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Dedicated to the 100th birthday of Academician N.N. Moiseev

Nonclassical Transonic Boundary Layers: Toward Overcoming Dead-End Situations in High-Speed Aerodynamics

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Abstract—Analytical models of unsteady free viscous-inviscid interaction of gas flows at transonic speeds, i.e., a transonic boundary layer with self-induced pressure (nonclassical boundary layer) are considered. It is shown that an adequate flow model can be constructed by applying methods of singular perturbations. The results of a comparative analysis of classical and regularized stability models for a boundary layer with self-induced pressure in the case of interaction at transonic speeds are overviewed.

Keywords: boundary layer, flow stability, transonic flow, asymptotic expansions.

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INTRODUCTION

Prandtl's boundary layer theory has facilitated the solution of numerous important problems in aeromechanics (see [1]), but, over time, its limitations have become clear, specifically, for flows with large spanwise variations of flow variables.

A natural step in the development of mathematical simulation of high-speed gas flows was the concept of free viscous-inviscid interaction (also known as the triple-deck model) proposed by Neiland [2] and Stewartson [3]. Essentially, the concept takes into account the influence exerted by the boundary layer on the pressure variation in the external flow (boundary layer with self-induced pressure; it is convenient to refer to it as nonclassical to distinguish it from the Prandtl boundary layer).

Proposed initially for supersonic flows, the model was extended to the transonic regime in [4]. A further analysis revealed that, in the derivation of the triple-deck equations, a choice has to be made between the preservation of nonlinearity of the equations in the upper deck and allowance for the unsteadiness of the flow in the lower deck (see [5]). The latter version was used in [5] to analyze the stability of the boundary layer with respect to small perturbations being (i.e., a linear problem was solved). An overview of works concerning the study of transonic boundary layers can be found in [6].

Based on the studies in [7, 8], it has been recognized that the classical model of unsteady transonic flow and the unsteady free viscous-inviscid interaction for the transonic regime are degenerate (see [9]). Specifically, in the study of the unsteady transonic interaction, the external inviscid flow (upper deck) was traditionally simulated using the Lin–Reissner–Tsien (LRT) equation [10]. This equation has undoubted advantages (describes both super- and subsonic regions of transonic flows and applies to multidimensional, unsteady, and nonlinear transonic flows), but also has limitations preventing the correct description of unsteady perturbations propagating in the flow: this equation is a degenerate hyperbolic one and describes unsteady perturbations in the flow field that propagate only upstream (see [7]). Physically, this phenomenon is not associated with viscous forces, but has a different nature, namely, the wide difference between the speeds of small perturbations propagating downstream and upstream. The traditional transonic expansion used in transonic flow problems leads to singular perturbations, a circumstance that seems to explain the later detection of the above degeneracy.

In this context, a modified (regularized) model was proposed for boundary layer problems with an interaction at transonic speeds in [9]. The modification means that a singular term of the transonic expansion with a second time derivative is retained (thus, appears naturally) in the LRT equation in its derivation from the complete potential equations. The equation thus obtained will be referred to as the modified LRT equation. This is a nondegenerate hyperbolic equation describing flow perturbations propagating in all directions.

The modified model leads not only to quantitative, but also to qualitative changes in the flow behavior associated with small perturbations developing in the flow field. The model provides an additional perturbation dropped from the classical model and allows one to determine its behavior. Known from the classical theory, the asymptotic representation for the lower branch of the neutral stability curve is an exceptional case. It is absent in a certain range of free-stream velocities and exists in two versions outside this range (see [11]). The complicated demarcation between the domains of flow stability and instability was earlier known for supersonic flows (see [12]).

The completion of the theory of nonclassical boundary layers is a task for the future. A number of questions regarding the behavior of perturbations remain open, although the first steps toward an understanding of these processes have been taken (see [13]).

The results obtained (see [14]) might be applied, for example, in the creation of devices for boundary layer control, although the conditions for their design appear qualitatively more complicated than it was earlier supposed (see [15]).

1. SIMULATION OF UNSTEADY TRANSONIC FLOW AND THE MODIFIED LIN–REISSNER–TSIEN EQUATION

The conventional transonic approximation is based on expanding flow variables in powers of the small deviation of the current Mach number from unity. In this case, shock waves (if any) remain weak and the vorticity arising in the flow is low; therefore, we can introduce the velocity potential Φ: $u = \Phi_X$, $v = \Phi_Y$, which is governed by an equation following from the equations of motion and continuity:

$$\left(a^2 - \Phi_X^2\right)\Phi_{XX} - 2\Phi_X\Phi_Y\Phi_{XY} + \left(a^2 - \Phi_Y^2\right)\Phi_{YY} - 2\Phi_X\Phi_{XT} - 2\Phi_Y\Phi_{YT} - \Phi_{TT} = 0;$$

here, the speed of sound a is determined by the energy conservation equation

$$\Phi_T + \frac{1}{2}\left(\Phi_X^2 + \Phi_Y^2\right) + \frac{a^2}{\gamma - 1} = \frac{U^2}{2} + \frac{a_\infty^2}{\gamma - 1},$$

where a_∞ and U are the characteristic values of the speed of sound and the free-stream velocity and γ is the ratio of specific heats.

