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Numerical Solution of Vector Sturm–Liouville Problems with Dirichlet Conditions and Nonlinear Dependence on the Spectral Parameter

L. D. Akulenko[a,b,c,] *, A. A. Gavrikov[c,] **, and S. V. Nesterov[c,] ***

[a] *Moscow Institute of Physics and Technology, Dolgoprudnyi, Moscow oblast, 141701 Russia*

[b] *Bauman Moscow State Technical University, Moscow, 105005 Russia*

[c] *Institute for Problems of Mechanics, Russian Academy of Sciences, Moscow, 119526 Russia*

** e-mail: l.akulenko@bk.ru*

*** e-mail: gavrikov@ipmnet.ru*

**** e-mail: kumak@ipmnet.ru*

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Abstract—A numerical-analytical iterative method is proposed for solving generalized self-adjoint regular vector Sturm–Liouville problems with Dirichlet boundary conditions. The method is based on eigenvalue (spectral) correction. The matrix coefficients of the equations are assumed to be nonlinear functions of the spectral parameter. For a relatively close initial approximation, the method is shown to have second-order convergence with respect to a small parameter. Test examples are considered, and the model problem of transverse vibrations of a hinged rod with a variable cross section is solved taking into account its rotational inertia.

Keywords: numerical solution of Sturm–Liouville problem, eigenvalues, eigenfunctions, boundary value problems, nonlinear dependence of coefficients on spectral parameter.

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INTRODUCTION

The interaction of distributed systems can be described by differential and integrodifferential equations, which lead to vector Sturm–Liouville problems (VSLPs) that are nonlinear with respect to the spectral parameter (SP).

An accelerated convergence method for finding eigenvalues and eigenfunctions of systems described by VSLPs with a classical SP dependence and by one-dimensional SLPs with a nonlinear SP dependence was developed in [1–3]. A natural continuation and generalization of [1–3] is to extend the method to problems with an arbitrary dependence of their coefficients on SP. The goal of this study is to describe such a method and test its performance. The existence of solutions, the isolation and multiplicity of eigenvalues, and the completeness and basis properties of eigen- and associated functions are not discussed.

Numerous methods have been developed for solving and studying the classical one-dimensional problem [1, 4–8]. Much attention has been given to problems with a nonlinear SP dependence of coefficients and with boundary conditions of various types in both one-dimensional [9–19] and vector [20, 21] setting. The basic difference of our method (including from those based on spectral correction [22]) is that the miss-distance (or mismatch) function is defined as the difference between the lengths of the problem's intervals for an eigenvalue approximation and its sought value (see below), and Newton-type spectral correction is used taking into account the error function.

By analogy with the one-dimensional version of the method for the classical SLP [23, 24], it can be expected that the method described will be convenient in experimental studies, since the basic varying parameter is the length of the problem's interval, which corresponds to a particular physical quantity (the length of a rod or string, an angular variable, etc.).

1. FORMULATION OF THE GENERALIZED VECTOR STURM–LIOUVILLE PROBLEM

Consider the Sturm–Liouville problem: find $\lambda \in \mathbb{R}$ such that there exists a nontrivial piecewise smooth vector function X that solves the following self-adjoint VSLP with Dirichlet conditions [25]:

$$\begin{aligned} &(P(x,\lambda)X')' + R(x,\lambda)X = 0, \quad 0 < x < l, \quad X(0) = X(l) = 0, \\ &X(x) = (X_1(x), \ldots, X_s(x))^{\mathrm{T}}, \quad X(x), P(x,\cdot)X'(x) \in (PC_1[0,l])^s. \end{aligned} \tag{1.1}$$

Assume that the SP $\lambda \in \Lambda \subset \mathbb{R}$ in problem (1.1) varies within a given domain Λ (specifically, $\Lambda = (0,\infty)$), the spectrum is discrete, and the problem has a solution. Further assumptions concerning the behavior of the coefficients are supposed to hold on some extended interval of x: $x \in [0,l^*]$, where $l^* = l + \delta_0$, $\delta_0 > 0$. The $s \times s$ matrices P and R are assumed to be symmetric for all admissible values of their arguments. Additionally, we assume that the elements of P and R are bounded for fixed λ, continuous with respect to x, and continuously differentiable with respect to λ and one of the matrices R_λ', P_λ' is not singular (see below). The eigenvalues λ are assumed to be continuously differentiable functions of the length l of the problem's interval, and the same it true for the inverse mapping $l(\lambda)$.

For some auxiliary (not mandatory) stages of the algorithm described below, the matrices P^{-1} and P are required to be differentiable with respect to x, while, for the proofs of the convergence theorems, this property is assumed to hold only for P. Additionally, we assume that the matrix functions P and R can be locally represented in a small δ-neighborhood of each point of the admissible domain of x, λ in the form

$$\begin{aligned} &P = P_0(x) + \zeta P_1(x,\lambda,\zeta), \quad R = \lambda R_0(x) - Q(x) + \zeta R_1(x,\lambda,\zeta), \quad 0 \le \zeta \ll 1, \\ &P_0 = P_0^{\mathrm{T}} > 0, \quad R_0 = R_0^{\mathrm{T}} > 0, \quad Q = Q^{\mathrm{T}} \ge 0, \quad \|P_0\| > \zeta\|P_1\|, \quad \|\lambda R_0 - Q\| > \zeta\|R_1\|, \end{aligned}$$

where $\|\cdot\|$ is a functional matrix norm.

2. DIFFERENTIAL PROPERTIES OF THE SOLUTION AND THE EIGENVALUE

Assuming that the sought eigenfunction X is a function of x, $\lambda(l)$, and l and treating l as an independent variable parameter, we have the representation

$$X = X(x,\lambda,l), \quad \lambda = \lambda(l), \quad X = \tilde{X}(x,\lambda(l),l) \equiv X(x,l), \quad 0 < l \le l^* < \infty.$$

Introducing the complete partial derivative $Y = \partial X/\partial l$, $x, l \in [0,l^*]$, $\lambda \in \Lambda$, we differentiate system (1.1) with respect to l (here, $\lambda' = d\lambda/dl$):

$$\lambda'(P_\lambda X')' + (PY')' + \lambda' R_\lambda X + RY = 0, \quad X_l(0) = Y(0) = 0, \quad X'(l) + Y(l) = 0. \tag{2.1}$$

The inner product in $\mathbb{L}_2(0,l)$ is defined as $(U,V)_l = \int_0^l U^{\mathrm{T}}(x)V(x)dx$. Then, integrating the scalar product of Eq. (2.1) and X by parts, we express the sought quantity $\lambda'(l)$ as a function of λ and l:

$$\begin{aligned} &\lambda' = -X^{\mathrm{T}}{}'(l)P(l,\lambda)X'(l)/N(l,\lambda) \equiv f(\lambda,l), \\ &N(l,\lambda) = -(X', P_\lambda' X')_l + (X, R_\lambda' X)_l \ne 0. \end{aligned} \tag{2.2}$$

For $R_\lambda' > 0$ and $P_\lambda' \le 0$, we have $N > 0$, which usually holds in applications.

