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Control of a Heat Conduction Process with a Quadratic Cost Functional

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Abstract—Two control problems with a quadratic cost functional for a parabolic equation with Robin boundary conditions are investigated.

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1. INTRODUCTION

We consider two optimal control problems for a process described by a boundary value problem for a linear parabolic equation with a linear Robin boundary condition involving the control p. One problem is concerned with minimum-energy control. It was previously studied in various special cases (see, e.g., [1–3]). However, its complete solution was not obtained even in those special cases. In the other problem, a usual quadratic functional is used as an optimality criterion. This problem has been addressed in numerous publications (see, e.g., [1–3]), but it has not been studied to a full extent either.

In this work, we perform a fairly complete analysis of both problems and analyze the relation between their solutions.

1.1. Description of the Process

Let D be a bounded domain in the n-dimensional Euclidean space $\mathbb{E}^n$, ∂D be the boundary of D, and $\bar{D} = D \cup \partial D$. A bounded cylinder in the $(n+1)$-dimensional Euclidean space is denoted by $Q_T = \{(t, x) : 0 < t < T,\ x \in D\}$, and its lateral surface, by $H_T = \{(t, x) : 0 \le t \le T,\ x \in \partial D\}$.

On D we define the differential operator

$$\mathcal{L}u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) - c(x)u, \tag{1}$$

where a_{ij} and c are given functions and c is a nonnegative function. Assume that the operator $\mathcal{L}$ is elliptic, i.e., the following conditions are satisfied: (i) $a_{ij} = a_{ji}$ for all $i, j = 1, \dots, n$; and (ii) there exists a positive constant γ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\alpha_i\alpha_j \ge \gamma\sum_{i=1}^{n}\alpha_i^2, \quad x \in \bar{D},$$

for any real numbers α_i ($i = 1, \dots, n$) satisfying $\alpha_1^2 + \dots + \alpha_n^2 \ne 0$.

Assume that the surface ∂D is continuous and piecewise smooth, the functions $a_{ij}(x)$ and $c(x)$ are continuous in $\bar{D}$, and $\partial a_{ij}/\partial x_i \in L_2(D)$, $i = 1, \ldots, n$. Let $\mathbf{n}$ denote the outward normal unit vector to the boundary ∂D of D. Define

$$\mathcal{P}u = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\cos(\mathbf{n}, x_i) + h(x)u. \tag{2}$$

Here, h is a given nonnegative function from $L_2(\partial D)$.

Under the indicated conditions, the boundary value problem (see [4])

$$\begin{aligned} \mathcal{L}\omega &= -\lambda^2\omega, \quad x \in D, \\ \mathcal{P}\omega &= 0, \quad x \in \partial D, \end{aligned} \tag{3}$$

has an orthonormal system of eigenfunctions $\{\omega_n\}_{n=1}^{\infty}$ that is complete in $L_2(D)$ and $\{\lambda_n\}_{n=1}^{\infty}$ is the corresponding sequence of eigenvalues such that $\lambda_{n+1} \geq \lambda_n$ and $\lambda_n \to \infty$ as $n \to \infty$.

Consider a thermal process described by the boundary value problem

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \mathcal{L}u(t,x), \quad (t,x) \in Q_T, \\ u(0,x) &= 0, \quad x \in D, \\ \mathcal{P}u(t,x) &= p(t)g(x), \quad (t,x) \in H_T, \end{aligned} \tag{4}$$

where g is a given function and the operators $\mathcal{L}$ and $\mathcal{P}$ are defined in (1) and (2), respectively.

1.2. Weak Solutions

Definition 1. The *weak solution* (see [1, 4, 5]) of the boundary value problem (4) is a function $u = u(t,x) \in L_2(Q_T)$ having generalized derivatives $\partial u/\partial x_i \in L_2(Q_T)$ for all $i = 1, \ldots, n$ that (i) satisfies the identity

$$\int_D (u\Psi)\Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2}\int_D \left[u\frac{\partial \Psi}{\partial t} - \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \Psi}{\partial x_i} - cu\Psi\right] dxdt - \int_{t_1}^{t_2}\int_{\partial D}[pg - hu]\Psi d\xi dt \equiv 0$$

for all functions $\Psi \in W_2^{1,1}(Q_T)$ and any $t_1, t_2 \in [0,T]$ and (ii) obeys the initial condition $u(0,x) = 0$ in the sense that

$$\int_D u(t,x)\Phi(x)dx \to 0, \quad t \to +0,$$

for any function $\Phi \in L_2(D)$.