The independent variables are nondimensionalized as follows: $Lx = X$, $Ly = \varepsilon^{1/3}Y$, and $t = \varepsilon^{2/3}\frac{U}{L}T$, where T and L are the characteristic time and length and $\varepsilon \ll 1$. The transonic flow is represented as a perturbation of the one-dimensional flow along the x axis with a constant velocity U:

$$\Phi = U\left(Lx + \varepsilon^{2/3}L\phi(x, y, t) + \ldots\right), \quad \frac{\Phi_X}{U} = 1 + \varepsilon^{2/3}\phi_x + \ldots,$$

$$\frac{\Phi_Y}{U} = \varepsilon\phi_y + \ldots, \quad \frac{\Phi_T}{U^2} = \varepsilon^{2/3}\phi_t + \ldots.$$

For higher order terms, i.e., terms of order $O\left(\varepsilon^{4/3}\right)$, we obtain

$$\left(K_\infty - (\gamma + 1)\phi_x\right)\phi_{xx} + \phi_{yy} - 2\phi_{xt} = 0, \quad K_\infty = \frac{1 - M_\infty^2}{\varepsilon^{2/3}}. \tag{1.1}$$

Equation (1.1) is known in gas dynamics as the LRT equation (see [10]). Although this equation retains many important features of transonic flows (nonlinearity, higher-than-one dimensionality, and the ability to describe the entire transonic range of speeds, i.e., the subsonic and supersonic flow regions), it has serious disadvantages. For example, the equation leads to infinite velocities of weak unsteady perturbations

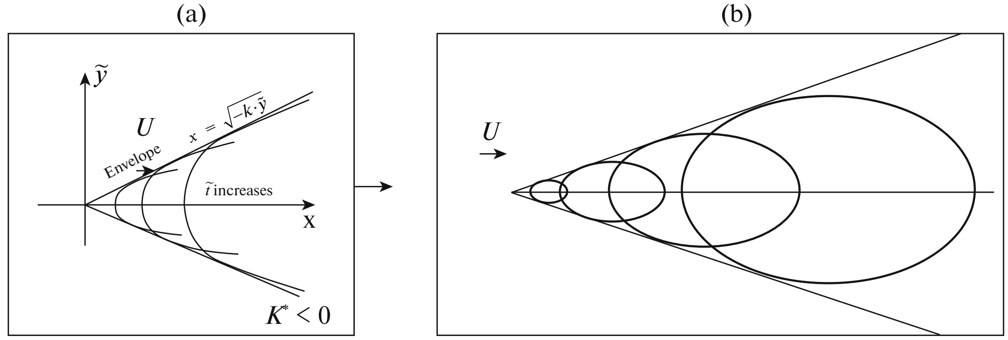

Fig. 1. Patterns of wavefronts of weak perturbations propagating from a point source carried away by the free stream.

propagating downstream, the wavefronts of perturbations generated by a point source represent open curves (parabolas) at all times, the equation fails to describe high-frequency unsteady perturbations, and the Cauchy problem for this equation is ill posed (see [16]). However, the LRT equation is a kind of limit: within the framework of regular expansions, nothing more general can be obtained. A way out of this situation was found by retaining a singular term of the transonic expansion (see [7]) (the resulting equation is called the modified LRT equation):

$$(K_\infty - (\gamma + 1)\phi_x)\phi_{xx} + \phi_{yy} - 2\phi_{xt} - \varepsilon\phi_{tt} = 0. \tag{1.2}$$

The characteristics in the one-dimensional problem have the form

$$\frac{dX}{dT} = u + a, \quad \frac{dX}{dT} = u - a,$$

the former corresponding to a perturbation propagating downstream, and the latter, to a perturbation propagating upstream. Using the expansions $u = U\left(1 + \varepsilon^{2/3}\phi_x\right)$ and $a = U\left(1 + \varepsilon^{2/3}K_\infty\right)$, where $\varepsilon \ll 1$, we have

$$u + a = U\left(2 + \varepsilon^{2/3}\phi_x + \varepsilon^{2/3}K_\infty\right), \quad u - a = \varepsilon^{2/3}U\left(\phi_x - K_\infty\right), \quad \frac{dx}{dt} = \varepsilon^{2/3}U.$$

It can be seen that the ratio of the downstream to upstream propagation speeds is a quantity on the order of $\varepsilon^{-2/3}$, which tends to infinity as $\varepsilon \to 0$. Accordingly, it becomes clear that only perturbations propagating upstream are considered when the transonic expansion is used.