Remark 1. Formula (2.2) is a generalization of the well-known expression [1, 8] for eigenvalues of classical scalar SLPs to the problems under consideration, and λ' can be treated as the Fréchet derivative of a functional defined on the Banach space of solutions to problem (2.1) and acting on $\mathbb{R}$.

3. CORRECTION PROCEDURE FOR EIGENVALUES AND EIGENFUNCTIONS

Let ε be a small quantity ($|\varepsilon| \ll 1$) corresponding to an arbitrarily small increment of l. Then λ' can be represented in the form $\lambda'(l) = \lim_{\varepsilon \to 0}(\lambda(l + \varepsilon l) - \lambda(l))/\varepsilon l$ or, for an arbitrary point η that is ε-close to l, we can write

$$\lambda(l) = \lambda(\eta) + \varepsilon l\lambda'(\eta) + O(\varepsilon^2), \quad \varepsilon = (l - \eta)/l, \quad |\varepsilon| \ll 1. \tag{3.1}$$

The derivative $\lambda'(\eta)$ can be found using formula (2.2) with $l = \eta$ and $\lambda = \lambda(\eta)$.

Let $\lambda^{(0)}$ be an initial approximation. The solution is constructed using successive approximations. To find $X^{(0)} = X(x, \lambda^{(0)})$, i.e., the generating solution of the boundary value problem satisfying system (1.1) for $\lambda = \lambda^{(0)}$ and some $l = \xi^{(0)}$, we consider s Cauchy problems

$$(PX'_{(i)})' + RX_{(i)} = 0, \quad X_{(i)}(0) = 0, \quad X'_{(i)}(0) = e_i, \quad i = 1, \ldots, s, \tag{3.2}$$

where e_i is the ith basis vector and $\lambda = \lambda^{(0)}$. The point $x = \xi^{(0)}$ is found as the sought mth (m is, for example, the eigenvalue index) or nearest-to-l root of the equation

$$\Delta(x, \lambda^{(0)}) = 0, \quad \xi^{(0)} = \xi(\lambda^{(0)}) \in (0, l^*]. \tag{3.3}$$

Here, Δ is the determinant of the matrix $W(x, \lambda^{(0)})$, whose columns are the solutions of Cauchy problems (3.2). In practice, $\xi^{(0)}$ can be found using a numerical-graphical method with simultaneously solving these problems, where Δ is calculated at every step of the numerical integration. Assuming for simplicity that $\Delta'_x(\xi^{(0)}, \lambda^{(0)}) \neq 0$, we obtain $\operatorname{rank} W(\xi^{(0)}, \lambda^{(0)}) = s - 1$. Then, representing the eigenfunction as a linear combination of the basis solutions $X_{(i)}$, i.e., $X^{(0)} = c_1 X_{(1)} + \ldots + c_s X_{(s)}$ and considering a linear homogeneous system for the constants c_j, namely, $X^{(0)}(\xi^{(0)}) = W(\xi^{(0)}, \lambda^{(0)})C = 0$, where $C = (c_1, \ldots, c_s)^{\mathrm{T}}$, we can use some c_i and the vector functions $X_{(j)}(\xi^{(0)})$ in order to express the other c_k, $k \neq i$: $X^{(0)}(x) = W(x, \lambda^{(0)})C(\xi^{(0)})$.

Then, according to (2.2), the next approximation (improved up to an $O(\varepsilon^2)$ error) is given by

$$\begin{gathered}\lambda^{(1)} = \lambda^{(0)} - \varepsilon^{(0)} l X^{(0)\mathrm{T}'}(\xi^{(0)}) P(\xi^{(0)}, \lambda^{(0)}) X^{(0)'}(\xi^{(0)}) / N(\xi^{(0)}, \lambda^{(0)}), \quad \varepsilon^{(0)} = \frac{l - \xi^{(0)}}{l}, \\ N(\xi^{(0)}, \lambda^{(0)}) = \left[-(X^{(0)'}, P'_\lambda X^{(0)'})_{\xi^{(0)}} + (X^{(0)}, R'_\lambda, X^{(0)})_{\xi^{(0)}} \right]\Big|_{\lambda = \lambda^{(0)}} \neq 0.\end{gathered} \tag{3.4}$$

Here, $N(\xi^{(0)}, \lambda^{(0)}) = V(\xi^{(0)})$ can be calculated by integrating the scalar Cauchy problem $V' = -X^{(0)\mathrm{T}'} P'_\lambda X^{(0)'} + X^{(0)\mathrm{T}} R'_\lambda X^{(0)}$, $V(0) = 0$ with $\lambda = \lambda^{(0)}$. However, the auxiliary problem in this formulation involves right-hand-side terms that are quadratic in the numerically found vector $X^{(0)}$, which requires high-order accurate integration, for example, by applying a higher order accurate scheme than that used for solving problems (3.2). Integration by parts yields

$$-(X', P'_\lambda X')_\xi = (X, (P_\lambda P^{-1})'(PX') - (P'_\lambda P^{-1})RX)_\xi$$

(here, $\lambda = \lambda^{(0)}$, $X = X^{(0)}$, $\xi = \xi^{(0)}$). We introduce a sensitivity vector function $K(x)$, $K(0) = 0$ such that

$$N(\xi, \lambda) = X^{\mathrm{T}'}(\xi) P(\xi, \lambda) K(\xi) = (P(\xi, \lambda) X'(\xi))^{\mathrm{T}} K(\xi). \tag{3.5}$$

Then expression (3.5) can be reduced to

$$N(\xi, \lambda) = \int_0^\xi (X^{\mathrm{T}'} PK)'\, dx = -(X, (PK')' + RK)_\xi.$$

Since X is a nontrivial function, $N(\xi, \lambda)$ can be represented in the form of (3.5), where the vector function K is the solution of the following Cauchy problem (linear with respect to K, X, and X') at $\lambda = \lambda^{(0)}$, $X = X^{(0)}$, and $\xi = \xi^{(0)}$:

$$(PK')' + RK = -(P'_\lambda P^{-1})' PX' - \left[R'_\lambda - (P'_\lambda P^{-1})R \right] X, \quad K(0) = K'(0) = 0. \tag{3.6}$$

Remark 2. The requirement $\Delta'_x \neq 0$ (which is equivalent to the condition that λ is simple) is caused by the necessity of calculating the number of zeros of $\Delta(x)$, whose sign does not change for eigenvalues of

even multiplicity; consequently, Cauchy problems (3.2) have to be integrated to high accuracy. If these problems are solved using a specialized numerical method that rather accurately takes into account even-order zeros of $\Delta(x)$ (e.g., by applying a method intended in the original form for calculating the number of eigenvalues [26] or the Prüfer vector method [27, 28], both of which automatically count zeros taking into account multiplicity, or by applying a semianalytical method of perturbing a constant reference potential of high order [7]), then modifications of the algorithm are minimal. Specifically, the multiplicity of λ is determined by the rank of $W(\xi,\lambda)$ and is equal to $s - \operatorname{rank} W(\xi,\lambda)$, and the corresponding eigenfunctions are also found as linear combinations of the constructed solutions $X_{(i)}$. For example, for a multiplicity of two, $X_l = c_{1,l}X_{(1)} + \ldots + c_{s,l}X_{(s)}$, $l = 1, 2$, where $C_l = (c_{1,l}, \ldots, c_{s,l})^{\mathrm{T}}$ are linearly independent solutions of the system of equations $W(\xi,\lambda)C = 0$, which are expressed in terms of some c_i, c_j, $i \neq j$, and, without loss of generality, we can set $c_i = 1$ and $c_j = 0$ for X_1 and $c_i = 0$ and $c_j = 1$ for X_2. Moreover, the multiplicity of λ can vary with varying l; hence, a situation is possible where the mth zero of $\Delta(x)$ is multiple for the given $\lambda^{(0)}$, but becomes simple for the refined $\lambda^{(1)}$.

