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A Splitting Algorithm for the Wavelet Transform of Cubic Splines on a Nonuniform Grid

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Abstract— For cubic splines with nonuniform nodes, splitting with respect to the even and odd nodes is used to obtain a wavelet expansion algorithm in the form of the solution to a three-diagonal system of linear algebraic equations for the coefficients. Computations by hand are used to investigate the application of this algorithm for numerical differentiation. The results are illustrated by solving a prediction problem.

Keywords: multiresolution analysis (MRA), wavelets, cubic splines, nonuniform measurements, orthogonality to second-order derivatives, expansion and reconstruction relations, numerical differentiation, prediction.

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1. INTRODUCTION

In the theory of multiresolution analysis (MRA), wavelets form a basis of the set that fills the gap between the approximating spaces on a fine and coarse grids (see [1]). In the classical case of approximations on uniform infinite in both directions grids, such a basis is generated by dilations and displacement of a single wave function that has the shape of a short rapidly damping wavelet. Due to dilation, the wavelets reveal to different degrees of detail differences in the characteristics of the measured signals, and due to displacements, they are able to analyze the signal properties at different points in the entire interval under examination. In the analysis of time-dependent signals, the locality property of wavelets gives them a significant advantage over the Fourier transform, which can reveal only global properties of the signal because the basis functions of the Fourier transform (sines and cosines) have an infinite support. Since the wavelets transform the system of basis functions with distributed parameters to a system with lumped parameters, the wavelet basis is much more effective from the viewpoint of conditioning and convergence. The construction of wavelets is based on the existence of scaling (or calibration) relations such that each basis function on a coarse grid can be represented as a linear combination of the basis functions on a fine grid. In particular, such relations are satisfied by splines, which are smooth functions composed of segments of polynomials of degree m on a sequence of embedded grids. In the case of uniform grids on the entire number line, these relations are well known, as well as some cases of approximation on a finite interval. However, the practically important case of measurements given on a nonuniform grid is less well studied. Some recommendations on deriving a system of equations for finding the scaling coefficients and methods for its solution can be found in [2]. In [3], the problem of adding nodes one-by-one for constructing a telescopic wavelet expansion was studied; in [4], another version—removing nodes one-by-one—was proposed. In [5], a special rational identity was used to obtain scaling relations for the interior grid points only. In [6], the scaling relations for cubic splines were formally obtained for all nodes. However, no explicit expressions for the scaling coefficients were written out. In the present paper, we propose an elementary method for solving this problem based on the use of a local approximation that is exact on splines and on the well-known de Boor—Fix lemma [7].

The second thing underlying any wavelet transform is a set of scaling relations for wavelets. Such relations are known for orthogonal and biorthogonal wavelets, which allows one to use an infinite iterative procedure to obtain their graphical representation but does not give an analytical expression that could be used as a trial function, e.g., in the Galerkin type method. In contrast, the semiorthogonal [8] and nonorthogonal [9] wavelets are determined explicitly by linear combinations of the basis splines on the fine grid. A distinctive feature of semiorthogonal wavelets, which is sometimes used as a basis for the corresponding numerical method (see [10]) for constructing the wavelet transform, is the fact that the wavelet expansion gives the best root-mean-square approximation of splines on the fine grid using splines on the coarse grid. This is a considerable advantage for compressing discrete numerical data. However, this advantage is lost when the resulting spline-wavelet expansions are differentiated. A progress in solving this problem was achieved by constructing nonorthogonal spline-wavelets for which a greater number of moments vanish (see [11–14]); however, the wavelet supports increase in this case. In our opinion, the optimal solution is to use spline-wavelets that provide the best root-mean-square approximation of the spline derivatives on the fine grid by spline derivatives on the coarse grid. First, such wavelets were studied in the case of cubic splines (see [15, 16]), where the cubic wavelets $\psi(x-i)\ \forall i$ were found for which the orthogonality conditions to the corresponding basis splines $\varphi(x-j)\ \forall j$ with respect to the scalar product of the second-order derivatives are fulfilled: $\int_{-\infty}^{\infty}\psi''(x-i)\varphi''(x-j)dx = 0\ \forall i, j$. It turns out that these wavelets have a simple organization; in particular, their support is smaller than the support of the classical semiorthogonal spline wavelets; namely, $[0,3]\subset[0,7]$. In addition, they proved to be useful in solving differential equations (see [17]), and they were numerically implemented in a standard program in MatLab (see [18]). A generalization of this construct for the case of a nonuniform grid was made in [6].

Finally, the last but not least problem in the theory of wavelets is the calculation of the coefficients of the wavelet expansion for a given function. In the case of orthonormal and biorthogonal wavelets, the solution is reduced to the application of averaging filters. In our opinion, this is a drawback because the information for the calculation of each coefficient on the coarse grid is used incompletely. In contrast, the coefficients of semiorthogonal (see [8]) and nonorthogonal (see [9]) wavelets are related by systems of linear algebraic equations; however, these systems are not guaranteed to be well conditioned. In the case of the measurements specified on a nonuniform grid, these difficulties are aggravated by the problem of stability with respect to the location of the grid points (see [19]). In [6], it was proposed to use in the computations the unique *point value vanishing* property of the constructed wavelets to obtain an algorithm for the discrete wavelet transform that requires the coefficients of an interpolation spline to be computed at each step. However, the resulting expressions turned out to be very cumbersome, and we did not found references to using this construct in practical computations. We propose to use the odd-even splitting technique (see [20]) based on finite implicit relating the basis functions of the set of splines on the coarse grid, the basis functions on the fine grid, and wavelets. A similar idea of constructing intermediate implicit finite spline schemes instead of implicit infinite (in the case of interpolation) and implicit finite (in the case of local approximation) spline schemes was earlier used in [21] to justify computationally convenient (decreased bandedness and diagonal dominance) spline approximation methods. In [22], the odd-even splitting technique of the matrix of the wavelet transform was used to prove its invertibility; however, the possibility to use this technique in practical computations was not explicitly mentioned.

