{n_i}^2} 0_{n_i}, \quad 2 \le i \le r, \end{gathered} \tag{11}$$

in which $u \in \mathbb{R}^m$.

Suppose that $n = n_1 + \dots + n^r$ and $v = [v_1; \dots; v_r]$. Denote the feasible set in problem (1) by $\mathcal{F}_P$ and the feasible set in problem (2) by $\mathcal{F}_D$.

In addition, define the product of the vectors x and v from $\mathbb{R}^n$ as follows:

$$x \circ v = [x_1 \circ v_1; \, x_2 \circ v_2; \, \dots; \, x_r \circ v_r], \tag{12}$$

where product (4) is used in the first component $x_1 \circ v_1$ and product (9) is used in all the others. Accoding to formulas (5), the equality $x \circ v = 0_n$ is rewritten as

$$x \circ v = [\mathrm{Diag}(x_1)v_1; \, \mathrm{Arr}(x_2)v_2; \, \dots; \, \mathrm{Arr}(x_r)v_r] = 0_n,$$

and the optimality conditions for problem (10) can be represented in the following form:

$$\begin{gathered} \mathrm{Diag}(x_1)v_1 = 0_{n_1}, \quad \mathrm{Arr}(x_i)v_i = 0_{n_i}; \quad 2 \le i \le r, \\ \sum_{i=1}^r A_i x_i = b, \quad v_i = c_i - A_i^\top u, \quad 1 \le i \le r, \\ x_1 \ge 0_{n_1}, \quad v_1 \ge 0_{n_1}, \quad x_i \succeq_{K_2^{n_i}} 0_{n_i}, \quad v_i \succeq_{K_2^{n_i}} 0_{n_i}, \quad 2 \le i \le r. \end{gathered} \tag{13}$$

Substituting v_i into the first two equalities and multiplying them by the corresponding matrices A_i, we obtain

$$\begin{gathered} A_1 \, \mathrm{Diag}(x_1) A_1^\top u = A_1 \, \mathrm{Diag}(x_1) c_1, \\ A_i \, \mathrm{Arr}(x_i) A_i^\top u = A_i \, \mathrm{Arr}(x_i) c_i, \quad 2 \le i \le r. \end{gathered}$$

Sum these equalities with each other and add to them the third equality from (13) previously multiplied by a certain coefficient $\tau > 0$. As a result, we arrive at an equation for the dual variable u:

$$\Gamma(x)u = A_1 \operatorname{Diag}(x_1)c_1 + \sum_{i=2}^{r} A_i \operatorname{Arr}(x_i)c_i + \tau\left[b - \sum_{i=1}^{r} A_i x_i\right], \tag{14}$$

where

$$\Gamma(x) = A_1 \operatorname{Diag}(x_1)A_1^\top + \sum_{i=2}^{r} A_i \operatorname{Arr}(x_i)A_i^\top.$$

If the matrix $\Gamma(x)$ is not singular, then, resolving Eq. (14), we find

$$u = u(x) = \Gamma^{-1}(x)\left[A_1 \operatorname{Diag}(x_1)c_1 + \sum_{i=2}^{r} A_i \operatorname{Arr}(x_i)c_i + \tau\left(b - \sum_{i=2}^{r} A_i x_i\right)\right],$$

and, for the weak dual variable $v = v(x) = c - A^\top u(x)$, we have

$$v = c - \mathcal{A}^\top\Gamma^{-1}(x)\left[A_1 \operatorname{Diag}(x_1)c_1 + \sum_{i=2}^{r} A_i \operatorname{Arr}(x_i)c_i + \tau\left(b - \sum_{i=2}^{r} A_i x_i\right)\right].$$

Then, the first two equalities in (13) after substituting $v(x)$ actually reduce to a system of n nonlinear equations with respect to the variable x, namely:

$$\operatorname{Diag}(x_1)v_1(x) = 0_{n_1}, \quad \operatorname{Arr}(x_i)v_i(x) = 0_{n_i}, \quad 2 \le i \le r. \tag{15}$$

Let us find the conditions under which the matrix $\Gamma(x)$ is not singular. Let $\mathcal{T}_{\mathcal{K}}(x)$ be the tangent space to the cone $\mathcal{K}$ at a point $x \in \mathcal{K}$. Since the cone $\mathcal{K}$ is the Cartesian product of the cones $\mathbb{R}^{n_1}_+$ and $K_2^{n_i}$, $2 \le i \le r$,

$$\mathcal{T}_{\mathcal{K}}(x) = \mathcal{T}_{\mathcal{K}_1} \times \mathcal{T}_{\mathcal{K}_2}, \quad \mathcal{T}_{\mathcal{K}_2} = \mathcal{T}_{K_2^{n_2}} \times \cdots \times \mathcal{T}_{K_2^{n_r}}.$$