Remark 3. In the SP-nonlinear case, it cannot be stated a priori that the number of zeros of $\Delta(x)$ increases with growing λ. As follows from (2.2), the number of zeros can increase or decrease depending on the behavior of the matrix functions P, P'_λ, and R'_λ: for example, for $P > 0$, if $N > 0$ on some interval $\Lambda_+ \subset \Lambda$ (which is possible, e.g., for $P'_\lambda < 0$, $R'_\lambda > 0$), then the number of zeros increases with growing λ, while, if $N < 0$ for $\lambda \in \Lambda_- \subset \Lambda$ (e.g., for $P'_\lambda > 0$, $R'_\lambda < 0$), then the number of zeros decreases with growing λ.

4. ITERATIVE PROCEDURE AND THE PROBLEM OF CONVERGENCE

To find the next approximation $\lambda^{(2)}$, we need (1) once again to find solutions of s Cauchy problems (3.4) at $\lambda = \lambda^{(1)}$; (2) determine $\xi^{(1)}$; (3) construct the eigenfunction $X^{(1)}$; and (4) calculate a refined value of $\lambda^{(2)}$ by using formula (3.4), etc.

Thus, the following assertions hold.

Theorem 1. (i) *An eigenvalue of problem* (1.1) *can be found with prescribed accuracy by applying the following iterative scheme with a quadratic error*:

$$\lambda^{(j+1)} = \lambda^{(j)} - \varepsilon^{(j)} l X^{(j)\mathrm{T}}(\xi^{(j)}) P(\xi^{(j)}, \lambda^{(j)}) X^{(j)\prime}(\xi^{(j)}) / N(\xi^{(j)}, \lambda^{(j)}),$$

$$N(\xi^{(j)}, \lambda^{(j)}) = [-(X', P'_\lambda X')_{\xi^{(j)}} + (X, R'_\lambda X)_{\xi^{(j)}}]\Big|_{\lambda=\lambda^{(j)}}, \quad \varepsilon^{(j)} = \frac{l - \xi^{(j)}}{l}.$$

Here, $\varepsilon^{(j)}, \xi^{(j)}$, *and* $X^{(j)}$ *for the given* $\lambda^{(j)}$ *are calculated by solving problems of type* (3.2) *at the preceding jth step*.

(ii) *The accuracy* $O(\varepsilon^{\theta_j})$ $(\theta_j = 2^j, \varepsilon = \varepsilon^{(0)})$ *of the computed eigenvalue* $\lambda(l)$ *is determined by a two-sided estimate following from formula* (2.2). *For* $P(\xi^{(j)})$, $P(\xi^{(k)}) > 0$, $P'_\lambda \leq 0$, *and* $R'_\lambda \geq 0$ (*where an equality is allowed for only one matrix*), *if* $\xi^{(j)} < l$ *at the jth iteration step, then* $\lambda^{(j)} > \lambda$, *and, if* $\xi^{(k)} > l$ *at the kth step, then* $\lambda^{(k)} < \lambda$. *For* $P(\xi^{(j)})$, $P(\xi^{(k)}) > 0$, $P'_\lambda \geq 0$, *and* $R'_\lambda \leq 0$ (*where an equality is also allowed for only one matrix*), *the reverse inequalities are valid. Namely, if* $\xi^{(j)} > l$ *at the jth iteration step, then* $\lambda^{(j)} < \lambda$, *and, if* $\xi^{(k)} < l$ *at the kth step, then* $\lambda^{(k)} > \lambda$. *Thus, the eigenvalue lies between the calculated* $\lambda^{(j)}$ *and* $\lambda^{(k)}$, *i.e.*, $\lambda^{(k)} < \lambda < \lambda^{(j)}$ *or* $\lambda^{(j)} < \lambda < \lambda^{(k)}$, $\left|\lambda^{(k)} - \lambda^{(j)}\right| = O(\varepsilon^{\theta_{\min(j,k)}})$. *If* $P(\xi^{(j)})$, $P(\xi^{(k)}) < 0$, *then the reverse inequalities are valid*.

Theorem 2. *According to* (3.1) *and* (3.4), *the procedure described leads to a uniform quadratic estimate with respect to the dimensionless parameter* ε:

$$\begin{gathered}\left|\lambda(l)-\lambda^{(1)}(\varepsilon,\lambda^0)\right|\le d(l,\lambda^0)|\varepsilon|^2,\quad \left\|X(x,\lambda,l)-X^{(1)}(x,\lambda^0,\xi)\right\|_C\le K_1|\varepsilon|^2,\\ \left\|P(x,\lambda)X'(x,\lambda,l)-P(x,\lambda^0)X^{(1)\prime}(x,\lambda^0,\xi)\right\|_C\le K_2|\varepsilon|^2,\quad \lambda^0=\lambda^{(0)},\quad \xi=\xi^{(0)},\end{gathered}\tag{4.1}$$

where $\varepsilon=(l-\xi)/l$ *is sufficiently small.*

In the generic case, the coefficient d in (4.1) is uniformly bounded in terms of the matrix functions $P(x,\lambda)$, $R(x,\lambda)$, and their derivatives with respect to x,λ.

Theorem 2 is proved by applying the perturbation method and majorant functions [29]. Assume that problem (1.1) has a solution at some λ. Consider s Cauchy problems (3.2) at $\lambda=\lambda^0\in\Lambda$. According to the assumptions made, the eigenvalues depend continuously on the interval length l. Therefore, there exists a point $x=\xi\leqslant l$ such that the determinant in (3.3) vanishes and there is a linear combination $X^{(0)}$ of solutions to these Cauchy problems for which

$$(P(x,\lambda^0)X^{(0)\prime})'+R(x,\lambda^0)X^{(0)}=0,\quad 0<x<\xi,\quad X^{(0)}(0)=X^{(0)}(\xi)=0.\tag{4.2}$$

Assuming that ξ is close to l, we introduce the small parameter $\varepsilon=(l-\xi)/l$, $\|\varepsilon\|\ll 1$, and make the substitution $y=x\xi/l$ in the original problem (1.1): $Y(y)=X(yl/\xi)$,

$$\left(P\left(\frac{y}{1-\varepsilon},\lambda\right)Y'(y)\right)'(1-\varepsilon)^2+R\left(\frac{y}{1-\varepsilon},\lambda\right)Y(y)=0,\quad Y(0)=Y(\xi)=0.$$