The characteristic surfaces for hyperbolic equation (1.2) have the form

$$K_\infty t^2 + 2t(x - x_0) - \varepsilon(x - x_0)^2 = (y - y_0)^2(1 + \varepsilon K_\infty). \tag{1.3}$$

Setting, for simplicity, $x_0 = y_0 = 0$ and neglecting εK_∞, we transform Eq. (1.3) to the form

$$y^2 + \varepsilon\left(x - \frac{t}{\varepsilon}\right)^2 = \left(K_\infty + \frac{1}{\varepsilon}\right)t^2,$$

which is similar to the equation for the characteristic front of a weak perturbation determined by the wave equation in a moving medium: $y^2 + (x - ut)^2 = a^2t^2$. In the case under consideration, it can be seen that the velocity of the basic flow carrying the weak perturbation downstream is $\frac{1}{\varepsilon}$, and the propagation velocity of the perturbation front corresponds to $\sqrt{K_\infty + \frac{1}{\varepsilon}}$. As $\varepsilon \to 0$, these velocities become infinite. Adding up the velocities of the upstream-moving perturbation front segment and the unperturbed flow carrying it away yields a finite velocity, while the perturbation front segments propagating downstream go to infinity and Eq. (1.3) degenerates into a parabola in the plane $t = \text{const}$ (Fig. 1a). For $K_\infty = \text{const}$, Eq. (1.3) at every fixed t represents an ellipse in the x, y plane (Fig. 1b).

In the case $\varepsilon = 0$, Eq. (1.3) (let $x_0 = y_0 = 0$) passes into the equation

$$K_\infty t^2 + 2tx = y^2, \tag{1.4}$$

whose solutions for a given t represent parabolas in the x, y plane and imply that the wavefronts of weak perturbations propagating downstream have infinite velocities, which are physically unreal. Note that, for $\varepsilon = 0$, the characteristic surfaces are also the planes $t = \text{const}$. A consequence of this fact is that the Cauchy problem for the LRT equation is ill posed.

From Eq. (1.4), we can derive an equation for the envelope of wavefronts with varying time t:

$$x = \pm\sqrt{-K_\infty y}. \tag{1.5}$$

Equation (1.5) is meaningful only for $K_\infty < 0$ (the flow velocity exceeds the speed of sound) and determines the Mach angle, which is well-known in gas dynamics (Fig. 1).

Let us analyze the stability of the steady transonic flow with slowly varying parameters with respect to linear harmonic perturbations $\phi \sim \exp(i(-\omega t + kx + ly))$ (where ω is the frequency and k is the wave number). For the modified LRT equation, the dispersion relation is given by

$$-K^{*0}k^2 - ik(\gamma+1)\phi^0_{xx} - l^2 - 2k\omega + \varepsilon\omega^2 = 0.$$

Its roots have the form

$$\omega_1 = 2\frac{k}{\varepsilon} + \frac{k}{2}\Lambda, \quad \omega_2 = -\frac{k}{2}\Lambda, \quad \Lambda = K_\infty + \frac{i}{k}(\gamma+1)\frac{\partial^2\phi^0}{\partial x^2} + \left(\frac{l}{k}\right)^2; \tag{1.6}$$

here, the superscript zero denotes the free-stream condition. As $\varepsilon \to 0$, the root ω_1 is dropped and perturbations propagating downstream cannot be analyzed. Meanwhile, it follows from (1.6) that the imaginary part of the roots, which corresponds to the growth (decay) of perturbations, depends on the character of the unperturbed flow (on $\partial^2\phi^0/\partial x^2$). The roots ω_1 and ω_2 have opposite signs, and, for $\partial^2\phi^0/\partial x^2 > 0$, the root dropped at $\varepsilon \to 0$ corresponds to a wave with an increasing amplitude ($\operatorname{Im}\omega_1 > 0$), while the remaining root corresponds to a damped wave ($\operatorname{Im}\omega_2 < 0$).

2. MODIFIED TRIPLE-DECK MODEL OF THE UNSTEADY FREE VISCOUS-INVISCID INTERACTION AT TRANSONIC SPEEDS

For a transonic boundary layer, the triple-deck equations were derived in [4] by applying the method of matched asymptotic expansions in the limit of infinitely high Reynolds numbers.

In the asymptotic theory of a self-induced boundary layer, the small parameter is $\varepsilon = \mathrm{Re}^{-1/8}$. This quantity corresponds to the order of the amplitude of a pressure perturbation arising somehow in the boundary layer. This choice is justified by an argument dating back to [17].

In Reynolds' original experiments (in 1883), the Reynolds number was defined using the characteristic spanwise size of the problem (the pipe diameter in Reynolds' experiments). In this paper, the Reynolds number is introduced using a streamwise size (the length of the flow segment under study). It is well known that the characteristic lengths for the boundary layer on a flat plate satisfy the relation

$$\frac{d}{L} = \frac{1}{\sqrt{\mathrm{Re}}}$$

(see [18]). As a result, the asymptotic expressions obtained below for the desired flow parameters differ from those based on the classical theory by a factor of $\sqrt{\mathrm{Re}}$. Due to the Reynolds number thus introduced, the domain of applicability of the approximation $\mathrm{Re} \gg 1$ is shifted into the range of flow parameters of interest (corresponding to conventionally introduced Reynolds numbers $\sim 10^5$).