2. CONSTRUCTING CUBIC SPLINE WAVELETS

Let V_L be the space of cubic splines that are continuous up to the third-order derivatives, inclusive, on the interval $[a,b]$ with the nonuniform grid of points $\Delta^L : a = x_0 < x_1 < x_2 < \ldots < x_{2^L} = b$ of size $2^L + 3$. To obtain the basis functions in the space V_L, one should add the fictitious grid points $x_{-3} < x_{-2} < x_{-1} < x_0$ and $x_{2^L} < x_{2^L+1} < x_{2^L+2} < x_{2^L+3}$ to the grid Δ^L and construct the fourth-order divided difference for the function $\varphi_3(x,t) = (x_{k+2} - x_{k-2})(\max\{x-t,0\})^3$ given the values of the argument $t = x_{k-2}, x_{k-1}, x_k, x_{k+1}, x_{k+2}$. Then, the functions

$$N_k^L(x) = \varphi_3[x; x_{k-2}, x_{k-1}, x_k, x_{k+1}, x_{k+2}], \quad k = -1, 0, \ldots, 2^L + 1,$$

are normalized B-splines (see [23; 24, pp. 18–23]). They are spline functions of degree 3 and defect 1. They are distinct from zero on the intervals (supports) (x_{k-2}, x_{k+2}) and are identically equal to zero outside these intervals. In other words, the derivatives $N_k^L(x)$ up to the second order vanish. This requirement imposes six conditions on the parameters of the resulting spline, and one more free parameter is determined so as to satisfy the normalization condition $\sum_{k=-1}^{2^L+1} N_k^L(x) = 1,\ a \le x \le b$. Every cubic spline

$S^L(x) \in V_L$ can be uniquely represented on the interval of the parameter x variation $[a, b]$ by a linear combination of B-splines:

$$S^L(x) = \sum_{i=-1}^{2^L+1} C_i^L N_i^L(x), \quad a \le x \le b, \tag{1}$$

where C_i^L are constant coefficients.

We do not impose any continuity conditions at the endpoints of the interval under examination. This corresponds to the fact that the basis consisting of B-splines ensures a valid representation of the elements of V_L only on the interval $[a, b]$. To take into account the defect $\nu > 1$, each point of the grid Δ should be assigned the multiplicity equal to the spline defect at this point and the grid points should be renumbered taking into account their multiplicity. For the further considerations, it is convenient to choose the first and the last grid points such that $a = x_{-1} = x_0$, $x_{2^L} = x_{2^L+1} = b$. Then, for $C_{-1}^L = C_0^L = C_{2^L}^L = C_{2^L+1}^L = 0$, the defined functions satisfy the homogeneous boundary conditions

$$S^L(a) = (S^L)'(a) = S^L(b) = (S^L)'(b) = 0.$$

Let us denote the space thus defined by V_L^0. Let the grid Δ^{L-1} $(L \ge 2)$ be obtained from Δ^L by removing every other grid point (apart from the fictitious points). Then, the corresponding space V_{L-1}^0 with the basis functions $N_i^{L-1}(x)$, which are distinct from zero on the doubled supports $[x_{2i-4}, x_{2i+4}]$, is embedded in V_L^0. For $i = 2, 3, \ldots, 2^{L-1} - 2$, the two-scale scaling relation between the basis functions in V_L^0 and in V_{L-1}^0 was obtained in [5]:

$$N_i^{L-1}(x) = \sum_{k=0}^{4} p_{i,k} N_{2i-2+k}^L(x), \tag{2}$$

where

$$p_{i,0} = \frac{(x_{2i-1} - x_{2i-4})(x_{2i-3} - x_{2i-4})}{(x_{2i+2} - x_{2i-4})(x_{2i} - x_{2i-4})}, \quad p_{i,1} = \frac{x_{2i-1} - x_{2i-4}}{x_{2i+2} - x_{2i-4}},$$

$$p_{i,2} = \frac{(x_{2i+1} - x_{2i+2})(x_{2i-1} - x_{2i-4})}{(x_{2i+2} - x_{2i-2})(x_{2i-4} - x_{2i+2})} + \frac{(x_{2i+1} - x_{2i-2})(x_{2i-1} - x_{2i+4})}{(x_{2i-2} - x_{2i+4})(x_{2i+2} - x_{2i-2})},$$

$$p_{i,3} = \frac{x_{2i+1} - x_{2i+4}}{x_{2i-2} - x_{2i+4}}, \quad p_{i,4} = \frac{(x_{2i+3} - x_{2i+4})(x_{2i+1} - x_{2i+4})}{(x_{2i} - x_{2i+4})(x_{2i-2} - x_{2i+4})}.$$

To obtain the scaling relation between the basis functions in V_L^0 and in V_{L-1}^0 at the left endpoint of the interval, we calculate the values of $N_1^{L-1}(x)$ at the points of the fine grid $x = x_1, x_2, x_3, x_4$ and substitute them into the formula (see [7, 23]) that represents the coefficients of the local approximation spline in terms of the values of the function to be approximated at the grid points

$$C_i^L(f) = f(x_i) + \frac{1}{3}(x_{i+1} - 2x_i + x_{i-1})f'(x_i) - \frac{1}{6}(x_{i+1} - x_i)(x_i - x_{i-1})f''(x_i).$$

Then, we obtain

$$p_{1,1} = \frac{x_1 - x_0}{x_4 - x_0}, \quad p_{1,2} = \frac{(x_3 - x_0)(x_4 - x_1)}{(x_0 - x_4)^2} + \frac{(x_0 - x_1)(x_3 - x_6)}{(x_6 - x_0)(x_4 - x_0)},$$

$$p_{1,3} = \frac{x_3 - x_6}{x_0 - x_6}, \quad p_{1,4} = \frac{(x_5 - x_6)(x_3 - x_6)}{(x_2 - x_6)(x_0 - x_6)},$$

and

$$N_1^{L-1}(x) = \sum_{k=1}^{4} p_{1,k} N_k^L(x). \tag{3}$$

The representation of the rightmost wide basis function is the mirror reflection of the above representation:

$$N^{L-1}_{2^{L-1}-1}(x) = \sum_{k=0}^{3} p_{2^{L-1}-1,k} N^{L}_{2^L-4+k}(x), \tag{4}$$

where

$$p_{2^{L-1}-1,0} = \frac{(x_{2^L-5} - x_{2^L-6})(x_{2^L-3} - x_{2^L-6})}{(x_{2^L-2} - x_{2^L-6})(x_{2^L} - x_{2^L-6})}, \quad p_{2^{L-1}-1,1} = \frac{x_{2^L-3} - x_{2^L-6}}{x_{2^L} - x_{2^L-6}},$$