Thus, we obtain the perturbed boundary value problem

$$(\tilde P(y,\lambda,\varepsilon)Y'(y,\varepsilon))'(1-\varepsilon)^2+\tilde R(y,\lambda,\varepsilon)Y(y,\varepsilon)=0,\quad Y(0,\varepsilon)=Y(\xi,\varepsilon)=0.\tag{4.3}$$

Expanding the desired solution λ, Y of problem (4.3) in powers of ε and omitting higher order terms yields the following representations for eigenvalues and eigenfunctions:

$$\lambda=\lambda(\varepsilon)=\lambda_0+\varepsilon\lambda_1+O(\varepsilon^2),\quad Y(y,\varepsilon)=Y_0(y)+\varepsilon Y_1(y)+O(\varepsilon^2).$$

The matrix coefficients are naturally represented in the form

$$\begin{gathered}\tilde P(y,\lambda,\varepsilon)=\tilde P(y,\lambda(0),0)+\varepsilon\left(\lambda'_\varepsilon(0)\tilde P'_\lambda(y,\lambda(0),0)+\tilde P'_\varepsilon(y,\lambda(0),0)\right)\\ +\varepsilon^2S(y,\lambda_0,\lambda_1)=P(y,\lambda_0)+\varepsilon\left(\lambda_1P'_\lambda(y,\lambda_0)+yP'_y(y,\lambda_0)\right)+\varepsilon^2S(y,\lambda_0,\lambda_1),\\ \tilde R(y,\lambda,\varepsilon)=R(y,\lambda_0)+\varepsilon\left(\lambda_1R'_\lambda(y,\lambda_0)+yR'_y(y,\lambda_0)\right)+\varepsilon^2T(y,\lambda_0,\lambda_1).\end{gathered}\tag{4.4}$$

According to (4.4), we obtain the boundary value problem

$$\begin{gathered}\left(\left(P+\varepsilon\left(\lambda_1P'_\lambda+yP'_y\right)+\varepsilon^2S\right)\left(Y'_o+\varepsilon Y'_1+O(\varepsilon^2)\right)\right)(1-\varepsilon)^2\\ +\left(R+\varepsilon\left(\lambda_1R'_\lambda+yR'_y\right)+\varepsilon^2T\right)\left(Y_0+\varepsilon Y_1+O(\varepsilon^2)\right)=0,\\ Y_0+\varepsilon Y_1+O(\varepsilon^2)\big|_{y=0}=Y_0+\varepsilon Y_1+O(\varepsilon^2)\big|_{y=\xi}=0\end{gathered}\tag{4.5}$$

(for brevity, the dependence of P, R, and their derivatives on y, λ_0, λ_1 was omitted). For the zero degree of the small parameter, the boundary value problem coincides with (4.2), whose solution λ^0, $X^{(0)}$ is known. It follows that $\lambda_0=\lambda^0$ and $Y_0=X^{(0)}$. In what follows, we assume that $\|Y_0\|_{\mathbb{L}_2(0,\xi)}=1$.

To apply further expansions in powers of ε, find the next expansion terms λ_i, Y_i, and estimate the remainder, we need the analyticity of the matrix coefficients in problem (4.3), which is a rather restrictive assumption for the generic case. For this reason, after finding the zeroth expansion term λ_0, Y_0, we solve problem (4.3) by applying perturbation theory [30], which does not require such assumptions (the method of successive approximations [29]).

Let $\lambda = \lambda_0 + \varepsilon\lambda_1(\varepsilon)$ and $Y = Y_0 + \varepsilon Y_1(y,\varepsilon)$. By analogy with (4.5), dividing through by $\varepsilon \neq 0$, we obtain (at the first and higher powers of the small parameter) an inhomogeneous boundary value problem that is quasilinear in λ_1 and linear in Y_1, namely,

$$(PY_1')' + RY_1 = \Big((2P - \lambda_1 P_\lambda' - yP_y')Y_0'\Big)' - \Big(\lambda_1 R_\lambda' + yR_y'\Big)Y_0 + F(y, Y_1, \lambda_1, \varepsilon), \quad Y_1(0,\varepsilon) = Y_1(\xi,\varepsilon) = 0, \quad (4.6)$$

where $\lambda = \lambda_0$ and the function F is linear in Y_1 and Lipschitz continuous in λ_1 and satisfies $|F| \le C\varepsilon$ and $F(\varepsilon)\big|_{\varepsilon=0} = 0$. Considering problem (4.6) without a perturbation of order ε (of F) and applying the Fredholm theorems [31], we conclude that a nontrivial solution of (4.6) exists if its right-hand side is orthogonal to the solution of the homogeneous equation (4.2), which is the given function Y_0:

$$\lambda_1\Big[-(Y_0', P_\lambda' Y_0')_\xi + (Y_0, R_\lambda' Y_0)_\xi\Big] + (Y_0', 2PY_0' - yP_y'Y_0')_\xi + (Y_0, yR_y'Y_0)_\xi = 0. \quad (4.7)$$

Here, the multiplier of λ_1 is equal to $N(\xi,\lambda^0)$ (see (3.4)). The other terms are transformed by integration by parts:

$$(Y_0', yP_y'Y_0')_\xi = -\xi Y_0^{\mathrm{T}\prime}(\xi)P(\xi)Y_0'(\xi) + 2(yY_0', (PY_0')')_\xi + (Y_0', PY_0')_\xi,$$
$$(Y_0, yR_y'Y_0)_\xi = -2(yY_0', RY_0)_\xi - (Y_0, RY_0)_\xi.$$

Thus, the expression outside square brackets in (4.7) can be reduced to the desired form

$$(Y_0', 2PY_0' - yP_y'Y_0')_\xi + (Y_0, yR_y'Y_0)_\xi = \xi Y_0^{\mathrm{T}\prime}(\xi)P(\xi)Y_0'(\xi),$$
$$\lambda_1 = -\xi Y_0^{\mathrm{T}\prime}(\xi)P(\xi)Y_0'(\xi)/N(\xi,\lambda^0).$$

Therefore, for the approximation $\lambda^{(1)}$ to the sought eigenvalue λ, we derive an expression equivalent to formula (3.4):

$$\lambda^{(1)} = \lambda_0 + \varepsilon\lambda_1 = \lambda^0 - \varepsilon l Y_0^{\mathrm{T}\prime}(\xi)P(\xi)Y_0'(\xi)/N(\xi,\lambda^0), \quad N \neq 0.$$

The solution $Y_1(y,\lambda^0,\lambda_1)$ of the inhomogeneous boundary value problem (4.6) with $\varepsilon = 0$ is found in the standard manner: replacing the boundary conditions by the initial conditions $Y_{1,(i)}(0) = 0$ and $Y_{1,(i)}'(0) = e_i$, we consider s Cauchy problems. The orthogonality condition (4.7) guarantees that there exists a linear combination Y_1 of solutions $Y_{1,(i)}$ such that the right boundary condition holds: $Y_1(\xi) = 0$. Next, Y_1 is normalized so that $\|Y_1\|_{\mathbb{L}_2(0,\xi)} = 1$ and we set $Y^{(1)} = Y_0 + \varepsilon Y_1$.