The weak solution of problem (4) can be represented in the form

$$u(t,x) = \sum_{n=1}^{\infty} u_n(t)\omega_n(x). \tag{5}$$

Here, $\{\omega_n\}_{n=1}^{\infty}$ is the complete orthonormal system of eigenfunctions of problem (3). Since

$$\int_D \mathcal{L}u\omega_n dx = \int_D \sum_{i,j=1}^{n}\left[\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\omega_n\right) - a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \omega_n}{\partial x_i}\right]dx - \int_D cu\omega_n dx$$

$$= \int\limits_{\partial D} [\mathcal{P}u - hu]\omega_n d\xi - \int\limits_D \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \omega_n}{\partial x_i} dx - \int\limits_D cu\omega_n dx$$

$$= \int\limits_{\partial D} \mathcal{P}u\omega_n d\xi - \int\limits_{\partial D} \mathcal{P}\omega_n u d\xi + \int\limits_D \mathcal{L}\omega_n u dx - \lambda_n^2 \int\limits_D u\omega_n dx + p(t)\int\limits_{\partial D} g\omega_n d\xi,$$

for the functions $u_n = u_n(t)$, $n = 1, 2, \ldots$, we obtain the Cauchy problem

$$\dot{u}_n(t) + \lambda_n^2 u_n(t) = \gamma_n p(t), \quad u_n(0) = 0.$$

Here,

$$\gamma_n = \int\limits_{\partial D} g(\xi)\omega_n(\xi) d\xi. \tag{6}$$

Thus, the functions $u_n = u_n(t)$ in (5) are given by

$$u_n(t) = \gamma_n \int\limits_0^t e^{-\lambda_n^2(t-\tau)} p(\tau) d\tau. \tag{7}$$

2. APPLICATION OF THE MAXIMUM PRINCIPLE

2.1. Formulation of the Control Problem

Let us set up the first optimal control problem.

Problem 1. Given a process described by problem (4) and admissible controls $p = p(t)$ belonging to the space $L_2[0, T]$, the task is to find a control $p = p(t)$ that minimizes the functional

$$J[p] = \int\limits_D [u(T, x) - \varphi(x)]^2 dx + \beta \int\limits_0^T p^2(t) dt. \tag{8}$$

Here, $\beta > 0$ and $\varphi \in L_2(D)$ is a given function.

Since functional (8) is strictly convex, it is easy to show that the formulated control problem cannot have more than one solution. The existence of at least one optimal control follows from [1].

2.2. Computation of an Increment of the Functional

Let $p = p(t, \beta)$ be an optimal control and $\Delta p = \Delta p(t)$ be an admissible increment of p. Denote by $u = u(t, x, \beta)$ the solution of problem (4) at $p = p(t, \beta)$, and let $u + \Delta u = u(t, x, \beta) + \Delta u(t, x)$ be the solution of problem (4) at $p = p(t, \beta) + \Delta p$. Then the function $\Delta u = \Delta u(t, x)$ solves the boundary value problem

$$\begin{gathered} \frac{\partial \Delta u(t, x)}{\partial t} = \mathcal{L}\Delta u(t, x), \quad (t, x) \in Q_T, \\ \Delta u(0, x) = 0, \quad x \in D, \\ \mathcal{P}\Delta u(t, x) = \Delta p(t) \cdot g(x), \quad (t, x) \in H_T. \end{gathered} \tag{9}$$

The solution of problem (9) satisfies the identity

$$\int\limits_D (\Delta u \Psi)\Big|_{t_1}^{t_2} dx - \int\limits_{t_1}^{t_2}\int\limits_D \left[\Delta u \frac{\partial \Psi}{\partial t} - \sum_{i,j=1}^n a_{ij} \frac{\partial \Delta u}{\partial x_j} \frac{\partial \Psi}{\partial x_i} - c\Delta u \Psi\right] dx dt - \int\limits_{t_1}^{t_2}\int\limits_{\partial D} [\Delta p \cdot g - h\Delta u]\Psi d\xi dt \equiv 0 \tag{10}$$

for all functions $\Psi \in W_2^{1,1}(Q_T)$ and any $t_1, t_2 \in [0, T]$. Here, the initial condition is understood in the sense that $\lim_{t \to +0} \int_D \Delta u(t, x)\Psi(x) dx = 0$ for an arbitrary function $\Psi \in L_2(D)$.

In identity (10), we set $t_1 = 0$ and $t_2 = T$. Then, in view of problem (9),

$$\int_D \Delta u(T,x)\Psi(T,x)dx - \int_0^T \int_D \left[[\Delta u \frac{\partial \Psi}{\partial t} - \sum_{i,j=1}^n a_{ij} \frac{\partial \Delta u}{\partial x_j} \frac{\partial \Psi}{\partial x_i} - c\Delta u \Psi \right] dxdt - \int_0^T \int_{\partial D} [\Delta p \cdot g - h\Delta u]\Psi d\xi dt \equiv 0. \quad (11)$$

The increment of the functional $J[p]$ defined by (8) is calculated as

$$\Delta J[p] = J[p(t,\beta) + \Delta p] - J[p(t,\beta)] = 2\int_D [u(T,x,\beta) - \varphi(x)]\Delta u(T,x)dx$$
$$+ 2\beta \int_0^T p(t,\beta)\Delta p(t)dt + o(\Delta p, \Delta u), \quad \Delta p \to 0, \quad \Delta u \to 0. \quad (12)$$