$$p_{2^{L-1}-1,2} = \frac{(x_{2^L-1} - x_{2^L-4})(x_{2^L} - x_{2^L-3})}{(x_{2^L} - x_{2^L-4})^2} + \frac{(x_{2^L} - x_{2^L-1})(x_{2^L-3} - x_{2^L-6})}{(x_{2^L-6} - x_{2^L})(x_{2^L-4} - x_{2^L})},$$

$$p_{2^{L-1}-1,3} = \frac{x_{2^L} - x_{2^L-1}}{x_{2^L} - x_{2^L-4}}.$$

The set of wavelets W_{L-1} is defined as the orthogonal complement of V^0_{L-1} of V^0_L in the Hilbert space

$$H^2_0[a,b] = \{f \in C[a,b];\ f'' \in L^2[a,b],\ f(a) = f'(a) = f(b) = f'(b) = 0\},$$

in which the scalar product is defined by

$$\langle f, g\rangle = \int_0^b f'' g''.$$

The characteristic property of the functions $g \in W_{L-1}$ is that $g(x_{2i}) = 0$ for $i = 1, 2, \ldots, 2^{L-1}$, which immediately implies that the set of basis wavelets satisfies the following scaling relations (see [6]):

$$\psi^{L-1}_k(x) = q_{k,0} N^L_{2k-2}(x) + N^L_{2k-1}(x) + q_{k,2} N^L_{2k}(x), \quad k = 2, 3, \ldots, 2^{L-1} - 1, \tag{5}$$

$$\psi^{L-1}_1(x) = N^L_1(x) + q_{1,2} N^L_2(x), \tag{6}$$

$$\psi^{L-1}_{2^{L-1}}(x) = q_{2^{L-1},0} N^L_{2^L-2}(x) + N^L_{2^L-1}(x), \tag{7}$$

where

$$q_{k,0} = -\frac{N^L_{2k-1}(x_{2k-2})}{N^L_{2k-2}(x_{2k-2})}, \quad k = 2, 3, \ldots, 2^{L-1},$$

and

$$q_{k,2} = -\frac{N^L_{2k-1}(x_{2k})}{N^L_{2k}(x_{2k})}, \quad k = 1, 2, \ldots, 2^{L-1} - 1.$$

For the case of a uniform grid, such wavelets were independently defined in [25]. It is clear that the support $[x_{2k-4}, x_{2k+2}] \cap [a,b]$ of these wavelets is fairly small—it is smaller than the support of the cubic B-spline.

3. CONSTRUCTION OF THE WAVELET TRANSFORM

It is convenient to write the spline coefficients and the basis functions in V^0_L as $C^L = [C^L_1, C^L_2, \ldots, C^L_{2^L-1}]^{\mathrm{T}}$ and $\varphi^L = [N^L_1, N^L_2, \ldots, N^L_{2^L-1}]$. Then, for every function $S^L_0(x) \in V^0_L$, Eq. (1) can be rewritten in the form $S^L_0(x) = \varphi^L(x) C^L$. Similarly, in the space W_L, we write the basis wavelet functions as the row matrix $\psi^L = [\psi^L_1, \psi^L_2, \ldots, \psi^L_{2^L}]$. The corresponding wavelet coefficients are written in the form of the vector $D^L = [D^L_1, D^L_2, \ldots, D^L_{2^L}]^{\mathrm{T}}$. For the grid Δ^{L-1} ($L \geq 2$), we can write the functions ϕ^{L-1} and ψ^{L-1} as linear combinations of the functions φ^L because each wide basis function inside the approximation interval can be constructed from five (respectively, three) and at the endpoints of the interval from four (respectively, two) narrow basis functions $\varphi^{L-1} = \varphi^L P^L$ and $\psi^{L-1} = \varphi^L Q^L$, where the elements of the columns of the matrix P^L are composed of the coefficients of the scaling relations (2)–(4), and the elements of the columns of the matrix Q^L are composed of the coefficients of the scaling relations for wavelets (5)–(7).

Since any function in V_L^0 can be written as the sum of a function in V_{L-1}^0 and a function in W_{L-1}, we have the equalities

$$\varphi^L(x)C^L = \varphi^{L-1}(x)C^{L-1} + \psi^{L-1}(x)D^{L-1} = \varphi^L(x)P^L C^{L-1} + \varphi^L(x)Q^L D^{L-1}.$$

Therefore, the coefficients C^L of the cubic spline on the grid Δ^L can be obtained from the coefficients C^{L-1} and D^{L-1} of the wavelet expansion on the grid Δ^{L-1} as $C^L = P^L C^{L-1} + Q^L D^{L-1}$ or, using the block notation for matrices (see [8]), as

$$C^L = \left[P^L | Q^L\right]\left[\frac{C^{L-1}}{D^{L-1}}\right]. \tag{8}$$

Here is an example of the matrix $[P^L | Q^L]$ corresponding to $L = 3$:

$$\left[P^3 | Q^3\right] = \left[\begin{array}{ccc|cccc} p_{1,1} & 0 & 0 & 1 & 0 & 0 & 0 \\ p_{1,2} & p_{2,0} & 0 & q_{1,2} & q_{2,0} & 0 & 0 \\ p_{1,3} & p_{2,1} & 0 & 0 & 1 & 0 & 0 \\ p_{1,4} & p_{2,2} & p_{3,0} & 0 & q_{2,2} & q_{3,0} & 0 \\ 0 & p_{2,3} & p_{3,1} & 0 & 0 & 1 & 0 \\ 0 & p_{2,4} & p_{3,2} & 0 & 0 & q_{3,2} & q_{4,0} \\ 0 & 0 & p_{3,3} & 0 & 0 & 0 & 1 \end{array}\right]. \tag{9}$$

4. AN ALGORITHM USING SPLITTING

The inverse procedure of decomposing the coefficients C^L into the coarser version C^{L-1} and the refining coefficients D^{L-1} consists of solving the system of linear equations (8). For this purpose, system (8) can be split with respect to the even and odd nodes. We prove the following result.