Following the method of successive approximations, the found corrections $\lambda_1^{(0)} = \lambda_1$ and $Y_1^{(0)} = Y_1$ are substituted into the right-hand side of Eq. (4.6):

$$\begin{aligned}(P(y,\lambda^0)Y_1^{(1)\prime})' + R(y,\lambda^0)Y_1^{(1)} &= ((2P(y,\lambda^0) - \lambda_1^{(1)}P_\lambda'(y,\lambda^0) - yP_y'(y,\lambda^0))Y_0')'\\ &- (\lambda_1^{(1)}R_\lambda'(y,\lambda^0) + yR_y'(y,\lambda^0))Y_0 + F(y, Y_1^{(0)}, \lambda_1^{(0)}, \varepsilon), \quad Y_1^{(1)}(0) = Y_1^{(1)}(\xi) = 0.\end{aligned} \quad (4.8)$$

The next correction approximation $\lambda_1^{(1)}$ is also determined using the orthogonality condition: $\lambda_1^{(1)} = \lambda_1^{(0)} - (F(y, Y_1^{(0)}, \lambda_1^{(0)}, \varepsilon), Y_0)/N(\xi,\lambda^0)$. Then the solution $Y_1^{(1)}$ of the inhomogeneous boundary value problem (4.8) can be found.

At the $(k+1)$th step, the boundary value problem and the correction term are given by

$$\begin{aligned}(P(y,\lambda^0)Y_1^{(k+1)\prime})' + R(y,\lambda^0)Y_1^{(k+1)} &= ((2P(y,\lambda^0) - \lambda_1^{(k+1)}P_\lambda'(y,\lambda^0) - yP'_y(y,\lambda^0))Y_0)'\\ &- (\lambda_1^{(k+1)}R_\lambda'(y,\lambda^0) + yR_y'(y,\lambda^0))Y_0 + F(y, Y_1^{(k)}, \lambda_1^{(k)}, \varepsilon); Y_1^{(k+1)}\big|_{y=0,\xi} = 0,\\ \lambda_1^{(k+1)} &= \lambda_1^{(k)} - (F(y, Y_1^{(k)}, \lambda_1^{(k)}, \varepsilon), Y_0)/N(\xi,\lambda^0), \quad N \neq 0.\end{aligned} \quad (4.9)$$

By the Schauder–Banach theorems on contraction operators [32], for sufficiently small $|\varepsilon|$, the successive approximations (4.9) converge to the unique solution $\lambda_1(\varepsilon)$, $Y_1(y,\varepsilon)$. The radius of convergence can be roughly estimated with the help of Shimanov's approach [29, 33].

Table 1

m_1, %	e_{rel}	e_{abs}	$\varepsilon^{(0)}l$	$\varepsilon^{(M)}l$	M
1	5.4×10^{-9}	1.1×10^{-7}	7.0×10^{-3}	-6.5×10^{-5}	2
10	6.1×10^{-9}	1.2×10^{-7}	6.4×10^{-2}	-5.5×10^{-5}	3
40	7.5×10^{-8}	1.6×10^{-6}	2.1×10^{-1}	-2.5×10^{-4}	4
m_5, %	e_{rel}	e_{abs}	$\varepsilon^{(0)}l$	$\varepsilon^{(M)}l$	M
1	6.6×10^{-7}	3.4×10^{-4}	6.9×10^{-3}	-6.4×10^{-5}	2
10	3.7×10^{-7}	1.9×10^{-4}	6.3×10^{-2}	-2.6×10^{-5}	3
40	1.6×10^{-7}	8.1×10^{-5}	2.0×10^{-1}	9.8×10^{-7}	5

Remark 4. The actual accuracy ε^2 of the computed eigenvalue depends on the accuracy of the numerical method used for integrating the auxiliary Cauchy problems (3.2): if $p$is the order of the numerical method used to solve the Cauchy problems, then the accuracy of λ is at most $O(h^{2p})$.

5. TEST EXAMPLES

The problem described in Sections 1–3 was computed using a C++ program. Singular value decomposition of matrices from the GNU Scientific Library [34] was used to monitor the rank of the matrix W and to find the constant vector $C(\xi)$. The initial grid was uniform, and the number of points in the partition of the integration interval was $n_l = 20$. On each of the grid subintervals, the Cauchy problems were integrated using the fourth/fifth-order Runge–Kutta–Fehlberg method with automatic step size selection implemented in RKF45 [35] with local relative and absolute errors of 10^{-6} and 10^{-7}, respectively. Following the technique described, if the sought zero satisfies $\xi < l$, then $n_\xi \le 20$, and if $\xi > l$, then $n_\xi > 20$ (the interval is expanded, and new integration points are added), while the step size is preserved. The point ξ was determined as the root of a third-degree Lagrange interpolating polynomial constructed using four points: two points in a uniform grid that are smaller than ξ and two points that are larger than ξ. To compute the eigenvalue correction term, the Cauchy problem was solved anew on the interval $[0, \xi]$ together with problem (3.6). All the examples presented below are considered on [0,1] with Dirichlet boundary conditions.

Example 1. Consider the classical SLP for the Euler-type scalar equation $u'' + \lambda u(1+x)^{-2} = 0$ [1].

An analytical solution is sought by making the substitution $u = (1+x)^p$:

$$u_n(x) = c_n\sqrt{1+x}\sin\left(\gamma_n \ln(1+x)\right), \quad c_n = \text{const},$$

$$\gamma_n = \sqrt{4\lambda_n - 1}/2, \quad \lambda_n = \pi^2 n^2 \ln^{-2} 2 + 1/4, \quad n = 1, 2, \ldots .$$

Assume that the nth eigenvalue λ_n differs from its estimate $\lambda_n^{(0)}$ by $m_n\%$, i.e., $\lambda_n^{(0)} = \lambda_n(1 + 10^{-2}m)$. For the given numerical integration method and a fixed integration grid, Table 1 presents the number M of iterations required for achieving a relative error $e_{\text{rel}} = \left|(\lambda_n^{(M+1)} - \lambda_n)/\lambda_n\right| \le 10^{-6}$ in the first and fifth eigenvalues ($n = 1, 5$): $\lambda_1 \approx 20.792288$, $\lambda_5 \approx 513.807212$. The table also gives the absolute error $e_{\text{abs}} = \left|\lambda_n^{(M+1)} - \lambda_n\right|$ and the difference $\varepsilon l = l - \xi$ for the initial $\varepsilon^{(0)}l$ and last $\varepsilon^{(M)}l$ approximations (i.e., $\varepsilon^{(0)}l$ corresponds to the mismatch of the eigenvalue approximation $\lambda_n^{(0)}$, and $\varepsilon^{(M)}l$, to the mismatch of $\lambda_n^{(M)}$ calculated at the penultimate $(M-1)$th step of the algorithm).

Example 2. Consider a Euler-type scalar equation with a nonlinear parameter dependence: $u'' + u(\lambda + x)^{-2} = 0$ (see [1]).