Here, $\mathcal{T}_{\mathcal{K}_1}$ is the tangent space to the cone $\mathcal{K}_1 = \mathbb{R}^{n_1}_+$. Accordingly, $\mathcal{T}_{K_2^{n_i}}$ is the tangent space to the cone $K_2^{n_i}$, $2 \le i \le r$. In addition, denote by $\mathcal{N}_{\mathcal{A}}$ the null-space of the matrix $\mathcal{A}$.

Definition 1 (see [12]). A point $x \in \mathcal{F}_P$ is said to be *nondegenerate* if $\mathcal{T}_{\mathcal{K}}(x) + \mathcal{N}_{\mathcal{A}} = \mathbb{R}^n$.

Let $x = [x_1; \bar{x}]$, where $\bar{x} = [x_2; \ldots; x_r]$. Divide the vector x_1 into two parts. The vector x_1^B includes the positive components of the vector x_1, and the vector x_1^N includes the remaining zero components. Divide the vector $\bar{x}$ into three parts, $\bar{x}^B$, $\bar{x}^I$, and $\bar{x}^Z$. Assign to $\bar{x}^B$ those nonzero components x_i of the vector $\bar{x}$ that belong to the boundary of the cone $K_2^{n_i}$. On the contrary, to $\bar{x}^I$, assign the components $x_i \in \operatorname{int} K_2^{n_i}$. All components $x_i = 0_{n_i}$ comprise the vector $\bar{x}^Z$. Without loss of generality, we assume that

$$x_1 = \left[x_1^B; x_1^N\right], \quad \bar{x} = \left[\bar{x}^B; \bar{x}^I; \bar{x}^Z\right], \quad A_1 = [A_1^B, A_1^N], \tag{16}$$

and the number of columns in the matrix A_1^B is equal to the number of components of the vector x_1^B.

If $x_i \in K_2^{n_1}$, then we can apply to it expansion (7) in which $\alpha_p \ge 0$ and $\alpha_q \ge 0$. If, in addition, x_i is a non-zero vector belonging to the boundary $\partial K_2^{n_i}$ of the cone $K_2^{n_i}$, then one component, α_p or α_q, is positive and the other is zero. The vectors $\sqrt{2}e_p$ and $\sqrt{2}e_q$ are eigenvectors of the matrix $\operatorname{Arr}(x_i)$. The eigenvalues corresponding to them are α_p and α_q. Let Q_i be an orthogonal matrix of the form

$$Q_i = \left[\sqrt{2}e_p, H_i, \sqrt{2}e_q\right], \tag{17}$$

where H_i is a $n_i \times (n_i - 2)$-matrix all columns of which are orthogonal to the vectors e_p and e_q. Then, any column of the matrix Q_i is an eigenvector of the matrix $\operatorname{Arr}(x_i)$ and the columns of the matrix H_i correspond to the same eigenvalue equal to x_i^0. Denote by Q_i^L the left submatrix of the $n_i \times (n_i - 1)$-matrix Q_i, i.e., in other words, the matrix Q_i from which the right column is removed.

Assume that $x_i \in \partial K_2^{n_i}$ and, for definiteness, that $\alpha_p > 0$ and $\alpha_q = 0$. Then, $\alpha_p = 2x_i^0$ and

$$\mathcal{T}_{K_2^{n_i}}(x_i) = \left\{y \in \mathbb{R}^{n_i} : \langle e_q, y\rangle = 0\right\}.$$

If $x_i \in \operatorname{int} K_2^{n_i}$, then $\mathcal{T}_{K_2^{n_i}}(x_i)$ coincides with the entire space $\mathbb{R}^{n_i}$.

Let n_1^B be the number of positive components of the vector x_1. Let, in addition, r_B be the number of vectors $x_i \in \partial K_2^{n_i}$ and r_I be the number of vectors $x_i \in \operatorname{int} K_2^{n_i}$. We have the following nondegeneracy criterion in problem (10) (see, e.g., [12]).