Теорема 1. *Let the values of the spline coefficients C_i^{L-1} on the coarse grid Δ^{L-1} be calculated by solving the tridiagonal system of linear equations*

$$\begin{bmatrix} N_1^{L-1}(x_2) & N_2^{L-1}(x_2) & 0 & \cdots & 0 & 0 \\ N_1^{L-1}(x_4) & N_2^{L-1}(x_4) & N_3^{L-1}(x_4) & \ddots & \vdots & \vdots \\ 0 & N_2^{L-1}(x_6) & N_3^{L-1}(x_6) & \ddots & 0 & 0 \\ 0 & 0 & N_3^{L-1}(x_8) & \ddots & N_{2^{L-1}-1}^{L-1}(x_{2^L-6}) & 0 \\ \vdots & \vdots & \vdots & \ddots & N_{2^{L-1}-1}^{L-1}(x_{2^L-4}) & N_{2^{L-1}}^{L-1}(x_{2^L-4}) \\ 0 & 0 & 0 & \cdots & N_{2^{L-1}-1}^{L-1}(x_{2^L-2}) & N_{2^{L-1}}^{L-1}(x_{2^L-2}) \end{bmatrix} \begin{bmatrix} C_1^{L-1} \\ C_2^{L-1} \\ \vdots \\ C_{2^{L-1}-1}^{L-1} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{2^{L-1}-1} \end{bmatrix}, \quad f_i = S^L(x_{2i}), \tag{10}$$

$$i = 1, \ldots, 2^{L-1} - 1.$$

Then, the values of the wavelet coefficients are

$$D_1^{L-1} = C_1^L - p_{1,1}C_1^{L-1}, \quad D_{2^{L-1}}^{L-1} = C_{2^L-1}^L - p_{2^{L-1}-1,3}C_{2^{L-1}-1}^{L-1}, \tag{11}$$

$$D_i^{L-1} = C_{2i-1}^L - p_{i-1,3}C_{i-1}^{L-1} - p_{i,1}C_i^{L-1}, \quad i = 2, 3, \ldots, 2^{L-1} - 1. \tag{12}$$

Proof. By construction, inside the interval $[a, b]$, on the support of each wide basis function, two wavelets and five narrow basis functions overlap. Therefore, upon renumbering the nodes to make them $x_0, x_1, \ldots, x_8$, we can write the following finite implicit relation on the interval $[x_0, x_8]$:

$$\sum_{i=0}^{4} a_{i+2} N_i^L(x) = \sum_{j=0}^{1} b_j \psi_{j+2}^{L-1}(x) + c_0 N_2^{L-1}(x). \tag{13}$$

Since both sides in (13) involve cubic splines, for these sides to be identical, it is sufficient that the corresponding coefficients in the B-spline basis on the fine grid coincide. For calculating the unknown coefficients in (13) using the scaling relations introduced above, we have for the corresponding numbers

$$\begin{aligned}
&i = 0: \ a_0 = b_0 q_{2,0} + c_0 p_{2,0},\\
&i = 1: \ a_1 = b_0 + c_0 p_{2,1},\\
&i = 2: \ a_2 = b_0 q_{2,2} + b_1 q_{3,0} + c_0 p_{2,2},\\
&i = 3: \ a_3 = b_1 + c_0 p_{2,3},\\
&i = 4: \ a_4 = b_1 q_{3,2} + c_0 p_{2,4}.
\end{aligned}$$

For $c_0 = 0$, the solutions to the system coincide with the coefficients of the scaling relations for the basis spline wavelets $\psi_2^{L-1}(x)$ for $b_1 = a_3 = a_4 = 0$, $b_0 = 1$ and $\psi_3^{L-1}(x)$ for $b_0 = a_0 = a_1 = 0$, $b_1 = 1$.

For $c_0 = 1$, we try to find a system of equations relating the expansion coefficients for the even nodes (the case when $a_1 = a_3 = 0$). It is easy to verify that the system has a unique nontrivial solution for which

$$a_0 = \frac{N_2^{L-1}(x_2)}{N_2^L(x_2)}, \quad a_2 = \frac{N_2^{L-1}(x_4)}{N_4^L(x_4)}, \quad a_4 = \frac{N_2^{L-1}(x_6)}{N_6^L(x_6)}.$$

Then, $b_0 = -p_{2,1}$, $b_1 = -p_{2,3}$, and $c_0 = 1$.

By construction, at the endpoints of the approximation interval, on the support of each extreme wide basis function two wavelets and four narrow basis functions overlap. In particular, at the left endpoint, we have

$$\sum_{i=1}^{4} a_i N_i^L(x) = \sum_{j=1}^{2} b_j \psi_j^{L-1}(x) + c_1 N_1^{L-1}(x). \tag{14}$$

Then, for finding the unknown coefficients in (14) using the scaling relations derived above, we have for the corresponding numbers

$$\begin{aligned}
&i = 1: a_1 = b_1 + c_1 p_{1,1},\\
&i = 2: \ a_2 = b_1 q_{1,2} + b_2 q_{2,0} + c_1 p_{1,2},\\
&i = 3: \ a_3 = b_2 + c_1 p_{1,3},\\
&i = 4: \ a_4 = b_2 q_{2,2} + c_1 p_{1,4}.
\end{aligned}$$

The solution to the resulting system of equations is

$$a_2 = \frac{N_1^{L-1}(x_2)}{N_2^L(x_2)}, \quad a_4 = \frac{N_1^{L-1}(x_4)}{N_4^L(x_4)}.$$

Then, $b_1 = -p_{1,1}$, $b_2 = -p_{1,3}$, and $c_1 = 1$.

There are two more solutions for $c_1 = 0$, which coincide with the coefficients of the scaling relations for the basis spline wavelets $\psi_1^{L-1}(x)$ for $b_2 = a_3 = a_4 = 0$, $b_1 = 1$ and $\psi_2^{L-1}(x)$ for $b_1 = a_1 = 0$, $b_2 = 1$.