Table 2

m_1, %	e_{rel}	e_{abs}	$\varepsilon^{(0)}l$	$\varepsilon^{(M)}l$	M
1	5.9×10^{-7}	1.6×10^{-8}	-1.0×10^{-2}	-1.0×10^{-2}	1
10	5.2×10^{-7}	1.4×10^{-8}	-0.10	4.2×10^{-5}	2
40	7.8×10^{-7}	2.1×10^{-8}	-0.40	8.4×10^{-3}	2
m_{-1}, %	e_{rel}	e_{abs}	$\varepsilon^{(0)}l$	$\varepsilon^{(M)}l$	M
1	3.1×10^{-7}	3.2×10^{-7}	-1.6×10^{-4}	1.4×10^{-6}	2
10	2.0×10^{-7}	2.1×10^{-7}	-0.12	-1.4×10^{-6}	3
40	2.3×10^{-7}	2.4×10^{-7}	-0.7	1.5×10^{-6}	5

Table 3

m_5^-, %	e_{rel}	e_{abs}	$\varepsilon^{(0)}l$	$\varepsilon^{(M)}l$	M
1	5.4×10^{-7}	1.6×10^{-5}	-3.4×10^{-3}	-2.1×10^{-6}	4
10	1.1×10^{-7}	3.1×10^{-6}	-4.7×10^{-2}	-4.5×10^{-7}	6
40	1.5×10^{-7}	4.3×10^{-6}	-0.49	-6.1×10^{-7}	8
m_5^+, %	e_{rel}	e_{abs}	$\varepsilon^{(0)}l$	$\varepsilon^{(M)}l$	M
1	9.7×10^{-7}	1.7×10^{-5}	2×10^{-3}	8.3×10^{-6}	2
10	9.5×10^{-8}	1.6×10^{-6}	1.7×10^{-2}	-2.6×10^{-7}	6
30	8.8×10^{-8}	1.5×10^{-6}	3.4×10^{-2}	-2.4×10^{-7}	13
40*	1.7×10^{-7}	2.9×10^{-6}	-0.16	-4.3×10^{-7}	7

Its solution is sought in the form $u = c(\lambda + x)^p$. In the domain of admissible values $\Lambda = \{\lambda < -1, \lambda > 0\}$, the analytical solution is given by

$$u_n(x) = c_n\sqrt{1 + x/\lambda_n}\sin\left(\frac{\pi}{\gamma}\ln(1 + x/\lambda_n)\right), \quad c_n = \text{const},$$
$$\lambda_n = (e^{\gamma n} - 1)^{-1}, \quad \gamma = 2\pi/\sqrt{3}, \quad \lambda_n + \lambda_{-n} = -1, \quad n = \pm 1, \pm 2, \ldots .$$

In this example, the eigenvalues tend exponentially to 0 from the right for $n \geq 1$ and to -1 from the left for $n \leq -1$. The problem is similar to a singular one, and its solution rapidly oscillates near $x = 0$ for $n > 1$ and near $x = 1$ for $n < -1$, so the construction of eigenfunction even for small n is a complicated computational problem. For simplicity, our consideration is restricted to the cases $n = \pm 1$, $\lambda_1 \approx 0.02731$, $\lambda_{-1} \approx -1.02731$, when the eigenfunction have no zeros inside the interval $[0,1]$. The function $r(\lambda, x) = 1/(\lambda + x)^2$ is extended by a constant to $x > 1$. Table 2 presents the corresponding results found as described in Example 1.

Example 3. Consider an SP-nonlinear vector equation with constant coefficients and a finite number of real eigenvalues (elementary case):

$$u_1'' - (k_1\lambda^2 + k_2\lambda)u_2 = 0, \quad u_2'' - (k_1\lambda^2 + k_2\lambda)u_1 = 0, \tag{5.1}$$

where $k_{1,2} > 0$ and $\Lambda = \{\lambda < 0, \lambda > -k_2/k_1\}$. Adding up the equations yields a boundary value problem [36] with the solution $u = u_1 + u_2 = \sin \pi n x$. The eigenvalues are the roots of the quadratic equation $k_1\lambda^2 + k_2\lambda + \pi^2 n^2 = 0$. For $k_1 = 1/2$ and $k_2 = 23$, the eigenvalues are real for $n \leq 5$. Table 3 presents results for two eigenvalues corresponding to $n = 5$: $\lambda_5^{\pm} = -k_2 \pm \sqrt{k_2^2 - 2\pi^2 n^2}$, $\lambda_5^- \approx -28.959847$,

Table 4

m_5, %	e_{rel}	e_{abs}	$\varepsilon^{(0)}l$	$\varepsilon^{(M)}l$	M
1	1.1×10^{-7}	1.1×10^{-5}	6.9×10^{-3}	-9.8×10^{-5}	2
10	7.3×10^{-8}	7.5×10^{-6}	6.4×10^{-2}	-5.6×10^{-5}	3
40	1.6×10^{-7}	1.7×10^{-5}	0.2	-2.4×10^{-4}	4

$\lambda_5^+ \approx -17.040153$. By using this example, it is easy to demonstrate the influence exerted by the dependence of the eigenvalues on the interval length l (or the point ξ). Obviously,

$$\lambda = \frac{-k_2 \pm \sqrt{k_2^2 - 4k_1\pi^2 n^2/l^2}}{2k_1}, \quad \lambda'(l) = \pm\frac{4k_1\pi^2 n^2}{l^2\sqrt{k_2^2 l^2 - 4k_1\pi^2 n^2}},$$

and $l = \pi n/\sqrt{-k_1\lambda^2 - k_2\lambda}$. Therefore, if the sought λ is sufficiently close to the critical value $\beta = -k_2/k_1$, then its insignificant variation can lead to a considerable variation in l. For example, for $n = 1$, we have $\lambda_1 \approx -45.566807$ and $\beta = -46$. If $\lambda^{(0)} = -45$, then $\xi^{(0)} \approx 0.662306$, $\varepsilon^{(0)} \approx 0.337694$, the correction term is $\varepsilon^{(0)}l\lambda' \approx -1.042929$, and the next approximation is $\lambda^{(1)} \approx -46.042929$ (for the accepted accuracy, the values produced by the program for system (5.1) are $\tilde{\xi}^{(0)} \approx 0.662303$, $\tilde{\varepsilon}^{(0)} \approx 0.337697$, $\tilde{\varepsilon}^{(0)}l\tilde{\lambda}' \approx -1.042937$, and $\tilde{\lambda}^{(1)} \approx -46.042937$, respectively), i.e., $\lambda^{(1)}$ lies farther away from the sought λ and leaves the domain of admissible SP values, so problem (5.1) has no solutions. Note that this behavior is not typical for most physical problems, but it has to be kept in mind in the application of the method.

Table 3 illustrates the case inverse to the preceding one, namely, the influence of a discontinuity in the derivative $\lambda'(l)$ for $m_5^+ = 30\%$. The initial approximation is $\lambda^{(0)} \approx -22.152199$, $\xi^{(0)} \approx 0.966501$, and the last value is close to the discontinuity point $l = 2k_1\pi n/k_2 \approx 0.965844$. The correction term is $\varepsilon^{(0)}l\lambda' \approx 21.597386$, and the next approximation $\lambda^{(1)} \approx -0.554812$ lies rather far away from the desired λ, so more iterations are required. Moreover, near the singularity, the algorithm exhibits a somewhat different behavior: $\tilde{\xi}^{(0)} \approx 0.965971$, $\tilde{\varepsilon}^{(0)}l\tilde{\lambda}' \approx 21.938739$, and $\tilde{\lambda}^{(1)} \approx -0.213459$. Further refinement requires 12 iterations.