As $\Psi = \Psi(t,x)$ in identity (11), we use a function satisfying the condition

$$\Psi(T,x) = 2[u(T,x,\beta) - \varphi(x)].$$

Then the increment of J in (12) is determined as

$$\Delta J[p] = \int_D \Delta u(T,x)\Psi(T,x)dx + 2\beta \int_0^T p(t,\beta)\Delta p(t)dt + o(\Delta p, \Delta u). \quad (13)$$

Assume that the function $\Psi = \Psi(t,x)$ satisfies the identity

$$\int_{t_1}^{t_2} \int_D \left[\frac{\partial \Psi}{\partial t}\omega - \sum_{i,j=1}^n a_{ij} \frac{\partial \Psi}{\partial x_i} \frac{\partial \omega}{\partial x_j} - c\Psi\omega \right] dxdt - \int_{t_1}^{t_2} \int_{\partial D} h(\xi)\Psi(t,\xi)\omega(t,\xi)d\xi dt \equiv 0, \quad (14)$$

where $0 \le t_1 < t_2 \le T$. This identity holds for any function $\omega = \omega(t,x)$ such that $\omega \in L_2(Q_T)$, $\omega_{x_i} \in L_2(Q_T)$ for $i = 1, \ldots, n$, and $\omega(t,x) \in L_2(H_T)$. Moreover, we have the limit relation

$$\lim_{t \to T-0} \int_{\partial D} [\Psi(t,\xi) - 2[u(t,\xi,\beta) - \varphi(\xi)]]\chi(\xi)d\xi = 0 \quad (15)$$

for an arbitrary function $\chi \in L_2(\partial D)$.

If the function Ψ belongs to $W_2^{1,2}(Q_T)$, then identity (14) becomes

$$\int_{t_1}^{t_2} \int_D \left[\frac{\partial \Psi}{\partial t} + \sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij} \frac{\partial \Psi}{\partial x_i} \right) - c\Psi \right] \omega dxdt - \int_{t_1}^{t_2} \int_{\partial D} \mathcal{P}\Psi(t,\xi)\omega(t,\xi)d\xi dt \equiv 0. \quad (16)$$

Therefore, it follows from (15) and (16) that $\Psi = \Psi(t,x)$ solves the boundary value problem

$$\begin{gathered} \frac{\partial \Psi(t,x)}{\partial t} + \mathcal{L}\Psi(t,x) = 0, \quad (t,x) \in Q_T, \\ \Psi(T,x) = 2[u(T,x,\beta) - \varphi(x)], \quad x \in D, \\ \mathcal{P}\Psi(t,x) \equiv 0, \quad (t,x) \in H_T. \end{gathered} \quad (17)$$

If $\Psi \in W_2^{1,1}(Q_T)$, then Ψ satisfying identity (14) is called a *weak solution* of problem (17).

In identity (14), we set $\omega = \Delta u$, $t_1 = 0$, and $t_2 = T$. Then (11) and (14) imply that

$$\int_D \Delta u(T,x)\Psi(T,x)dx = \int_0^T \int_{\partial D} \Delta p g \Psi d\xi dt. \tag{18}$$

Combining (13) with (18) and using the Taylor formula yields

$$\Delta J[p] = \int_0^T \left[\int_{\partial D} g\Psi d\xi + 2\beta p(t,\beta) \right] \Delta p(t)dt + o(\Delta p, \Delta u). \tag{19}$$

Define the function

$$H = H(\Psi, p) = -\left[\beta p^2 + p \int_{\partial D} g\Psi d\xi \right].$$

Then it follows from (19) that

$$\Delta J[p] = -\int_0^T \Delta_p H(\Psi, p(t,\beta))dt + o(\Delta p, \Delta u).$$

Therefore, an optimal control $p = p(t,\beta)$ minimizes the functional J if $p = p(t,\beta)$ maximizes H.

Thus, the following result is true.

Theorem 1. *An admissible control* $p = p(t,\beta)$ *in the boundary value problem* (4) *minimizes functional* (8) *if and only if it maximizes the function*

$$H(\Psi, p) = -\left[\beta p^2 + p \int_{\partial D} g\Psi d\xi \right], \tag{20}$$

where Ψ *is the solution of the boundary value problem* (17).

2.3. Integral Equation for the Optimal Control

The optimality condition obtained above is used to construct an optimal control. For this purpose, we first derive an integral equation for this control. Then we prove the uniqueness of its solution and, finally, describe methods for constructing the optimal control and its approximations.