The solutions at the right endpoint of the interval $[a, b]$ are the mirror reflections of the solutions obtained above. We have

$$a_{2^L-2} = \frac{N_{2^{L-1}-1}^{L-1}(x_{2^L-2})}{N_{2^L-2}^{L}(x_{2^L-2})}, \quad a_{2^L-4} = \frac{N_{2^{L-1}-1}^{L-1}(x_{2^L-4})}{N_{2^L-4}^{L}(x_{2^L-4})}.$$

Then, $b_{2^{L-1}-1} = -p_{2^{L-1}-1,3}, b_{2^{L-1}-2} = -p_{2^{L-1}-1,1}$, and $c_{2^{L-1}-1} = 1$. Furthermore, we have two scaling identities for the basis spline wavelets $\psi^{L-1}_{2^{L-1}-1}(x)$ and $\psi^{L-1}_{2^{L-1}}(x)$.

Define sequences of matrices G^L and R^L whose blocks are composed, respectively, of the coefficients of the left-hand and right-hand sides of the expansions obtained above:

$$G^L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ q_{1,2} & \dfrac{N_1^{L-1}(x_2)}{N_2^L(x_2)} & q_{2,0} & \dfrac{N_2^{L-1}(x_2)}{N_2^L(x_2)} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \dfrac{N_1^{L-1}(x_4)}{N_4^L(x_4)} & q_{2,2} & \dfrac{N_2^{L-1}(x_4)}{N_4^L(x_4)} & q_{3,0} & \dfrac{N_3^{L-1}(x_4)}{N_4^L(x_4)} & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dfrac{N_2^{L-1}(x_6)}{N_6^L(x_6)} & q_{3,2} & \dfrac{N_3^{L-1}(x_6)}{N_6^L(x_6)} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \dfrac{N_{2^{L-1}-1}^{L-1}(x_{2^L-4})}{N_{2^L-4}^L(x_{2^L-4})} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \dfrac{N_{2^{L-1}-1}^{L-1}(x_{2^L-2})}{N_{2^L-2}^L(x_{2^L-2})} & q_{2^{L-1},0} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

$$R^L = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \ddots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 1 & -p_{1,1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -p_{1,3} & 1 & -p_{2,1} & 0 & 0 & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -p_{2,3} & 1 & -p_{3,1} & \ddots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & -p_{2^{L-1}-2,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -p_{2^{L-1}-2,3} & 1 & -p_{2^{L-1}-1,1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -p_{2^{L-1}-1,3} & 1 \end{bmatrix}.$$

As a result, we conclude that the basis functions of the space of splines on the fine grid, the basis functions on the coarse grid, and the wavelets satisfy the equality

$$\varphi^L G^L = \left[\varphi^{L-1} | \psi^{L-1}\right] R^L, \quad L \geq 2.$$

Hence, using the property of the complement of the space of wavelets, we find that

$$\left[\varphi^{L-1} | \psi^{L-1}\right] \left[\frac{C^{L-1}}{D^{L-1}}\right] = \varphi^L C^L = \left[\varphi^{L-1} | \psi^{L-1}\right] R^L \left(G^L\right) C^L.$$

Now, we can write the solution to system (8) in matrix form as (see [20])

$$\left[\frac{C^{L-1}}{D^{L-1}}\right] = \left[P^L | Q^L\right]^{-1} C^L = R^L \left(G^L\right)^{-1} C^L;$$

after splitting with respect to the even and odd nodes, we obtain the assertions of Theorem 1.

The solvability of system (10) follows from the uniqueness of the corresponding interpolation spline (see [24, p. 141]).

Let

$$\rho = \max(h_i/h_j, |i-j| = 1),$$

where $h_i = x_{2i+2} - x_{2i}$.

It is known (see [24, p. 143]) that, if $\rho < (1+\sqrt{13})/2$, then system (10) possesses the strong diagonal dominance property, and it is recommended to solve it using the tridiagonal matrix algorithm. Otherwise the non-monotonic elimination method, which is valid for any well-conditioned matrix, should be used. Unfortunately, the well conditioning of the matrix in the case $\rho \geq (3+\sqrt{5})/2$ cannot be guaranteed.

For this reason, in the general case we will solve the following interpolation system for the unknowns $(S^{L-1})''(x_{2i}) = M_i$ $(i = 0, 1, \ldots, 2^{L-1})$ (see [24, p. 100]):

$$\begin{gathered} 2M_0 + M_1 = \frac{6f_1}{h_0^2}, \\ \mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = d_i, \quad i = 1, \ldots, 2^{L-1}-1, \\ M_i + 2M_{i+1} = \frac{6f_i}{h_i^2}, \quad i = 2^{L-1}-1. \end{gathered} \tag{15}$$

Here

$$\lambda_i = h_i/(h_{i-1}+h_i), \quad \mu_i = 1-\lambda_i, \tag{16}$$

$$d_i = \frac{6}{h_{i-1}+h_i}\left(\frac{f_{i+1}-f_i}{h_i} - \frac{f_i - f_{i-1}}{h_{i-1}}\right). \tag{17}$$

The matrix of system (15) possesses the diagonal dominance property (see [26]). Such matrices are nonsingular; therefore, the system of equations for the spline coefficients on the coarse grid Δ^{L-1} always has a unique solution. The solution to the system of equations for M_i is found by the tridiagonal matrix algorithm (TDMA). After finding M_i the coefficients C_i^{L-1} of the spline are calculated as described above using the local approximation formula, which is exact on the splines, but on the grid Δ^{L-1}:

$$C_i^{L-1} = f_i + \frac{1}{3}(h_i - h_{i-1})m_i - \frac{1}{6}h_i h_{i-1} M_i, \tag{18}$$

$$m_i = \frac{f_{i+1}-f_i}{h_i} - \frac{h_i}{6}(2M_i + M_{i+1}) = \frac{f_i - f_{i-1}}{h_{i-1}} + \frac{h_{i-1}}{6}(M_{i-1} + 2M_i). \tag{19}$$

5. EXAMPLES OF THE CALCULATION OF THE WAVELET DECOMPOSIYION BY HAND

5.1. The Numerical Differentiation Problem

Let $L = 3$, and let the discrete signal be represented by nine values of the analytic function $f(x) = (x^2 - 100)^2$ at the given points $\Delta^3 : x = -10, 1, 2, \ldots, 7, 10$. At the points ± 10, all the homogeneous boundary conditions that are required for constructing the wavelet expansion are satisfied; therefore, we can investigate the application of the constructed algorithm for numerical differentiation. Note that, at the first step of the wavelet expansion algorithm, the right-hand sides of Eqs. (7) may be assumed to be equal to the values of the interpolation spline; i.e., they are equal to the values of the given function at the even nodes of the fine grid. In reality, the data are typically noisy—they can be the values of a smoothing spline (see [27; 24, p. 154; 3]).