Following [5], we formulate the triple-deck model for the unsteady free viscous-inviscid interaction at transonic speeds. The equations describing the flow in the lower (near-wall) deck have the form of the usual unsteady boundary layer equations for an incompressible fluid:

$$u_x + v_y = 0, \quad p_y = 0, \quad u_t + uu_x + vu_y = -p_x + u_{yy}. \tag{2.1}$$

The middle deck consists of transitional (inviscid rotational) flow, the conditions in which are limiting for the solutions of the boundary layer equations and the solutions of the equations modeling the external inviscid potential flow and are used to match them. The matching conditions have the form (see [4, 5])

$$\phi_x(t,x,0) = -p(t,x), \quad \phi_{y_1}(t,x,0) = -A_x(t,x) \quad \text{as} \quad y_1 \to 0, \tag{2.2}$$

$$u \to y + A(t,x) \quad \text{as} \quad y \to \infty, \tag{2.3}$$

where $\phi(t,x,y)$ is the perturbed velocity potential and $A(t,x)$ is a function signifying the instantaneous displacement of streamlines in the transitional flow. Note that, in the transition to the triple-deck model, the coordinates and time are transformed, so that x and t become identical for all decks, while the spanwise coordinate y_1 in the external flow and the spanwise coordinate y in the boundary layer and the transitional flow have different scales, since these flow regions have different thicknesses.

The inviscid transonic flow far away from the plate can approximately be assumed to be irrotational, so it can be described by the modified linear LRT equation

$$\varepsilon\phi_{tt} + 2\phi_{xt} + K_\infty\phi_{xx} - \phi_{y_1y_1} = 0, \quad \varepsilon \ll 1. \tag{2.4}$$

Equations (2.1)–(2.4) make up the triple-deck model. Since (2.4) is the modified LRT equation, this model will be referred to as *modified*, in contrast to the classical model [4, 5]. By using this model, it is possible to investigate various problems of viscous-inviscid interaction with initial and boundary conditions depending on the problem under consideration.

3. STABILITY OF A TRANSONIC BOUNDARY LAYER WITH SELF-INDUCED PRESSURE

The stability of the boundary layer arising in the flow over a flat plate was studied using the triple-deck model (system (2.1)–(2.4)) under the following conditions. In the near-wall domain, the unperturbed flow was assumed to be one-dimensional and directed along the plate surface, the velocity profile in the unperturbed flow was linear in the spanwise coordinate, and the pressure was set to a constant:

$$u_0 = y, \quad v_0 = 0, \quad p_0 = \text{const}.$$

Additionally, we set the no-slip and impermeability conditions

$$u(t,x,0) = 0, \quad v(t,x,0) = 0,$$

and the absence of incoming perturbations:

$$x \to -\infty: \quad u \to y, \quad v \to 0, \quad p \to \text{const}.$$

The solution was required to be bounded at the exit from the domain under study: $\phi_{y_1} \to 0$, $y_1 \to \infty$.

The initial conditions were specified as

$$t = 0: \quad u = y, \quad v = 0, \quad p = \text{const}.$$

The perturbations of the boundary layer were assumed to be periodic in time t and the coordinate x aligned with the free-stream velocity along the plate surface. The dependence on the spanwise coordinate y was not preliminarily specified: $f(y)\exp(\omega t + ikx)$, where ω is generally a complex number and k is assumed to be a positive real number. Since $\text{Real}\,\omega > 0$, the flow is unstable.

The dispersion relation (DR) in this problem has the form (see [9])

$$\text{Ai}'(\Omega)\frac{1}{\int_\Omega^\infty \text{Ai}(\zeta)d\zeta} = -\frac{(ik)^{5/3}}{\sqrt{\varepsilon\Omega^2 + (ik)^{1/3}\left(\Omega + (ik)^{1/3}K_\infty\right)}}, \quad \Omega = \omega(ik)^{-2/3}, \tag{3.1}$$

where $\text{Ai}(\Omega)$ is the Airy function and the prime denotes the derivative of the function with respect to its argument.

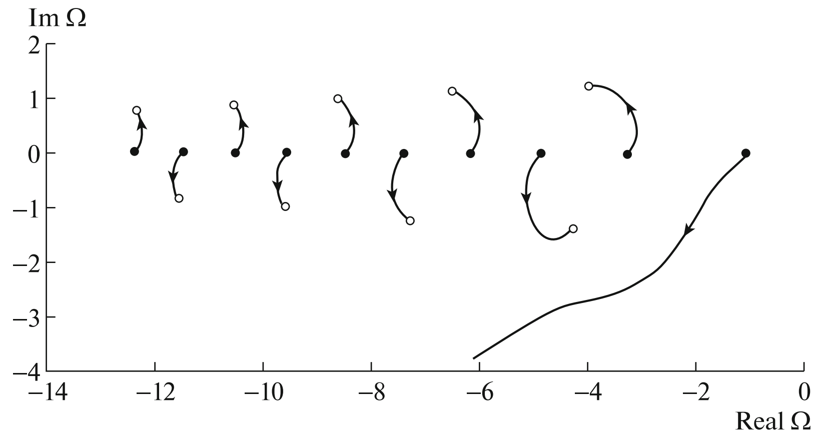

Fig. 2. Typical pattern of dispersion curves for a transonic boundary layer.