Assertion 1. *A point $x = [x_1; \overline{x}] \in \mathcal{F}_P$ in which x_1 and $\overline{x}$ have form* (16) *is nondegenerate if and only if the rows of the matrix*

$$\left[A_1^B,\ A_2Q_2^L,\ \ldots,\ A_{1+r_B}Q_{1+r_B}^L,\ A_{2+r_B},\ \ldots,\ A_{1+r_B+r_I}\right] \tag{18}$$

are linearly independent.

According to this criterion, at a nondegenerate point $x \in \mathcal{F}_P$, we must have the following inequality: $m \le \sum_{i=2}^{1+r_B+r_I} n_i + n_1^B - r_B$.

Assertion 2. *If a point $x \in \mathcal{F}_P$ is nondegenerate, then the matrix $\Gamma(x)$ is nonsingular.*

Proof. Substitute instead of the matrices $\operatorname{Arr}(x_i)$, $2 \le i \le r_I$, their factorizations using the orthogonal matrices Q_i from (17). Then, taking into account Assertion 1, we find that $\Gamma(x)$ is a Gram matrix composed for linearly independent vectors. Therefore, it is nonsingular. The assertion is proven.

Assertion 3. *Let x_* be a nondegenerate solution of problem* (10) *and $[u_*, v_*]$ be the solution of dual problem* (11). *Then, $u(x_*) = u_*$ and $x_* \circ v(x_*) = 0_n$.*

Proof. From the duality condition, it follows that $x_* \circ v_* = 0_n$. On the other hand, the vector $u = u(x_*)$ must satisfy system (14) in which $x = x_*$. Substituting $u = u_*$, we find that the given vector satisfies this system. Since the matrix of the system is nonsingular, its solution is unique. Hence, $u(x_*) = u_*$ and $v(x_*) = v_*$. Therefore, we have the equality $x_* \circ v(x_*) = 0_n$. The assertion is proven.

3. ITERATIVE PROCESS

Suppose that problem (10) is nondegenerate, i.e., all points $x \in \mathcal{F}_P$ are nondegenerate. Then, by Assertion 2, the matrix $\Gamma(x)$ is nonsingular at any point $x \in \mathcal{F}_P$. Due to its continuity, it will remain nonsingular also in some neighborhood of the feasible set $\mathcal{F}_P$. By Assertion 3, the solution of problem (10), the point x_* satisfies the system of equations (15).

Let us apply Newton's method to solving system (15). Let $W(x)$ be the block-diagonal matrix with blocks

$$\operatorname{Diag}(x_1), \quad \operatorname{Arr}(x_2), \ldots, \operatorname{Arr}(x_r).$$

Then we come to the following iterative process:

$$x_{k+1} = x_k - \Lambda^{-1}(x_k)W(x_k)v(x_k), \tag{19}$$

where $\Lambda(x)$ is the Jacoby matrix of the vector function $F(x) = W(x)v(x)$.

Assertion 4. *Let $x \in \mathcal{F}_P$ be a nondegenerate point. Then,*

$$\Lambda(x) = \left(I_n - \mathcal{P}(x)\right)W(v(x)) + \tau\mathcal{P}(x), \tag{20}$$

where $P(x) = W(x)\mathcal{A}^\top\Gamma^{-1}(x)\mathcal{A}$.

Proof. Equality (14) is actually satisfied as an identity with respect to x and can be represented in the form

$$\mathcal{A}W(x)\left(c - \mathcal{A}^\top u(x)\right) \equiv \tau\mathcal{A}. \tag{21}$$

Differentiating (21) with respect to x and taking into account Assertion 2, we obtain

$$u_x(x) = \Gamma^{-1}\left[\mathcal{A}W(v(x)) - \tau\mathcal{A}\right].$$

After substituting $u_x(x)$ into the Jacobi matrix $\Lambda(x) = W(v(x)) - W(x)\mathcal{A}^\top u_x(x)$, we arrive at (20). The assertion is proven.

If x_* and $[u_*, v_*]$ are the solutions of the primal and dual problems (1) and (2), then, due to (3), we have the complementarity condition $x_* \circ v_* = 0_n$, where the product of vectors is understood in the sense of (12). The strict complementarity condition means that $x_* + v_* \in \operatorname{int} K$. The matrices $\operatorname{Arr}(x_{*,i})$ and $\operatorname{Arr}(v_{*,i})$, $2 \le i \le 1 + r_B$, are mutually commutative. Therefore, in their factorizations, one can use the same orthogonal matrix Q_i.