Theorem 2. *Let*

$$r_n(t) = \gamma_n e^{-\lambda_n^2(T-t)}, \quad K(t,\tau) = \sum_{n=1}^{\infty} r_n(t) r_n(\tau), \quad f(t) = \sum_{n=1}^{\infty} \varphi_n r_n(t), \tag{21}$$

where the numbers γ_n *are given by* (6), φ_n *are the Fourier coefficients of* $\varphi = \varphi(x)$ *expanded in terms of the complete orthonormal system of eigenfunctions of problem* (3), *and* λ_n *are the eigenvalues of this problem. Then the optimal control* $p = p(t,\beta)$ *satisfies the integral equation*

$$\beta p(t,\beta) + \int_0^T p(\tau,\beta) K(t,\tau) d\tau = f(t). \tag{22}$$

Proof. The stationary point of the function $H = H(\Psi, p)$ defined by (20) is determined by the equation $2\beta p + \int_{\partial D} g\Psi d\xi = 0$. Therefore,

$$p = p(t,\beta) = -\frac{1}{2\beta} \int_{\partial D} g(\xi)\Psi(t,\xi)d\xi. \tag{23}$$

Substituting function (23) into (4) yields the boundary value problem

$$\frac{\partial u(t,x)}{\partial t} = \mathcal{L}u(t,x), \quad (t,x) \in Q_T,$$
$$u(0,x) = 0, \quad x \in D, \tag{24}$$
$$\mathcal{P}u(t,x) = -\frac{g(x)}{2\beta}\int_{\partial D} g(\xi)\Psi(t,\xi)d\xi, \quad (t,x) \in H_T.$$

The function $u = u(t,x,\beta)$ corresponding to the optimal control $p = p(t,\beta)$ is a weak solution of this problem. Problem (24) is solved together with problem (17).

The solution of problem (24) has the form of (5), where the functions $u_n = u_n(t)$ are given by (7) and $\{\omega_n\}_{n=1}^{\infty}$ is the complete orthonormal system of eigenfunctions of problem (3).

To solve problem (17), the function $\Psi = \Psi(t,x)$ is represented as a series: $\Psi(t,x) = \sum_{n=1}^{\infty}\Psi_n(t)\omega_n(x)$. In view of the obvious identity $\int_D \left[\frac{\partial\Psi}{\partial t} + L\Psi\right]\omega_n dx \equiv 0$ and the terminal conditions, the following Cauchy problems are obtained for Ψ_n:

$$\dot{\Psi}_n(t) - \lambda_n^2\Psi_n(t) = 0, \quad \Psi_n(T) = 2[u_n(T) - \varphi_n], \quad n = 1, 2, \ldots.$$

Thus, we find the functions $\Psi_n(t)$:

$$\Psi_n(t) = 2[u_n(T) - \varphi_n]e^{\lambda_n^2(t-T)}, \quad n = 1, 2, \ldots, \tag{25}$$

where $\varphi(x) = \sum_{n=1}^{\infty}\varphi_n\omega_n(x)$.

Next, from equality (23) and expressions (25) and (7) for Ψ_n and $u_n(T)$, respectively, we derive an integral equation for $p = p(t,\beta)$:

$$\beta p(t,\beta) + \sum_{n=1}^{\infty}\left[\int_0^T p(\tau,\beta)\gamma_n e^{-\lambda_n^2(T-\tau)}d\tau\right]\gamma_n e^{-\lambda_n^2(T-t)} = \sum_{n=1}^{\infty}\varphi_n\gamma_n e^{-\lambda_n^2(T-t)}.$$

Combining this relation with (21) yields Eq. (22).

2.4. Existence and Uniqueness of a Solution to the Integral Equation

Let us prove that Eq. (22) is uniquely solvable in $L_2[0,T]$. Straightforward calculations show that

$$\int_0^T\int_0^T K^2(t,\tau)d\tau dt = \int_0^T\int_0^T\left[\sum_{n=1}^{\infty} r_n(t)r_n(\tau)\right]^2 dtd\tau \le \int_0^T\int_0^T\sum_{n=1}^{\infty} r_n^2(t)\sum_{k=1}^{\infty} r_k^2(\tau)dtd\tau$$
$$= \left[\sum_{n=1}^{\infty}\int_0^T r_n^2(t)dt\right]^2 = \left[\sum_{n=1}^{\infty}\frac{\gamma_n^2}{2\lambda_n^2}(1 - e^{-2\lambda_n^2 T})\right]^2 < \infty.$$

The last inequality follows from $\sum_{n=1}^{\infty}\gamma_n^2 < \infty$ (see [4]) and the fact that $\lambda_n \to \infty$ as $n \to \infty$.

For any function $p = p(t,\beta)$ from $L_2[0,T]$, we have

$$\int_0^T\int_0^T K(t,\tau)p(t,\beta)p(\tau,\beta)dtd\tau = \left[\int_0^T\sum_{n=1}^{\infty} r_n(t)p(t,\beta)dt\right]^2 \ge 0.$$

The operator $\mathcal{A} = \beta\mathcal{I} + \mathcal{K}$ defined by the formula

$$\mathcal{A}p(t,\beta) = (\beta\mathcal{I} + \mathcal{K})p(t,\beta) = \beta p(t,\beta) + \int_0^T K(t,\tau)p(\tau,\beta)d\tau$$

is positive definite, since the equality $(\mathcal{A}p, p)_{L_2} = 0$ holds if and only if $p = p(t,\beta)$ vanishes for almost all t from the interval $[0,T]$.