DR (3.1) includes, as special cases, the DRs for the earlier studied subsonic and supersonic interactions (in a steady external flow, when Ω on the right-hand side of (3.1) vanishes) with a suitable choice of the parameter K_∞, namely, for $K_\infty = -1$, we obtain DR for supersonic flow (see [15]):

$$\mathrm{Ai}'(\Omega)\frac{1}{\int_\Omega^\infty \mathrm{Ai}(\zeta)d\zeta} = -(ik)^{4/3}, \tag{3.2}$$

while, for $K_\infty = 1$, (3.1) yields DR for a subsonic external flow if $k > 0$ (see [15]):

$$\mathrm{Ai}'(\Omega)\frac{1}{\int_\Omega^\infty \mathrm{Ai}(\zeta)d\zeta} = i^{1/3}k^{4/3}. \tag{3.3}$$

A result of the classical theory (i.e., with $\varepsilon = 0$ in (3.1)) is that the pattern of dispersion curves (for positive, negative, and zero values of K_∞) in the plane Ω represents a countable set of branches going (with k ranging from 0 to $+\infty$) out of zeros of the derivative of the Airy function lying on the negative part of the real axis Real Ω and, after a short path, entering (with the only exception) the zeros of the integral of this function (see [5]). The only branch starts at the first zero of the Airy function derivative and goes to infinity (see Fig. 2). This pattern is common for interacting boundary layers (but, for the supersonic regime, no roots corresponding to growing perturbations were initially found [15]). For the transonic regime, this pattern is supplemented with another feature. On the complex plane of Ω, there are singular lines determined by the factor $\left[\sqrt{\varepsilon\Omega^2 + (ik)^{1/3}\left(\Omega + (ik)^{1/3}K_\infty\right)}\right]^{-1}$, and the values of $(\mathrm{Real}\,\Omega, \mathrm{Im}\,\Omega)$ for which this denominator vanishes have to be excluded from the solutions. In a more specific case, the root $i^{1/3}$ gives six variants satisfying the equation $\varepsilon\Omega^2 + (ik)^{1/3}\left(\Omega + (ik)^{1/3}K_\infty\right) = 0$. On the plane $(\mathrm{Real}\,\Omega, \mathrm{Im}\,\Omega)$, they correspond to three straight lines: $\mathrm{Real}\,\Omega = \pm\sqrt{3}\,\mathrm{Im}\,\Omega$ and $\mathrm{Real}\,\Omega \equiv 0$. Moreover, this takes place for both $\varepsilon = 0$ and $\varepsilon \neq 0$ irrespective of the value and sign of K_∞. A consequence is that there are no dispersion curves passing through the origin $\Omega = 0$.

When the roots of the DR are analyzed in the Ω plane, it should be kept in mind that, since $\omega \equiv (ik)^{2/3}\Omega$, the third-degree root of the imaginary unit plays an important role in determining stable and unstable roots of DR. Specifically,

$$i^{1/3} = \left\{\left(\sqrt{3} + i\right)/2, \left(-\sqrt{3} + i\right)/2, -i\right\},$$

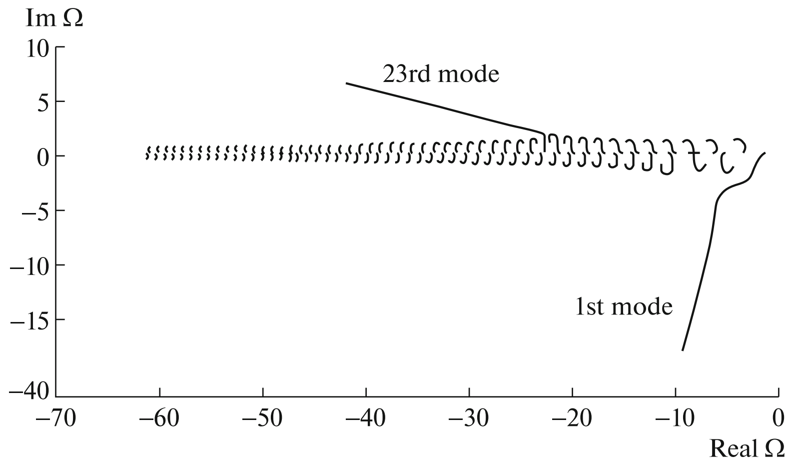

Fig. 3. Hundred modes of dispersion relation (3.1) for $K_\infty = -1$, $\varepsilon = 0.2$.

and the specification of the root values covers all possible combinations of the signs of the real and imaginary parts of the complex roots Ω. Thus, with a suitable choice of the sign of the root $i^{1/3}$, any solution $(\mathrm{Real}\,\Omega, \mathrm{Im}\,\Omega)$ of DR becomes growing.