Below, we will need the definition of nondegeneracy in dual problem (11).

Definition 2 (see [12]). A point $[u, v] \in \mathcal{F}_D$ is said to be *nondegenerate* if $\mathcal{T}_{\mathcal{K}}(v) + \mathcal{R}_{\mathcal{A}^\top} = \mathbb{R}^n$, where $\mathcal{R}_{\mathcal{A}^\top}$ is the space of columns of the matrix $\mathcal{A}^\top$.

Suppose that $v \in \partial\mathcal{K}$ and, for $2 \le i \le 1 + r_B$, v_i admits the representation $v_i = \alpha_p e_p + \alpha_q e_q$, similar to (7) where $\alpha_p = 0$ and $\alpha_q > 0$. We also assume that the matrix $\operatorname{Arr}(v_i)$ admits the decomposition $\operatorname{Arr}(v_i) = Q_i \operatorname{Diag}(\theta_i) Q_i^\top$ in which the orthogonal matrix Q_i has form (17) and θ_i is the vector of eigenvalues of $\operatorname{Arr}(v_i)$. Define $A_i^p = A_i e_{p,i}$, $2 \le i \le 1 + r_B$, and form from vectors A_i^p the matrix $\mathcal{A}_B^p = \left[A_2^p, \ldots, A_{1+r_B}^p\right]$. The matrix $\mathcal{A}_B^p$ has the dimensions $m \times r_B$.

Assertion 5 (see [12]). *A point* $[u, v] \in \mathcal{F}_D$ *at which* $v = [v_1; \bar{v}]$ *and* $\bar{v} = [\bar{v}^B; \bar{v}^Z; \bar{v}^I]$ *is nondegenerate if and only if the columns of the matrix* $\mathcal{A}^p = [A_1^B, \mathcal{A}_B^p, \mathcal{A}_I]$ *are linearly independent.*

Assertion 5 implies the inequality

$$m \ge n_1^B + r_B + \sum_{i=2+r_B}^{1+r_B+r_Z} n_i.$$

Theorem 1. *Suppose that the solutions* x_* *and* $[u_*, v_*]$ *of the primal and dual problems* (10) *and* (11) *are nondegenerate. Suppose, in addition, that* x_* *and* $[u_*, v_*]$ *satisfy the strict complementarity condition. Then, iterative process* (19) *converges locally to* x_* *with a superlinear rate.*

Proof. Let us show that a homogeneous system of linear equations $\Lambda(x)y = 0_n$, where $x = x_*$, has only the trivial solution $y = 0_n$.

Without loss of generality, suppose that the components x_1 and $\bar{x}$ of the vector x admit representation (16) and the numbers of components x_1^B, $\bar{x}^B$, $\bar{x}^I$, and $\bar{x}^Z$ are n_1^B, r_B, r_I, and r_Z, respectively. If we introduce a similar representation for the vector $v = [v_1; \bar{v}]$, where $v = v_*$, then both subvectors v_1 and $\bar{v}$ can be represented as $v_1 = [v_1^B; v_1^N]$ and $\bar{v} = [\bar{v}^B; \bar{v}^Z; \bar{v}^I]$. According to the strict complementarity condition, the number of zero components of the vector v_1 that compose the subvector v_1^B is equal exactly to n_1^B. For the same reason, the number of vectors entering into the blocks $\bar{v}^B$, $\bar{v}^Z$, and $\bar{v}^I$ are equal to r_B, r_I, and r_Z, respectively. In a similar way, divide the vector y into the subvectors $y = [y_1; \bar{y}]$ and $\bar{y} = [\bar{y}^B; \bar{y}^I; \bar{y}^Z]$ and set

$$n_B = \sum_{i=2}^{1+r_B} n_i, \quad n_I = \sum_{i=2+r_B}^{1+r_B+r_I} n_i, \quad n_Z = \sum_{i=2+r_B+r_I}^{r} n_i.$$

Henceforward, in order to simplify the proof, assume that all components of the vector x_1 are zero; i.e., $x_1 = x_1^N = 0_{n_1}$. Then, $v_1 > 0_{n_1}$. Under the assumptions made, the first and last rows in the block diagonal matrix $W(x)$ are zero but, in the block-diagonal matrix $W(v)$, the first and last diagonal blocks are positive-definite matrices. Since the matrix $\Lambda(x)$ has form (20), this immediately implies that $y_1 = 0_{n_1}$ and $\bar{y}^Z = 0_{n_Z}$.