The following result is well known in functional analysis (see, e.g., [6]).

Theorem 3. *If $\mathcal{A}$ is a linear symmetric operator mapping an element $p \in L_2$ to an element $f \in L_2$ and $\mathcal{A}$ is positive definite, then, for any function $f \in L_2$, the equation $\mathcal{A}p = f$ has a unique solution p in L_2.*

Therefore, for any function f from $L_2[0,T]$, Eq. (22) is uniquely solvable in $L_2[0,T]$.

2.5. Reduction of the Integral Equation to an Infinite Linear System of Equations

Integral equation (22) can be written as

$$\beta p(t,\beta) + \sum_{n=1}^{\infty} c_n r_n(t) = f(t), \quad c_n = \int_0^T r_n(\tau)p(\tau,\beta)d\tau. \tag{26}$$

Multiplying both sides of Eq. (26) by $r_m(t)$ and integrating the result from zero to T, we obtain the system of equations

$$\beta c_m + \sum_{n=1}^{\infty} c_n \int_0^T r_n(t)r_m(t)dt = \int_0^T r_m(t)f(t)dt, \quad i = 1, 2, \ldots. \tag{27}$$

Since (see (21))

$$\int_0^T r_n(t)r_m(t)dt = \frac{\gamma_n\gamma_m}{\lambda_n^2 + \lambda_m^2}(1 - e^{-(\lambda_n^2+\lambda_m^2)T}),$$

we introduce the notation

$$M_{nm} = \frac{\gamma_n\gamma_m}{\lambda_n^2 + \lambda_m^2}(1 - e^{-(\lambda_n^2+\lambda_m^2)T}). \tag{28}$$

Obviously, $M_{nm} = M_{mn}$. Therefore, system (27) can be represented in the form

$$\beta c_m + \sum_{n=1}^{\infty} M_{nm}c_n = \sum_{n=1}^{\infty} M_{nm}\varphi_n, \quad i = 1, 2, \ldots. \tag{29}$$

Setting $c = \{c_1, c_2, \ldots\}$, $\phi = \{\varphi_1, \varphi_2, \ldots\}$, and $\mathcal{M} = (M_{nm})$, we write system (29) in operator form:

$$\beta c + \mathcal{M}c = \mathcal{M}\phi. \tag{30}$$

The operator $\beta\mathcal{I} + \mathcal{M}$ is linear and symmetric; moreover, as before, it can be proved that it is positive definite in the space l_2. It is easy to show that φ belongs to l_2. Therefore, system (30) has a unique solution $c(\beta) = \{c_1(\beta), c_2(\beta), \ldots\} \in l_2$ for any $\phi \in l_2$ and a positive β.

3. CONTROL WITH MINIMAL ENERGY

The second optimal control problem is stated as follows.

Problem 2. Given a process described by the boundary value problem (4) and admissible controls $p = p(t)$ belonging to $L_2[0,T]$, the task is to find a control p that minimizes the functional $J[p] = \int_0^T p^2(t)dt$ and such that the corresponding solution $u = u(t,x)$ of problem (4) satisfies the condition

$$u(T,x) = \varphi(x), \quad 0 \le x \le 1. \tag{31}$$

In Problem 2, the given object is driven from the initial state $u(0,x) = 0$ to the given state $u(T,x) = \varphi(x)$ by applying a boundary control p having the minimal energy $J[p]$.

Problem 2 was completely solved in [2]. Let us describe the main results of [2], which will be used in a comparative analysis of the two problems considered in this paper. It was shown in [2] that the following theorems hold for controls with minimal energy.

Theorem 4. *A control with minimal energy* (*if any*) *can be represented in the form*

$$p(t) = \sum_{m=1}^{\infty} \alpha_m \gamma_m e^{-\lambda_m^2 (T-t)}, \tag{32}$$

where the constants γ_n *are defined by* (6) *and* α_m ($m = 1, 2, \ldots$) *solve the system of equations*

$$\sum_{m=1}^{\infty} M_{mn} \alpha_m = \varphi_n, \quad n = 1, 2, \ldots. \tag{33}$$

Formula (32) implies that the optimal control belongs to $L_2[0, T]$ if and only if the sequence $\alpha = \{\alpha_1, \alpha_2, \ldots\}$ satisfies the condition

$$\sum_{m,n=1}^{\infty} M_{mn} \alpha_n \alpha_m < \infty. \tag{34}$$

This follows directly from

$$\int_0^T p^2(t) dt = \int_0^T \left[\sum_{m=1}^{\infty} \alpha_m \gamma_m e^{-\lambda_m^2 (T-t)} \right]^2 dt = \sum_{m,n=1}^{\infty} M_{mn} \alpha_n \alpha_m.$$

System (33) is rewritten in the operator form

$$\mathcal{M} \alpha = \phi, \tag{35}$$

where the operator $\mathcal{M}$ is defined by the matrix M with elements M_{nm}, $\alpha = \{\alpha_1, \alpha_2, \ldots\}$, and the element $\phi = \{\varphi_1, \varphi_2, \ldots\}$ belongs to l_2 since $\sum_{k=1}^{\infty} \varphi_k^2 < \infty$. Therefore, the following result is valid.