The high-frequency asymptotic representation of the first (growing) root of DR (3.1) with $\varepsilon = 0$ was obtained (see [5]) in the form

$$\omega = -i\left(k^{5/3} + \frac{1}{3}K_\infty k + \frac{1}{3}K_\infty^2 k^{1/3}\right) + \frac{\sqrt{2}}{3}(1-i)k^{1/6} + \dots.$$

The high-frequency asymptotic expansion of solutions to DR (3.1) can be determined using a relation following from this DR for higher order terms (for $\omega, k \gg 1$), namely,

$$\varepsilon\omega^4 + ik\omega^3 - k^2K_\infty\omega^2 + k^6 = 0.$$

For flows at the speed of sound ($K_\infty = 0$), the classical model ($\varepsilon = 0$) yields the explicit dependence

$$\omega = \sqrt[3]{i}\,k^{5/3}, \tag{3.4}$$

which corresponds to increasing perturbations, since roots (3.4) include ones with $\mathrm{Real}\,\omega = k^{5/3}\cos\frac{\pi}{6} > 0$.

DR (3.1) has a root that is not described by the classical theory; its asymptotic representation for $\Omega \gg 1$ is given by

$$\omega = -\frac{ik}{\varepsilon} + (1-i)\sqrt{\frac{k\varepsilon}{2}}.$$

This root is growing, since $\mathrm{Real}\,\omega = \sqrt{\frac{k\varepsilon}{2}} > 0$. For comparison, we give the asymptotics (as $k \to \infty$) of the unstable root in the case of an incompressible fluid (see [19]):

$$\omega = -ik^2 + (1-i)\frac{\sqrt{2}}{2} + \dots.$$

It can be seen that the free viscous-inviscid interaction in the transonic regime has a difference. Namely, as $k \to \infty$, the rate of increase in the perturbations in this regime grows indefinitely, while remaining a constant in the subsonic boundary layer.

In [20] Chernyshev used the modified model to compute the field of dispersion curves presented in Fig. 3. This pattern differs from the general one for self-induced boundary layers (see Fig. 2) only in the behavior of the 23rd mode.

For large Ω, the asymptotic behavior of the 23rd mode can be examined using expansions of the Airy function for large argument values (see [21]):

$$\frac{d\,\mathrm{Ai}}{d\zeta}\frac{1}{\int_{\Omega}^{\infty}\mathrm{Ai}(\zeta)d\zeta} = -\Omega - \frac{1}{\sqrt{\Omega}} + O\left(\frac{1}{\Omega^2}\right). \tag{3.5}$$

Retaining only the first term of (3.5), we obtain, instead of (3.1), the fourth-order equation $\varepsilon\Omega^4 = (ik)^{10/3}$, whence $\Omega = \pm(ik)^{5/6}$. In the $(\mathrm{Real}\,\Omega, \mathrm{Im}\,\Omega)$ plane, these are the straight lines $(\sqrt{3}-1)\mathrm{Real}\,\Omega = (\sqrt{3}+1)\mathrm{Im}\,\Omega$, $(\sqrt{3}+1)\mathrm{Real}\,\Omega = (\sqrt{3}-1)\mathrm{Im}\,\Omega$, $\mathrm{Real}\,\Omega = -\mathrm{Im}\,\Omega$. The behavior of the last variant is qualitatively similar to the numerically found behavior of the asymptotics of the 23rd mode. Specifically, this curve passes through the second quadrant of the plane $(\mathrm{Real}\,\Omega, \mathrm{Im}\,\Omega)$, but has different slopes with respect to the coordinate axes. No branches of DR (3.1) (analogues of the 23rd mode) have been known previously.

4. ASYMPTOTICS OF THE NEUTRAL STABILITY CURVE FOR A TRANSONIC BOUNDARY LAYER

The use of the modified triple-deck model, instead of the classical one, as applied to nonclassical boundary layer problems leads to corrections in previously obtained asymptotics of the lower branch of the neutral stability curve at large Reynolds numbers (see [11]).

By applying the method from [22], we study neutral perturbations of a transonic boundary layer with the use of modified DR (3.1) written as

$$\mathrm{Ai}'(\Omega)\frac{1}{\int_{\Omega}^{\infty}\mathrm{Ai}(\zeta)d\zeta} = \frac{i^{1/3}k^{4/3}}{\sqrt{\varepsilon c^2 + ic - K_\infty}}, \quad c = \omega/k. \tag{4.1}$$

In [22] the characteristics of a neutral perturbation were determined by comparing the DR obtained for the transonic range with the DR for an incompressible boundary layer. The latter is given by

$$\mathrm{Ai}'(\Omega)\frac{1}{\int_{\Omega}^{\infty}\mathrm{Ai}(\zeta)d\zeta} = |k|(ik)^{1/3} \tag{4.2}$$

(see [23]).