Now suppose that $\tilde{x}$, $\tilde{v}$, and $\tilde{y}$ are the vectors x, v, and y without the first and last component. Suppose, in addition, that $\tilde{W}(\tilde{x})$ and $\tilde{W}(\tilde{v})$ are the block-diagonal matrices $W(x)$ and $W(v)$ from which the first and last diagonal blocks are removed and $\tilde{\mathcal{A}}$ is the submatrix of $\mathcal{A}$ that does not have submatrices A_1 and $\mathcal{A}_Z = [A_{2+r_B+r_I}, \dots, A_r]$. Then it only remains to show that the system

$$\left[\left(I_{n_B+n_I} - \tilde{\mathcal{P}}(\tilde{x})\right)\tilde{W}(\tilde{v}) + \tau\tilde{\mathcal{P}}(\tilde{x})\right]\tilde{y} = 0_{n_B+n_I}, \tag{22}$$

where $\tilde{\mathcal{P}}(\tilde{x}) = \tilde{W}(\tilde{x})\tilde{\mathcal{A}}^\top\tilde{\Gamma}^{-1}(\tilde{x})\tilde{\mathcal{A}}$ and $\tilde{\Gamma}(\tilde{x}) = \tilde{A}\tilde{W}(\tilde{x})\tilde{\mathcal{A}}^\top$, has only the trivial solution.

The matrix $\tilde{\mathcal{P}}(\tilde{x})$ is idempotent; therefore, multiplying (22) from the left by $\tilde{\mathcal{P}}(\tilde{x})$, we obtain $\tilde{\mathcal{P}}(\tilde{x})\tilde{y} = 0_{n_B+n_I}$, i.e., the vector $\tilde{y}$ must belong to the null-space $\mathcal{N}_{\tilde{\mathcal{P}}}$ of the matrix $\tilde{\mathcal{P}}(\tilde{x})$. Equation (22) reduces to

$$\left(I_{n_B+n_I} - \tilde{\mathcal{P}}(\tilde{x})\right)\tilde{W}(\tilde{v})\tilde{y} = 0_{n_B+n_I}. \tag{23}$$

Let us select in the matrix $\tilde{W}(\tilde{x})$ block-diagonal submatrices $\tilde{W}_B(\bar{x}^B)$ and $\tilde{W}_I(\bar{x}^I)$ including the first, r_B, and subsequent, r_I, diagonal blocks. We do the same with the block-diagonal matrix $\tilde{W}(\tilde{v})$, denoting the selected blocks as $\tilde{W}_B(\bar{v}^B)$ and $\tilde{W}_Z(\bar{v}^Z)$.

The matrix $\tilde{W}_Z(\bar{v}^Z)$ is zero. Thus, (23) splits into two equalities:

$$\left(I_{n_B} - \tilde{W}_B(\bar{x}^B)\tilde{\mathcal{A}}_B^\top\tilde{\Gamma}(\tilde{x})\tilde{\mathcal{A}}_B\right)\tilde{W}_B(\bar{v}^B)\bar{y}^B = 0_{n_B}, \quad \tilde{\mathcal{A}}_I^\top\tilde{\Gamma}(\tilde{x})\tilde{\mathcal{A}}_B\tilde{W}_B(\bar{v}^B)\bar{y}^B = 0_{n_I}, \tag{24}$$

in which $\tilde{\mathcal{A}}_B = [A_2, \dots, A_{1+r_B}]$ and $\tilde{\mathcal{A}}_I = [A_{2+r_B}, \dots, A_{1+r_B+r_I}]$. Here, we also took into account that the matrix $\tilde{W}_I(\bar{x}^I)$ is nonsingular.

Further on, for $2 \le i \le 1 + r_B$, let us use the factorizations

$$\operatorname{Arr}(x_i) = Q_i \operatorname{Diag}\left(\eta_i^1, \dots, \eta_i^{n_i}\right)Q_i^\top, \quad \operatorname{Arr}(v_i) = Q_i \operatorname{Diag}\left(\theta_i^1, \dots, \theta_i^{n_i}\right)Q_i^\top,$$