Recall that

$$N_1^{L-1}(x_2) = \frac{(x_2-x_0)(x_4-x_2)}{(x_4-x_0)(x_4-x_0)} + \frac{(x_6-x_2)(x_2-x_0)}{(x_6-x_0)(x_4-x_0)} = \left[\frac{2}{14} + \frac{4}{16}\right]\frac{12}{14} = \frac{33}{98},$$

$$N_1^{L-1}(x_4) = \frac{(x_6 - x_4)^2}{(x_6 - x_0)(x_6 - x_2)} = \frac{4}{16\times 4} = \frac{1}{16},$$

$$N_2^{L-1}(x_2) = \frac{(x_2 - x_0)^2}{(x_6 - x_0)(x_4 - x_0)} = \frac{12^2}{16\times 14} = \frac{9}{14},$$

$$N_2^{L-1}(x_4) = \frac{(x_4 - x_0)(x_6 - x_4)}{(x_6 - x_0)(x_6 - x_2)} + \frac{(x_8 - x_4)(x_4 - x_2)}{(x_8 - x_2)(x_6 - x_2)} = \frac{14\times 2}{16\times 4} + \frac{6\times 2}{8\times 4} = \frac{13}{16},$$

$$N_2^{L-1}(x_6) = \frac{(x_8 - x_6)^2}{(x_8 - x_2)(x_8 - x_4)} = \frac{4^2}{8\times 6} = \frac{1}{3},$$

$$N_3^{L-1}(x_4) = \frac{(x_4 - x_2)^2}{(x_8 - x_2)(x_6 - x_2)} = \frac{4}{8\times 4} = \frac{1}{8},$$

$$N_3^{L-1}(x_6) = \frac{(x_6 - x_2)(x_8 - x_6)}{(x_8 - x_2)(x_8 - x_4)} + \frac{(x_8 - x_6)(x_6 - x_4)}{(x_8 - x_4)(x_8 - x_4)} = \left[\frac{4}{8} + \frac{2}{6}\right]\frac{4}{6} = \frac{5}{9}.$$

Thus, there is no diagonal dominance in the first row:

$$\frac{33}{98} - \frac{9}{14} = -\frac{15}{49} < 0.$$

For this reason, we will use system (15), which is guaranteed to have the diagonal dominance property; for this system, (16) and (17) have the form

$$\lambda_1 = \frac{x_4 - x_2}{x_4 - x_0} = \frac{2}{14} = \frac{1}{7}, \quad \mu_1 = 1 - \lambda_1 = \frac{6}{7},$$

$$\lambda_2 = \frac{x_6 - x_4}{x_6 - x_2} = \frac{2}{4} = \frac{1}{2}, \quad \mu_2 = 1 - \lambda_2 = \frac{1}{2},$$

$$\lambda_3 = \frac{x_8 - x_6}{x_8 - x_4} = \frac{4}{6} = \frac{2}{3}, \quad \mu_3 = 1 - \lambda_3 = \frac{1}{3},$$

$$d_1 = \frac{6}{x_4 - x_0}\left(\frac{f(x_4) - f(x_2)}{x_4 - x_2} - \frac{f(x_2) - f(x_0)}{x_2 - x_0}\right) = \frac{6}{14}\left(\frac{7056 - 9216}{2} - \frac{9216}{12}\right) = -792,$$

$$d_2 = \frac{6}{x_6 - x_2}\left(\frac{f(x_6) - f(x_4)}{x_6 - x_4} - \frac{f(x_4) - f(x_2)}{x_4 - x_2}\right) = \frac{6}{4}\left(\frac{4096 - 7056}{2} - \frac{7056 - 9216}{2}\right) = -600,$$

$$d_3 = \frac{6}{x_8 - x_4}\left(\frac{f(x_8) - f(x_6)}{x_8 - x_6} - \frac{f(x_6) - f(x_4)}{x_6 - x_4}\right) = \frac{6}{6}\left(\frac{-4096}{4} - \frac{4096 - 7056}{2}\right) = 456.$$

First, we find the elimination coefficients v_i, u_i (the forward course of the TDMA [24, p. 337]):

$$v_0 = -\frac{1}{2}, \quad u_0 = \frac{6\times 9216}{2\times 12^2} = 192,$$

$$v_1 = \frac{-1/7}{2 + (6/7)(-1/2)} = \frac{-1/6}{3/4} = -\frac{1}{11}, \quad u_1 = \frac{-792 - 6/7\times 192}{2 + (6/7)(-1/2)} = \frac{112.5}{3/4} = -\frac{6696}{11},$$

$$v_2 = \frac{-1/2}{2 + (1/2)(-1/11)} = \frac{-1/2}{2/3} = -\frac{11}{43}, \quad u_2 = \frac{-600 + 6696/22}{2 + (1/2)(-1/11)} = \frac{139}{2/3} = -\frac{6504}{43},$$

$$v_3 = \frac{-2/3}{2 + (1/3)(-11/43)} = \frac{-1/2}{2/3} = -\frac{86}{247}, \quad u_3 = \frac{456 - 6504(-1/129)}{2 + (1/3)(-11/43)} = \frac{139}{2/3} = \frac{65328}{247},$$

$$v_4 = 0, \quad u_4 = \frac{6\times 4096/4^2 - 65328/247}{2 - 86/247} = \frac{13086}{17}.$$

It is clear that $M_4 = 13086/17 = 769.765$. The other unknowns are found using the backward TDMA formulas:

$$M_3 = -86/247 \times 13086/17 + 65328/247 = -3.529,$$
$$M_2 = -11/43 \times (-3.529) - 6504/43 = -150.353,$$
$$M_1 = 1/11 \times 150.353 - 6696/11 = -595.059,$$
$$M_0 = 1/2 \times 595.059 + 192H9.529.$$

The auxiliary quantities m_i are found by formulas (19):

$$m_1 = 9216/12 + 12/6(489.529 + 2 \times (-595.059)) = -633.178,$$
$$m_3 = -4096/4 - 4/6(2 \times (-3.529) + 769.765) = -1532.47.$$