For real $k > 0$, DR (4.2) has a single neutral root (see [23]):

$$k = k_0 = 1.0005, \quad \omega = \omega_0 = 2.298. \tag{4.3}$$

Let the desired neutral value c for the transonic boundary layer differs from $c_0 = \omega_0/k_0$ based on (4.3) by σ times (σ is a positive constant; otherwise, k will be negative). By applying the transformations $c = \sigma c^*$, $k = k^*/\sigma^3$ (note that $\Omega = \Omega^*$), DR (4.1) can be brought to the form

$$|k^*|(ik^*)^{1/3}\frac{1}{\mathrm{Ai}'(\Omega^*)}\int_{\Omega^*}^{\infty}\mathrm{Ai}(\zeta)d\zeta = \sigma^4\sqrt{\varepsilon\sigma^2 c^{*2} + i\sigma c^* - K_\infty}. \tag{4.4}$$

To derive an equation for σ with real coefficient, we make the transformation $c^* = -i\bar{c}^*$. Then

$$\sigma^4\sqrt{\varepsilon\sigma^2 c^{*2} + i\sigma c^* - K_\infty} = \sigma^4\sqrt{-\varepsilon\sigma^2\bar{c}^{*2} + \sigma\bar{c}^* - K_\infty}.$$

By substituting the neutral values of k_0 and c_0 into (4.4), it is easy to obtain the relation

$$\left(-\varepsilon\sigma^2 c_0^{\,2} + \sigma c_0 - K_\infty\right)\sigma^8 = 1, \tag{4.5}$$

which implies that $\sigma = \sigma(K_\infty)$ (c_0 is fixed by (4.3)) is a 10th-order algebraic equation for σ. The classical theory ($\varepsilon = 0$) yields a 9th-order algebraic equation. The order of the equation has increased by one;

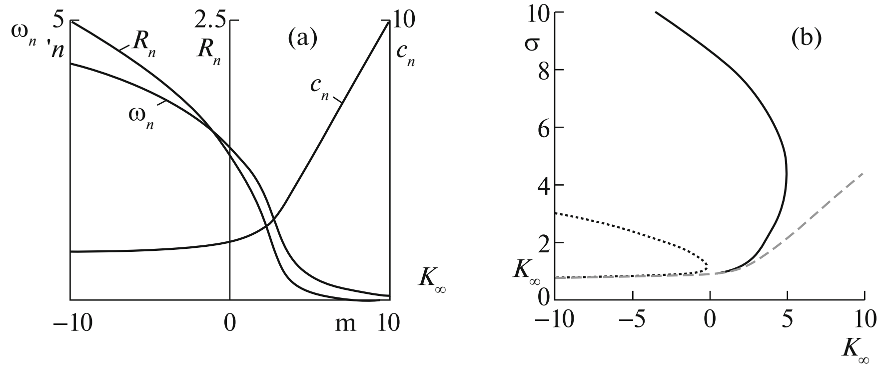

Fig. 4. (a) Neutral values of ω, k, c as functions of the transonic parameter K_∞ (numerical computation) [22] for perturbations of the form $\exp(ik_nx - i\omega_nt)$ and (b) comparison of the results obtained by the numerical integration of (4.6): the classical model, $\varepsilon = 0$ (dashed curve) and the modified model at $\varepsilon = 0.05$ and $\varepsilon = 0.1$ (dotted and solid curves, respectively).

therefore, the modified model give another neutral value $\sigma = \sigma_\varepsilon$ (however many values the classical theory produces). However, this is important only if σ_ε is not imaginary.

It is convenient to rewrite (4.5) as a dependence $K_\infty = K_\infty(\sigma)$, namely,

$$K_\infty = -\varepsilon\sigma^2 c_0{}^2 + \sigma c_0 - \frac{1}{\sigma^8}, \tag{4.6}$$

which, for large σ, approximately yields the parabola $K_\infty = -\varepsilon\sigma^2 c_0{}^2 + \sigma c_0$ with branches directed to the left, because the coefficient of σ^2 is negative. For $\varepsilon = 0$, relation (4.6) approximately gives the linear dependence $K_\infty = \sigma c_0$ everywhere, except for the neighborhood of the axis $\sigma = 0$.

Inspection of Fig. 4b shows that the modified model with K_∞ varying in certain ranges of $K_\infty < K_\infty^0$ yields two values of σ, while a single value of σ exists at the only value $K_\infty = K_\infty^0$. For $K_\infty > K_\infty^0$, there are no roots σ at all.

Quantitatively, the form of the upper parabola branch in Fig. 4b depends strongly on the value of ε. As ε decreases, the solution covers an increasingly larger domain of positive values of K_∞ (corresponding to supersonic speeds). Thus, the modified model shows that the onset of the instability of a transonic boundary layer differs from that predicted by the classical model.