where Q_i is an orthogonal matrix and $\eta_i = \left[\eta_i^1, \dots, \eta_i^{n_i}\right]$ and $\theta_i = \left[\theta_i^1, \dots, \theta_i^{n_i}\right]$ are the vectors of eigenvalues of the matrices $\operatorname{Arr}(x_i)$ and $\operatorname{Arr}(v_i)$, respectively. Then, denoting by $\mathfrak{Q}_B$ the orthogonal block-diagonal matrix with the blocks $Q_2, \dots, Q_{1+r_B}$ and, by Ω_x and Ω_v, the diagonal matrices with the vectors of eigenvalues $[\eta_2; \dots; \eta_{1+r_B}]$ and $[\theta_2; \dots; \theta_{1+r_B}]$ on the diagonal, respectively, we find that the first equality in (24) can be written in the form

$$\mathfrak{Q}_B\left(I_{n_B} - \Omega_x\mathfrak{Q}_B^\top\tilde{\mathcal{A}}_B^\top\tilde{\Gamma}(\tilde{x})\tilde{\mathcal{A}}_B Q_B\right)\Omega_v\mathfrak{Q}_B^\top\bar{y}^B = 0_{n_B}. \tag{25}$$

Define $\hat{y}^B = \mathfrak{Q}_B^\top\bar{y}^B$. Since $\mathfrak{Q}_B$ is a nonsingular matrix, equality (25) can hold if and only if

$$\left(I_{n_B} - \Omega_x\mathfrak{Q}_B^\top\tilde{\mathcal{A}}_B^\top\tilde{\Gamma}(\tilde{x})\tilde{\mathcal{A}}_B\mathfrak{Q}_B\right)\Omega_v\hat{y}^B = 0_{n_B}. \tag{26}$$

Let, for definiteness, the points $x_i \in \partial K_i$, $2 \le i \le 1 + r_B$, be such that $x_i = \alpha_{p,i}e_{p,i}$, where $\alpha_{p,i} = 2x_i^0 > 0$. Then, $\alpha_{q,i} = 0$ and, for the corresponding eigenvalues of the matrices $\operatorname{Arr}(x_i)$ and $\operatorname{Arr}(v_i)$, by the strict complementarity condition, we have $\eta_i^{n_i} = 0$, $\theta_i^{n_i} > 0$. On the contrary, $\eta_i^1 > 0$, $\theta_i^1 = 0$. Therefore, it follows from (26) that the last components of the vectors $\hat{y}_i$, $2 \le i \le 1 + r_B$, are zero. Henceforward, we will assume that these components are absent in the vector $\hat{y}^B$ and define $\hat{y} = [\hat{y}^B; y^I]$.

Consider a submatrix $\tilde{\mathcal{A}}^Q$ of matrix (18), obtained by removing from it the matrix A_1^B and denoted by $\mathcal{N}_{\mathcal{A}^Q}$ its null-space. The case when $\hat{y} \notin \mathcal{N}_{\mathcal{A}^Q}$ is impossible, since otherwise, from the membership $\hat{y} \in \mathcal{N}_{\tilde{\mathcal{P}}}$, it follows that the rows of the matrix $\tilde{\mathcal{A}}^Q$ are linearly dependent. This contradicts the nondegeneracy of the point x_*. Therefore, $\hat{y} \in \mathcal{N}_{\mathcal{A}^Q}$.

Since $\tilde{y} \in \mathcal{N}_{\tilde{\mathcal{P}}}$, based on (23), we have the equalities

$$\left\langle \tilde{y}, \left(I_{n_B+n_I} - \tilde{\mathcal{P}}(\tilde{x})\right)\tilde{W}(\tilde{v})\tilde{y}\right\rangle = \left\langle \tilde{y}, \tilde{W}(\tilde{v})\tilde{y}\right\rangle = \left\langle \hat{y}^B, \Omega_v \hat{y}^B\right\rangle = 0.$$

Hence, since the components θ_i^1, $2 \le i \le r_B$, of the diagonal matrix Ω_v are equal to zero and all others are strictly positive, we find that, in the vectors y_i, $2 \le i \le r_B$, only the first components can be nonzero.

Thus, if we assume that the vector $\hat{y}$ is nonzero, then the columns of the matrix $[\mathcal{A}_B^p, \mathcal{A}_I]$ are linearly dependent. We arrived at a contradiction with the requirement that the solution $[u_*, v_*]$ be nondegenerate in dual problem (11). Therefore, necessarily, $y = 0_n$. Hence, the matrix $\Lambda(x_*)$ is nonsingular and, by Theorem 10.2.2 from [13], iterative process (19) locally converges to the point x_* with a superlinear rate. The theorem is proven.

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