For the level $L = 2$, we find the coefficients C_i^2 using the formulas of the local spline approximation (18):

$$C_1^2 = 9216 + (4 - 4 - 10)/ \times (-633.178) - 2 \times 12/6 \times (-595.059) = 13707,$$
$$C_2^2 = 7056 - 2 \times 2/6 \times (-150.353) = 7156,$$
$$C_3^2 = 4096 + (10 - 12 + 4)/3 \times (-1532.47) - 4 \times 2/6 \times (-3.53) = 3079.$$

To calculate the odd coefficients C_i^3, we have to repeat the same operation for the original grid Δ^3. Then, the wavelet coefficients D_i^2 are obtained by (11), (12):

$$D_1^2 = 12025.21 - \frac{x_1 - x_0}{x_4 - x_0} \times 13707 = 12025.21 - 11/14 \times 13707 = 1255.56,$$

$$D_2^2 = 8332.61 - \frac{x_3 - x_6}{x_0 - x_6} \times 13707 - \frac{x_3 - x_0}{x_6 - x_0} \times 7156$$
$$= 8332.61 - 3/16 \times 13707 - 13/16 \times 7156 = -51.86,$$

$$D_3^2 = 5642.324 - \frac{x_5 - x_8}{x_2 - x_8} \cdot 7156 - \frac{x_5 - x_2}{x_8 - x_2} \times 3079$$
$$= 5642.324 - 5/8 \times 7156 - 3/8 \times 3079 = 15.03,$$

$$D_4^2 = 1559.6 - \frac{x_8 - x_7}{x_8 - x_4} \times 3079 = 1559.6 - 3079/2 = 20.07.$$

Following [6], we will not encumber the presentation by constructing the wavelets of level $L = 1$. We restrict ourselves to discarding three relatively small coefficients D_2^2, D_3^2, and D_4^2 of the wavelet expansion thus compressing the input digital signal with the coefficient $K = 9/4 = 2.25$.

The coefficients of the approximation spline are reconstructed using the complete matrix of the direct wavelet transform (9):

$$\begin{bmatrix} C_1^3 \\ C_2^3 \\ \vdots \\ C_7^3 \end{bmatrix} = \begin{bmatrix} \frac{11}{14} & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{543}{1568} & \frac{143}{224} & 0 & -\frac{3}{62} & -\frac{13}{62} & 0 & 0 \\ \frac{3}{16} & \frac{13}{16} & 0 & 0 & 1 & 0 & 0 \\ \frac{3}{64} & \frac{23}{64} & \frac{3}{32} & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{5}{8} & \frac{3}{8} & 0 & 0 & 1 & 0 \\ 0 & \frac{5}{16} & \frac{29}{48} & 0 & 0 & -\frac{5}{22} & -\frac{3}{22} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 13707 \\ 7156 \\ 3079 \\ 1255 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 12025.21 \\ 9254.424 \\ 8384.471 \\ 7081.059 \\ 5627.294 \\ 4096.588 \\ 1539.529 \end{bmatrix}.$$

Using the vector notation of the cubic spline, we obtain the filtered values

$$\begin{bmatrix} \varphi^3(x_1) \\ \varphi^3(x_2) \\ \vdots \\ \varphi^3(x_7) \end{bmatrix} C^3 = [9792.57, 9216, 8312.23, 7056, 5614.47, 4095, 2593.99]^{\mathrm{T}}$$

and the resulting values of the second-order derivative at the endpoints of the interval

$$(S^3)''(x_0) = (N_1^3)''(x_0)C_1^3 = \frac{6}{(x_0 - x_1)(x_0 - x_2)} C_1^3 = \frac{6 \times 12025.21}{11 \times 12} = 546.6,$$

$$(S^3)''(x_8) = (N_7^3)''(x_8)C_7^3 = \frac{6}{(x_8 - x_7)(x_8 - x_6)} C_1^3 = \frac{6 \times 1539.529}{3 \times 4} = 769.764,$$

which is only 1.3% less than the value 779.8 at the right endpoint obtained using the interpolation spline. The root-mean-square deviation is

$$\mathrm{rms} = \left(\frac{1}{7} \sum_{i=1}^{7} \left(f(x_i) - S^3(x_i) \right)^2 \right)^{1/2} \approx 4.96.$$

5.2. The Prediction Problem

In this section, we perform the wavelet analysis of the function $f(x) = (x^2 - 100)^2$ given on the interval $[0, 8]$ at the nodes of the uniform grid $\Delta^3 : x = 0, 1, \ldots, 8$. Since the homogeneous boundary conditions are not completely satisfied ($f(a) = 10\,000$, $f'(a) = 0$, $f(b) = 1296$, and $f'(b) = -1152$), we must subtract from the given signal the values of the cubic interpolation polynomial $f(a) + (x - a)[f'(a) + t(B + tA)]$ before processing, where (see [24, p. 59])

$$A = -2(f(b) - f(a))/(b - a) + f'(a) + f'(b),$$
$$B = -A + (f(b) - f(a))/(b - a) - f'(a),$$
$$t = (x - a)/(b - a).$$

After the wavelet analysis of the differences and the reconstruction of the cubic approximation spline by the wavelet coefficients, the values of this polynomial are added to this spline. Note that there are no computational difficulties in the case of the uniform grid; therefore, the direct matrix approach can be used.

For the level $L = 3$, we find the coefficients C^3 by solving the interpolation problem (see [24, p. 141]):

$$\begin{bmatrix} C_1^3 \\ C_2^3 \\ \vdots \\ C_7^3 \end{bmatrix} = \begin{bmatrix} \frac{7}{12} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & 0 & 0 & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{7}{12} \end{bmatrix}^{-1} \begin{bmatrix} 9801 - 9752 \\ 9216 - 9072 \\ \vdots \\ 2601 - 2552 \end{bmatrix} = \begin{bmatrix} 42 \\ 147 \\ 234 \\ 267 \\ 234 \\ 147 \\ 42 \end{bmatrix}.$$

For the level $L = 2$, the coefficients C_i^2 and D_i^2 are found by the forward course of the wavelet transform:

$$\begin{bmatrix} C_1^2 \\ C_2^2 \\ C_3^2 \\ D_1^2 \\ D_2^2 \\ D_3^2 \\ D_4^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{11}{16} & \frac{1}{8} & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{8} & \frac{11}{16} & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 42 \\ 147 \\ 234 \\ 267 \\ 234 \\ 147 \\ 42 \end{bmatrix} = \begin{bmatrix} 160 \\ 304 \\ 160 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}.$$