Theorem 5. *System* (33) *has a unique solution satisfying condition* (34) *if and only if there exists a constant* N *such that an arbitrary sequence* $c = \{c_1, c_2, \ldots\}$ *from* l_2 *satisfies the inequality*

$$|(c, \phi)| \le N \sum_{n,m=1}^{\infty} M_{mn} \varphi_n \varphi_m, \tag{36}$$

where $\phi = \{\varphi_1, \varphi_2, \ldots\}$.

This theorem implies a key result that can be stated as follows.

Theorem 6. *The control problem with minimal energy has a solution if and only if there is a constant* N *such that any element* $c = \{c_1, c_2, \ldots\}$ *from* l_2 *satisfies inequality* (36).

Let ρ_n ($n = 1, 2, \ldots$) be the sequence of eigenvalues of the matrix M and $v^n = \{v_1^n, v_2^n, \ldots\}$ be the corresponding set of orthonormal (in l_2) eigenvectors, i.e.,

$$M v^k = \rho_k v^k, \quad \sum_{n=1}^{\infty} v_n^k v_n^i = \delta_{k,i}, \quad i, k = 1, 2, \ldots.$$

Remark 1. For the eigenvalues ρ_n of the operator $\mathcal{M}$, Vieta's theorem yields $\operatorname{Sp} M = \sum_{n=1}^{\infty} \rho_n$ (the trace of M is equal to the sum of its eigenvalues). For $\mathcal{M}$, by virtue of (28), we have

$$\operatorname{Sp} M = \sum_{n=1}^{\infty} \rho_n = \sum_{n=1}^{\infty} \frac{\gamma_n^2}{2\lambda_n^2} (1 - e^{-2\lambda_n^2 T}).$$

The series on the right-hand side of this equality converges (see the substantiation in Subsection 2.4). Therefore, $\sum_{n=1}^{\infty} \rho_n$ converges and $\rho_n \to 0$ as $n \to \infty$.

Let $H_{\mathcal{M}}$ denote the set of sequences $\{\alpha_1, \alpha_2, \ldots\}$ satisfying condition (34). On this set, we define the Hilbert space $H_{\mathcal{M}}$ (energy space of the operator $\mathcal{M}$) by introducing the inner product and the norm

$$[\alpha^1, \alpha^2] = \sum_{n,m=1}^{\infty} M_{nm}\alpha_n^1\alpha_m^2 = (\alpha^1, M\alpha^2), \quad [\alpha]^2 = (\alpha, M\alpha).$$

Theorem 7. *If*

$$\sum_{n=1}^{\infty}\frac{\phi_n^2}{\rho_n} < \infty, \quad \phi_n = (\phi, v^n), \tag{37}$$

then Eq. (35) *has a unique solution in the energy space* $H_{\mathcal{M}}$ *of sequences* $\{\gamma_1, \gamma_2, \ldots\}$ *satisfying condition* (34).

It is of interest to determine the conditions on the right-hand side ϕ of Eq. (35) under which (35) is unsolvable in the space $H_{\mathcal{M}}$ of sequences $\{\gamma_1, \gamma_2, \ldots\}$ satisfying condition (34).

Theorem 8. *If the sequence* $\{\varphi_n\lambda_n\}_{n=1}^{\infty}$ *is unbounded, then Eq.* (35) *has no solution in the space* $H_{\mathcal{M}}$ *of sequences* $\{\gamma_1, \gamma_2, \ldots\}$ *satisfying condition* (34).

4. APPROXIMATE SOLUTION METHODS

All the theorems presented above were taken from [2]. They are proved by applying variational methods of mathematical physics (see, e.g., [6]) to the linear operator equation (35) with a positive (but not positive definite) operator. For a more complete comparative analysis of the optimal control problems under study, we state another two theorems. They are also proved by applying variational methods (see [1, pp. 201–202]).

Together with Eq. (35), we consider the equation

$$\mathcal{M}\alpha = \phi^m, \tag{38}$$

where $\phi^m = (\varphi_1, \ldots, \varphi_m)$.

Theorem 9. *Under condition* (37), *the solution of Eq.* (38) *converges, as* $m \to \infty$, *to the solution of Eq.* (35) *in the metric of the energy space* $H_{\mathcal{M}}$.