In the last line in Table 3 for $m_5^+ = 40\%$, the first approximation was specified as $\lambda^{(0)} = 0.6\lambda_5^+$, since, with the use of $\lambda^{(0)} = 1.4\lambda_5^+ = -23.856214$, the algorithm converged to the eigenvalue $\lambda_5^- = -28.959847$.

Example 4. Consider the SP-linear Euler-type vector equation

$$u_1'' + \frac{\lambda}{(1+x)^2}(u_1 + 2u_2) = 0, \quad u_2'' + \frac{\lambda}{(1+x)^2}(2u_1 + 4u_2) = 0.$$

By analogy with the above examples, the solution is sought in the form $U = C(1+x)^p$, where $C = (C_1, C_2)^{\mathrm{T}}$ is a constant vector:

$$U_n(x) = C\sqrt{1+x}\sin\left(\gamma_n \ln(1+x)\right), \quad \gamma_n = \sqrt{20\lambda_n - 1}/2, \quad \lambda_n = \frac{\pi^2 n^2}{5\ln^2 2} + \frac{1}{20}, \quad n = 1, 2, \ldots.$$

Table 4 presents the results obtained for $n = 5$.

Example 5. Consider the SP-nonlinear Euler-type vector equation

$$u_1'' + \frac{\lambda}{(1+x)^2}(\lambda u_1 + 15u_2) = 0, \quad u_2'' + \frac{\lambda}{(1+x)^2}(15u_1 + 4\lambda u_2) = 0.$$

Its solution is found as described in Example 4:

$$U_n(x) = C\sqrt{1+x}\sin\left(\gamma_n^{\pm}\ln(1+x)\right), \quad \gamma_n^{\pm} = \sqrt{10\lambda_n^2 - 1 \pm 6\sqrt{\lambda_n^4 + 100\lambda_n^2}}/2.$$

Table 5

m_{1+}^{+}, %	e_{rel}	e_{abs}	$\varepsilon^{(0)}l$	$\varepsilon^{(M)}l$	M
1	2.5×10^{-8}	2.9×10^{-8}	8.1×10^{-3}	-7.9×10^{-5}	2
10	2.1×10^{-8}	2.4×10^{-8}	7.5×10^{-2}	-6.9×10^{-5}	3
40	2.1×10^{-7}	2.5×10^{-7}	0.24	-3.1×10^{-4}	4
m_{5-}^{-}, %	e_{rel}	e_{abs}	$\varepsilon^{(0)}l$	$\varepsilon^{(M)}l$	M
1	1.5×10^{-7}	1.2×10^{-6}	1.2×10^{-2}	-1.2×10^{-4}	2
10	1.9×10^{-7}	1.5×10^{-6}	0.11	-1.6×10^{-4}	3
40	9.3×10^{-8}	7.7×10^{-7}	0.53	-1.4×10^{-3}	6

The corresponding eigenvalues are sought by solving the equations $\gamma_n^{\pm} = \pi n/\ln 2$. Table 5 shows the numerical results obtained for $\lambda_{1+}^{+} \approx 1.155809$ (the positive root of the equation $\gamma_1^{+} = \pi/\ln 2$) and for $\lambda_{4-}^{-} \approx -8.225792$ (the negative root of the equation $\gamma_4^{-} = 4\pi/\ln 2$). To correctly calculate the zeros of the function $\Delta(x)$ for λ_{4-}^{-}, we set $n_l = 30$.

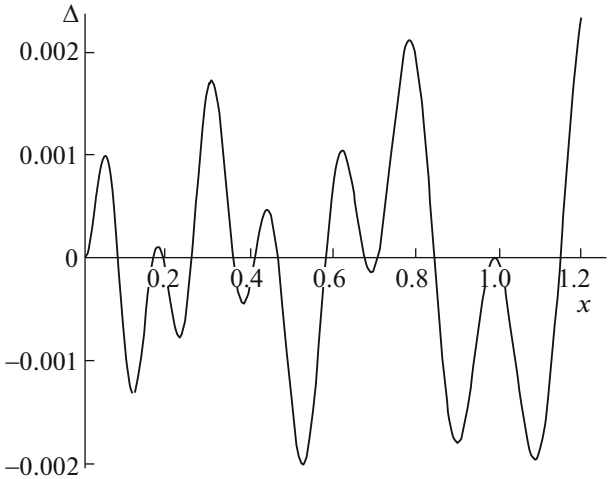
For $\lambda^{(0)} = 20.1$, the figure shows the plot of $\Delta(x, \lambda^{(0)})$ in this example, which demonstrates the possibility of appearing multiple zeros and the importance of accurately solving the auxiliary Cauchy problems: near the point $x = 1$, there appear two closely spaced roots (differing by a value on the order of 10^{-3}), which may be ignored or taken into account as multiple roots in the case of an inaccurately constructed Δ.

Example 6. Consider a system with both matrices P and R depending on the SP:

$$\left(\frac{1}{(1+x)^2}(u_1' + 5\lambda u_2')\right)' + \frac{\lambda}{(1+x)^4}(\lambda u_1 + 15u_2) = 0,$$
$$\left(\frac{1}{(1+x)^2}(5\lambda u_1' + 4u_2')\right)' + \frac{\lambda}{(1+x)^4}(15u_1 + 4\lambda u_2) = 0.$$

Its solution is sought as described in Example 4:

$$U_n(x) = C(1+x)^{\frac{3}{2}} \sin\left(\gamma_n^{\pm} \ln(1+x)\right), \quad \gamma_n^{\pm} = \frac{\sqrt{\pm 40\lambda_n^3 + 91\lambda_n^2 \mp 60\lambda_n - 36}}{10\lambda_n \pm 4}.$$



Behavior of the function $\Delta(x)$.

Table 6

m_{1+}^{+}, %	e_{rel}	e_{abs}	$\varepsilon^{(0)}l$	$\varepsilon^{(M)}l$	M
1	2.7×10^{-8}	1.4×10^{-6}	6.7×10^{-3}	-5.8×10^{-5}	2
10	2.3×10^{-8}	1.1×10^{-6}	6.2×10^{-2}	-4.6×10^{-5}	3
40	3.7×10^{-8}	1.9×10^{-6}	0.2	-1.8×10^{-4}	4

Table 7

m_1, %	e_{rel}	e_{abs}	$\varepsilon^{(0)}l$	$\varepsilon^{(M)}l$	M
1	1.9×10^{-9}	8.1×10^{-10}	-1×10^{-2}	-1.2×10^{-6}	2
10	8×10^{-9}	3.4×10^{-9}	-0.1	1×10^{-4}	2
40	1.9×10^{-9}	8.1×10^{-10}	-0.39	-1.2×10^{-6}	3
m_{-2}, %	e_{rel}	e_{abs}	$\varepsilon^{(0)}l$	$\varepsilon^{(M)}l$	M
1	2.9×10^{-7}	3.2×10^{-7}	-9×10^{-3}	-2×10^{-6}	5
10	4.7×10^{-7}	5.1×10^{-7}	-0.12	-3.2×10^{-6}	6
40	6.4×10^{-7}	7.1×10^{-7}	-0.72	-1×10^{-4}	5

Table 6 presents the numerical results obtained for $\lambda_{1+}^{+} \approx 49.93714$, i.e., for the positive root of the equation $\gamma_1^{+} = \pi/ln(2)$.