In this case, all four wavelet coefficients are equally insignificant. In the backward course of the wavelet transform, we obtain

$$\begin{bmatrix} C_1^3 \\ C_2^3 \\ \vdots \\ C_7^3 \end{bmatrix} = \begin{bmatrix} \frac{11}{14} & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{543}{1568} & \frac{143}{224} & 0 & -\frac{3}{62} & -\frac{13}{62} & 0 & 0 \\ \frac{3}{16} & \frac{13}{16} & 0 & 0 & 1 & 0 & 0 \\ \frac{3}{64} & \frac{23}{64} & \frac{3}{32} & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{5}{8} & \frac{3}{8} & 0 & 0 & 1 & 0 \\ 0 & \frac{5}{16} & \frac{29}{48} & 0 & 0 & -\frac{5}{22} & -\frac{3}{22} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 160 \\ 304 \\ 160 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 40 \\ 148 \\ 232 \\ 268 \\ 232 \\ 148 \\ 40 \end{bmatrix}.$$

Then, the reconstructed values of the approximation spline are

$$\begin{bmatrix} \frac{7}{12} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & 0 & 0 & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{7}{12} \end{bmatrix} \begin{bmatrix} C_1^3 \\ C_2^3 \\ \vdots \\ C_7^3 \end{bmatrix} = \begin{bmatrix} 48 \\ 144 \\ 224 \\ 256 \\ 224 \\ 144 \\ 48 \end{bmatrix}.$$

Taking into account the corrections by the values of the cubic polynomial, we obtain the filtered values $[9800, 9216, 8280, 7056, 5624, 4096, 2600]^{\mathrm{T}}$ and the resulting rms ≈ 0.286.

In the wavelet analysis of actual discrete signals, the homogeneous boundary conditions required for constructing the wavelet expansion are known to be unsatisfied. Therefore, the above algorithm can be interpreted as the solution of the problem of the point forward and backward prediction given a time series of the measured values of the quantity to be predicted (see [28]).

For the above example, consider an alternative hypothesis that the homogeneous boundary conditions are fulfilled on the interval $[0, 8]$ but only the zero value of the first-order derivative at the left endpoint is actually true; in this case, the input data are processed without corrections. For the level $L = 3$, we find the coefficients C^3:

$$\begin{bmatrix} C_1^3 \\ C_2^3 \\ \vdots \\ C_7^3 \end{bmatrix} = \begin{bmatrix} \frac{7}{12} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & 0 & 0 & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{7}{12} \end{bmatrix}^{-1} \begin{bmatrix} 9801 \\ 9216 \\ \vdots \\ 2601 \end{bmatrix} = \begin{bmatrix} 14507.325 \\ 8030.361 \\ 8667.231 \\ 6986.714 \\ 5721.912 \\ 3875.639 \\ 3351.532 \end{bmatrix}.$$

For the level $L = 2$, we find the coefficients C_i^2 and D_i^2 using the forward course of the wavelet transform:

$$\begin{bmatrix} C_1^2 \\ C_2^2 \\ C_3^2 \\ D_1^2 \\ D_2^2 \\ D_3^2 \\ D_4^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{11}{16} & \frac{1}{8} & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{8} & \frac{3}{4} & \frac{1}{8} & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{8} & \frac{11}{16} & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 14507.325 \\ 8030.361 \\ 8667.231 \\ 6986.714 \\ 5721.912 \\ 3875.639 \\ 3351.532 \end{bmatrix} = \begin{bmatrix} 14172.5715 \\ 5691.9997 \\ 5395.4287 \\ 10964.1821 \\ -1265.0546 \\ 178.1978 \\ 2002.6749 \end{bmatrix}.$$

In the backward course of the wavelet transform, we can try to eliminate the minimal wavelet coefficient $D_3^2 = 178.1978$ and obtain in this case rms ≈ 14.925. The attempt to eliminate the next (in the absolute value) wavelet coefficient $D_2^2 = -1265.0546$ gives rms ≈ 109.074. The relation between the prediction error and the number of discarded wavelet coefficients immediately shows that the alternative hypothesis is not true.

If we subtract from the input data the values of the linear function $10\,000 - 1088x$ thus making an error only in the satisfaction of the zero condition for the first-order derivative at the right endpoint, then the algorithm yields the following results:

$$C^3 = [1122.318, 1405.887, 1606.135, 1439.571, 1083.679, 616.113, 195.968]^{\mathrm{T}},$$

$$C^2 = [1970.2862, 1455.9994, 653.7142]^{\mathrm{T}},$$

and

$$D^2 = [629.7465, -107.0078, 28.7222, 32.5395]^{\mathrm{T}}.$$

In this case, we may neglect the two insignificant coefficients D_3^2 and D_4^2, which yields the reconstructed coefficients

$$C^3 = [1122.318, 1405.887, 1606.135, 1446.752, 1054.857, 631.428, 163.429]^{\mathrm{T}},$$

the filtered values $[9801, 9216, 8282.197, 7056, 5609.601, 4096, 2584.572]^{\mathrm{T}}$, and $\mathrm{rms} \approx 3.221$. The attempt to eliminate the next (in the absolute value) wavelet coefficient $D_2^2 = -107.0078$ yields $\mathrm{rms} \approx 9.826$. The relation between the prediction error and the number of discarded wavelet coefficients shows that this solution is also significantly inferior to the solution obtained by substituting into the prediction all the exact values of the function and its first-order derivative. The choice of the prediction step size and the development of a numerical method for finding the best predicted values of the function and (or) its first-order derivative at the points $x = a, b$ require further studies.

6. CONCLUSIONS

The procedure for constructing the spline wavelets that are semiorthogonal with respect to the scalar product with the derivatives proposed in this paper and the corresponding procedure for obtaining for them implicit relations for the expansion with splitting with respect to the even and odd modes provides new capabilities for creating computationally efficient algorithms of constructing and using spline wavelets on nonuniform grids. The numerical results confirm the existence of the optimal prediction values of the constructed spline wavelets.

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