Theorem 10. *The solution of Eq.* (35) *is unstable in* $H_{\mathcal{M}}$ *with respect to small* (*in* l_2) *variations in* ϕ. *Specifically, if the right-hand sides* ϕ *and* $\tilde{\phi}^m$ *of Eq.* (35) *and*

$$\mathcal{M}\alpha = \tilde{\phi}^m \tag{39}$$

satisfy condition (37), *then the fact that* $\left\|\phi - \tilde{\phi}^m\right\|_{l_2} \to 0$ *as* $m \to \infty$ *does not necessarily imply that* $[\alpha - \tilde{\alpha}^m] \to 0$ *as* $m \to \infty$, *where* α *and* $\tilde{\alpha}^m$ *are the solutions of Eqs.* (35) *and* (39), *respectively, in the energy space* $H_{\mathcal{M}}$.

However, if there exist ψ *and* $\tilde{\psi}^m$ *from* l_2 *such that* $\phi_n = \sqrt{\rho_n}\psi_n$ *and* $\tilde{\phi}_n^m = \sqrt{\rho_n}\tilde{\psi}_n^m$, *then* $[\alpha - \alpha^m] \to 0$ *as* $m \to \infty$ *as soon as* $\left\|\psi - \tilde{\psi}^m\right\|_{l_2} \to 0$ *as* $m \to \infty$.

This theorem states that the constructed control with minimal energy is unstable with respect to small variations in the objective function φ in condition (31). An issue of much greater interest is to construct an approximate solution of Eq. (35) with the help of the equation

$$\mathcal{M}_m\alpha = \phi. \tag{40}$$

Here, the following two cases have to be considered.

1. $\mathcal{M}_m$ is the projection of $\mathcal{M}$ onto the subspace generated by the first m eigenelements of $\mathcal{M}$.

2. $\mathcal{M}_m$ is the same projection, but calculated with some error.

Consider the first case. Let v^i be the ith eigenelement of the operator $\mathcal{M}$ (i.e., $\mathcal{M}v^i = \rho_i v^i$) and l_2^i be the one-dimensional space generated by this element. The operator P_i defined by the formula $P_i c = (c, v^i)v^i$ for arbitrary $c \in l_2$ is called the *projector* of l_2 onto l_2^i. The following properties of this operator are obvious: (a) $P_i P_j = \delta_{i,j}$ and (b) $\mathcal{M} = \sum_{i=1}^{\infty} \rho_i P_i$.

Therefore, $\mathcal{M}_m$ can be represented in the form

$$\mathcal{M}_m = \sum_{n=1}^{m} \rho_n P_n,$$

and the condition $\|\mathcal{M} - \mathcal{M}_m\| \to 0$ is satisfied if

$$\sum_{n=1}^{\infty} \rho_n < \infty. \tag{41}$$

Under this condition, the solutions of Eqs. (35) and (40) are obtained in the form

$$\alpha = \sum_{n=1}^{\infty} \frac{\varphi_n}{\rho_n}, \quad \alpha_m = \sum_{n=1}^{m} \frac{\varphi_n}{\rho_n}.$$

The sum of the series in (41) is equal to the trace of the matrix M. Therefore, in view of Remark 1, we find that $[\alpha - \alpha_m] \to 0$ as $m \to \infty$.

Now, let $\mathcal{M}_m$ be obtained by approximately computing the projector onto the subspace spanned by $v^1, \ldots, v^m$, i.e., $\mathcal{M}_m = \mathcal{N}_m + \mathcal{G}_m$, where $\mathcal{N}_m$ is the projector and $\mathcal{G}_m$ is the operator determining the computational error made in the construction of $\mathcal{N}_m$. If the right-hand side of Eq. (35) is also calculated with an error δ_m, then, instead of Eq. (40), we solve the equation

$$(\mathcal{N}_m + \mathcal{G}_m)\alpha = \phi_m + \delta_m. \tag{42}$$

Accordingly, it is natural to examine how far the solution $\tilde{\alpha}_m$ of this equation differs from the solution α_m of Eq. (35). In doing this, we will use the following definition (see [7]).

Definition 2. The computation of solutions to Eq. (42) is called o-*stable* if the following conditions are satisfied: (a) $\|G_m\|/\|\delta_m\|$ tends to zero as $m \to \infty$, which ensures the existence of an operator $(\mathcal{N}_m + \mathcal{G}_m)^{-1}$ (at least, for sufficiently large m); and (b) $\|G_m\|/\|\delta_m\|$ and $\|\delta_m\|/\|\phi_m + \delta_m\|$ tend to zero, which ensures that $\|\alpha_m - \tilde{\alpha}_m\|/\|\tilde{\alpha}_m\|$ tends to zero as well.

Below is the main result concerning this method for finding an approximate solution of the considered equation (see [1]).

Theorem 11. *A process of finding* $\{\alpha_1, \alpha_2, \ldots\}$ *is* o-*stable if and only if the sequence* $\{\mu(\mathcal{N}_m)\} = \left\{\|\mathcal{N}_m\|/\left\|\mathcal{N}_m^{-1}\right\|\right\}$ *is bounded*.