Example 7. Consider the SP-nonlinear Euler-type vector equation

$$u_1'' + \frac{1}{(\lambda + x)^2}(u_1 + 2\sqrt{3}u_2) = 0, \quad u_2'' + \frac{1}{(\lambda + x)^2}(2\sqrt{3}u_1 + 5u_2) = 0.$$

The solution of the problem has the form $u_n(x) = C\sqrt{1 + x/\lambda_n}\sin\left(\frac{\pi}{\gamma}\ln(1 + x/\lambda_n)\right)$, where $\lambda_n = (e^{\gamma n} - 1)^{-1}$, $\gamma = 2\pi/3\sqrt{3}$, $\lambda_n + \lambda_{-n} = -1$, $n = \pm 1, \pm 2, \ldots$.

Table 7 gives the results for $n = 1$, $\lambda_1 \approx 0.425387$ and $n = -2$, $\lambda_{-2} \approx -1.097772$.

6. TRANSVERSE VIBRATIONS OF A ROD OF VARIABLE CROSS SECTION WITH ALLOWANCE FOR ITS ROTATIONAL INERTIA

As a model example, we consider the classical problem of free transverse vibrations of a hinged thin rod with allowance for the rotational inertia of its cross section [37, 38]. The dynamics of the rod is described by the equation

$$\rho(x)S(x)\ddot{U} - \left(\rho(x)I(x)\ddot{U}'\right)' = -\left(E(x)I(x)U''\right)'', \quad U\big|_{x=0,l} = U''\big|_{x=0,l} = 0,$$

where $U(x,t)$ are transverse displacements, $\rho(x)$ is the density of the rod material, $S(x)$ is the cross-sectional area, $E(x)$ is Young's modulus, $I(x)$ is the moment of inertia of the cross section, and l is the length of the rod. Passing to the harmonic vibrations $U(x,t) = e^{i\omega t}u(x)$ and introducing $\lambda = \rho_0\omega^2 l^2/E_0 > 0$ (where ω is the eigenfrequency and ρ_0, E_0 are characteristic constant values of the density and Young's modulus), we obtain the boundary value problem

$$\left(\frac{E(x)I(x)}{E_0 l^4}u''\right)'' + \lambda\left(\left(\frac{\rho(x)I(x)}{\rho_0 l^4}u'\right)' - \frac{\rho(x)S(x)}{\rho_0 l^2}u\right) = 0, \quad 0 < x < 1, \tag{6.1}$$
$$u(0) = u(1) = u''(0) = u''(1) = 0.$$

Table 8

m_1, %	e_{rel}	e_{abs}	$\varepsilon^{(0)}l$	$\varepsilon^{(M)}l$	M
1	7.6×10^{-9}	1.2×10^{-10}	2.5×10^{-3}	-1.5×10^{-5}	2
10	2×10^{-9}	3×10^{-11}	2.4×10^{-2}	-6×10^{-6}	3
40	1.3×10^{-9}	2×10^{-11}	8×10^{-2}	-4.6×10^{-6}	4
m_3, %	e_{rel}	e_{abs}	$\varepsilon^{(0)}l$	$\varepsilon^{(M)}l$	M
1	8.4×10^{-7}	1×10^{-6}	2.5×10^{-3}	-2.4×10^{-6}	4
10	8.4×10^{-7}	1.0×10^{-6}	2.4×10^{-2}	-2.4×10^{-6}	4
40	2.1×10^{-7}	2.6×10^{-7}	8.1×10^{-2}	-5.7×10^{-7}	6

Table 9

n	λ_n^E	λ_n^R	m, %	M^E	M^R	$\varepsilon^{(M)}l$
1	0.00756158	0.00755396	0.1	5	2	-5.8×10^{-6}
2	0.132408	0.131952	0.3	6	6	-3.3×10^{-5}
3	0.664790	0.659782	0.7	5	5	-4.1×10^{-5}
4	2.08971	2.06212	1.3	5	5	-1.1×10^{-6}
5	5.08577	4.98206	2	5	8	4.5×10^{-6}

If the coefficients of the equation are constant, then the solution has the form

$$u_n(x) = c_n \sin \gamma_n x, \quad \gamma_n = (\lambda_n/2 + \sqrt{(\lambda_n/2)^2 + \lambda SI^{-1}})^{1/2}, \quad c_n = \text{const},$$

where $\lambda_n = (\pi n/l)^4/(SI^{-1} + (\pi n/l)^2)$ are the eigenvalues.

Let $p = I\rho/(\rho_0 l^4)$, $r_1 = \rho S/(\rho_0 l^2)$, $r_2 = E_0 l^4/(EI)$, $u_1 = u$, and $u_2 = -EIu''/(E_0 l^4)$. Then problem (6.1) can be reformulated as the VSLP

$$-(p\lambda u_1')' + u_2'' + \lambda r_1 u = 0, \quad u_1'' + r_2 u_2 = 0, \quad u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0.$$

Table 8 presents the numerical results obtained as described in the above examples for $n = 1, 3$, $\lambda_1 \approx 1.519674 \times 10^{-2}$, $\lambda_3 \approx 1.215957$, $l = 1$ m, $S = \pi r^2$, $I = \pi r^4/4$, $r = 0.025$ m, $p = I/l^4$, $r_1 = S/l^2$, and $r_2 = l^4/I$ (for comparison, without allowance for rotational inertia, $\lambda_1 \approx 1.522017 \times 10^{-2}$ and $\lambda_3 \approx 1.232834$).

Additionally, Table 8 gives the smallest five eigenvalues found for a rod of variable cross section with $r(x) = r_0 - xr_0/(2l)$, where $r_0 = 0.025$ m. Specifically, λ_n^R and λ_n^E are the eigenvalues calculated with and without allowance for rotational inertia, respectively, with a relative error on the order of 10^{-6}; M^R and M^E are the respective numbers of iterations required for achieving the required accuracy; and m_n is the difference between λ_n^R and λ_n^E in percent. In both cases, the first approximation was specified as the eigenvalue $\lambda_1^{(0)}$ for a rod of constant cross section without allowance for rotational inertia (see above) for $n = 1$ and as $\lambda_{n-1}^{E,R}$ for $n > 1$. The difference $\varepsilon^{(M)}l$ is given to compute λ_n^R. The relative error was determined using the two-sided estimate from Theorem 1: if $\lambda^{(i)} < \lambda < \lambda^{(j)}$, then $e_{\text{rel}} = (\lambda^{(j)} - \lambda^{(i)})/0.5(\lambda^{(i)} + \lambda^{(j)})$. The solutions of the auxiliary Cauchy problems grew exponentially, so the uniform grid for the integration of the auxiliary problems was refined to $n_l = 128$ in order to achieve sufficient accuracy.

The results demonstrate the capabilities and efficiency of the proposed method.

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