In the case $K_\infty = 0$, since the roots are multiple, the classical model ($\varepsilon = 0$) gives the explicit dependence $\sigma = \dfrac{1}{\sqrt[9]{c_0}}$. For $\varepsilon \neq 0$, some of the roots are no more multiple. The degree of the equation for σ becomes even, but no question arises concerning the sign of the roots: since k is chosen positive, σ cannot be negative.

The modified model ($\varepsilon \neq 0$) gives an additional root $\sigma_\varepsilon \sim \dfrac{1}{\varepsilon c_0}$, which is absent in the case $\varepsilon = 0$. For $K_\infty \neq 0$, we have $\sigma_\varepsilon \sim \dfrac{1 - \varepsilon K_\infty}{\varepsilon c_0}$. It can be seen that $\sigma_\varepsilon \to \infty$ as $\varepsilon \to 0$.

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REFERENCES

1. L. Prandtl, "Über Flüssigkeitsbewegungen bei sehr kleiner Reibung," Verh. Int. Math. Kongr. Heidelberg (1904).
2. V. Ya. Neiland, "Supersonic viscous gas flow near the separation point," *Abstract of Papers of the 3rd All-Union Conference on theoretical and Applied Mechanics, January 25–February 1, 1968* (Nauka, Moscow, 1968), p. 224.
3. K. Stewartson and P. G. Williams, "Self-induced separation," Proc. R. Soc. London, Ser. A **312** (1509), 181–206 (1969).
4. O. S. Ryzhov, "Unsteady boundary layer with self-induced pressure in transonic flow," Dokl. Akad. Nauk SSSR **236** (5), 1091–1094 (1977).
5. O. S. Ryzhov and I. V. Savenkov, "Stability of a boundary layer with transonic external-flow velocities," J. Appl. Mech. Tech. Phys. **31** (2), 222–227 (1990).
6. V. I. Zhuk, *Tollmien–Schlichting Waves and Solitons* (Nauka, Moscow, 2001) [in Russian].
7. A. N. Bogdanov, "Higher approximations of the transonic expansion in problems of unsteady transonic flow," J. Appl. Math. Mech. **61** (5), 775–786 (1997).
8. A. N. Bogdanov, "Modeling the transient regime of a transonic nozzle," Mat. Model. **7** (9), 117–126 (1995).
9. A. N. Bogdanov and V. N. Diesperov, "The simulation of unsteady transonic flow and the stability of a transonic boundary layer," J. Appl. Math. Mech. **69** (3), 358–365 (2005).
10. C. C. Lin, E. Reissner, and H. S. Tsien, "On two-dimensional non-steady motion of a slender body in a compressible fluid," J. Math. Phys. **27** (3), 220–231 (1948).
11. A. N. Bogdanov, V. N. Diesperov, and V. I. Zhuk, "Asymptotics of the lower branch of the neutral stability curve for the transonic interacting boundary layer on a flat plate," Dokl. Phys. **58** (3), 104–106 (2013).
12. S. A. Gaponov and A. A. Maslov, *Development of Perturbations in Compressible Flows* (Nauka, Novosibirsk, 1980) [in Russian].
13. A. N. Bogdanov, V. N. Diesperov, and V. I. Zhuk, "Special points of the field of dispersion curves when investigating the stability of the transonic boundary layer," Dokl. Phys. **58** (4), 141–142 (2013).
14. A. N. Bogdanov and V. N. Diesperov, "Tollmien–Schlichting waves in a transonic boundary layer: Excitation from the outer flow and from the surface," J. Appl. Math. Mech. **71** (2), 258–268 (2007).
15. E. D. Terent'ev, Doctoral Dissertation in Mathematics and Physics (Computing Center, Russian Academy of Sciences, Moscow, 1986).
16. J. D. Cole and L. P. Cook, *Transonic Aerodynamics* (North-Holland, Amsterdam, 1986).
17. V. Ya. Neiland, "Asymptotic problems in the theory of supersonic viscous flows," in *Proceedings of Central Institute of Aerohydrodynamics* (Moscow, 1974), No. 1529 [in Russian].
18. H. Schlichting, *Boundary Layer Theory* (McGraw-Hill, New York, 1968).
19. O. S. Ryzhov and E. D. Terent'ev, "On the transition mode characterizing the triggering of a vibrator in the subsonic boundary layer on a plate," J. Appl. Math. Mech. **50** (6), 753–762 (1986).
20. A. V. Chernyshev, Candidate's Dissertation in Mathematics and Physics (Computing Center, Russian Academy of Sciences, Moscow, 2010).
21. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables* (National Bureau of Standards, Washington, 1972).
22. S. N. Timoshin, "Asymptotic form of the lower branch of the neutral curve in transonic boundary layer," Uch. Zap. TsAGI **21** (6), 50–57 (1990).
23. V. V. Mikhailov, "Asymptotic behavior of the neutral curves of the linear stability problem for a laminar boundary layer," Fluid Dyn. **16** (5), 669–674 (1981).

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