To conclude, we analyze the relation between the optimal control problems under consideration. In the first problem, the optimality criterion is the quadratic functional (see (8))

$$J[p] = \int_D [u(T,x) - \varphi(x)]^2 dx + \beta \int_0^T p^2(t) dt,$$

where $\varphi(x)$ is a given function from $L_2(D)$ and β is a positive parameter. No constraints are imposed on the state of the system at the terminal time $t = T$. A unique optimal control exists and is the solution of the integral equation (see Theorem 2)

$$\beta p(t,\beta) + \int_0^T p(\tau,\beta) K(t,\tau) d\tau = f(t).$$

It can be represented in the form (see (26))

$$\beta p(t,\beta) + \sum_{n=1}^{\infty} c_n r_n(t) = f(t),$$

where the constants c_n are uniquely determined by solving the system of equations

$$\beta c_m + \sum_{n=1}^{\infty} M_{nm} c_n = \sum_{n=1}^{\infty} M_{nm}\varphi_n, \quad i = 1, 2, \ldots.$$

Since $f(t) = \sum \varphi_n \gamma_n r_n(t)$ (see Theorem 2), the control $p(t,\beta)$ can be represented in the form

$$p(t,\beta) = \sum_{n=1}^{\infty} \gamma_n \alpha_n(\beta) e^{-\lambda_n^2 (T-t)}, \quad \alpha_n(\beta) = \frac{\varphi_n - c_n(\beta)}{\beta}.$$

It has the same structure as the control with minimal energy (see (32)). However, the latter belongs to $L_2(0,T)$ only under additional conditions (see Theorem 4). The sequence $\{\alpha_1(\beta), \alpha_2(\beta), \ldots\}$ is determined by the system of equations

$$\beta \alpha_n(\beta) + \sum_{m=1}^{\infty} M_{mn} \alpha_m(\beta) = \varphi_n, \quad n = 1, 2, \ldots.$$

By using the notation introduced in Eq. (35), this system can be written in operator form:

$$(\beta \mathcal{I} + \mathcal{M})\alpha(\beta) = \phi; \tag{43}$$

here, $\mathcal{I}$ denotes the identity operator and $\alpha(\beta)$ is the sequence $\{\alpha_1(\beta), \alpha_2(\beta), \ldots\}$.

Thus, for sufficiently small β, Eq. (43) can be treated as an approximation to Eq. (35).

Theorem 12. *If Eq.* (35) *satisfies the conditions of Theorem* 7 *and, hence, has a unique solution in the energy space* $H_{\mathcal{M}}$, *then the solutions of Eqs.* (35) *and* (43) *satisfy the condition*

$$\lim_{\beta \to 0} [\alpha(\beta) - \alpha] = 0.$$

Proof. The condition of the theorem implies that the solutions of the indicated equations can be represented in the form

$$\alpha = \sum_{n=1}^{\infty} \frac{\phi_n}{\rho_n} v^n, \quad \alpha(\beta) = \frac{\phi_n}{\beta + \rho_n} v^n, \quad \phi_n = (\phi, v^n),$$

where v^n ($n = 1, 2, \ldots$) is the complete orthonormal system of eigenelements of the operator $\mathcal{M}$ and ρ_n are the corresponding eigenvalues. Therefore, in view of $\rho_n \geq \rho_{n+1}$, we obtain

$$[\alpha - \alpha(\beta)]^2 = (\alpha - \alpha(\beta), \mathcal{M}(\alpha - \alpha(\beta))) = \left(\sum_{n=1}^{\infty} \phi_n \left(\frac{1}{\rho_n} - \frac{1}{\rho_n + \beta} \right) v^n, \right.$$

$$\left. \sum_{m=1}^{\infty} \phi_m \rho_m \left(\frac{1}{\rho_m} - \frac{1}{\rho_m + \beta} \right) v^m \right) = \beta \sum_{n=1}^{\infty} \frac{1}{(\rho_n + \beta)^2} \frac{\phi_n^2}{\rho_n} \leqslant \frac{\beta}{(\rho_N + \beta)^2} \sum_{n=1}^{N} \frac{\phi_n^2}{\rho_n} + \sum_{n=N+1}^{\infty} \frac{\phi_n^2}{\rho_n}.$$

Let ε be an arbitrarily small number. We choose N so large that

$$\sum_{n=N+1}^{\infty} \frac{\phi_n^2}{\rho_n} \leq \frac{\varepsilon}{2}.$$

Fixing N, we choose β small enough to satisfy the inequality

$$\frac{\beta}{(\rho_N+\beta)^2}\sum_{n=1}^{N}\frac{\phi_n^2}{\rho_n}\le\frac{\varepsilon}{2},$$

whence $[\alpha(\beta)-\alpha]^2\le\varepsilon$.

To analyze the o-stability of finding an approximate solution of the problems in the case under consideration, we use Theorem 11. Specifically, the operator $\mathcal{G}_m$ is specified as $\mathcal{I}\beta_m$, where β_m is a sequence of numbers vanishing as $m\to\infty$ (which is used in the computations).

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