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Parabolic Equations with Unknown Time-Dependent Coefficients

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Abstract—The solvability of inverse problems of finding the coefficients of a parabolic equation together with solving this equation is studied. In these problems, certain additional conditions on the boundary are used as overdetermination conditions. Existence and uniqueness theorems for regular solutions of such problems are proven.

Keywords: parabolic equations, nonlinear inverse problems, unknown coefficients, boundary overdetermination, regular solutions, existence, uniqueness.

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INTRODUCTION

In this paper, we study the solvability of initial boundary value problems for parabolic equations when one of the coefficients of the equation is unknown. As a rule, in such situations it is assumed that the unknown coefficient has some special form. Here, we assume that the unknown coefficient is a function of the time variable only.

In mathematics and mathematical modeling, the problems considered in this work are called inverse problems (see [1, 2]). The presence of additional unknown functions in inverse problems makes it necessary to impose some additional conditions: overdetermination conditions, in addition to the boundary conditions that are natural for a particular class of differential equations. For inverse problems with an unknown coefficient being a function of the time variable, the required overdetermination conditions are either conditions of integral overdetermination, or conditions of boundary overdetermination, or conditions of internal overdetermination. Conditions of integral overdetermination assume that additional information is specified as the values of certain integrals of the solution over the domain of variation of the spatial variables for all t. In problems with boundary overdetermination, it is assumed that, in addition to the natural boundary conditions, some additional conditions whose supports are certain manifolds from the lateral boundary of the corresponding cylindrical region are specified for all t. Finally, in problems with internal overdetermination, additional information is specified as the values of solutions and (or) its derivatives at certain interior points of the spatial domain at all t.

Inverse problems, both linear and nonlinear, with an unknown coefficient depending only on the time variable with integral, boundary, or internal overdetermination conditions have been more or less studied for different classes of differential equations: second-order parabolic and hyperbolic equations, high-order parabolic equations, pseudoparabolic and pseudohyperbolic equations, and equations with multiple characteristics; for more detail, see [1–24]. At the same time, it is worth noting that, in most of these papers and monographs, inverse problems with an unknown coefficient depending on the time variable and with boundary overdetermination were studied either in the one-dimensional case or in special domains such as a parallelepiped; only in [18, 19], inverse problems with boundary integral overdetermination and an unknown right-hand side for second-order parabolic equations in multidimensional domains of arbitrary spatial geometry were studied.

The aim of this work is to study the solvability of new nonlinear inverse problems with the boundary overdetermination. The method of study are based on replacing the inverse problem with a new direct boundary value problem for a "loaded" [25, 26] differential equation, proving its solvability, and constructing the solution of the original inverse problem.

Below, we will consider some model situation. Possible extensions and comments will be given in the end of the paper.

1. PROBLEM STATEMENT

Let Ω be a bounded domain of space $\mathbb{R}^n$ with a smooth compact (for simplicity, infinitely differentiable) boundary Γ, Q be a cylinder $\Omega \times (0,T)$ of a finite height T, $S = \Gamma \times (0,T)$ be the lateral side of Q, and $c(x,t)$, $f(x,t)$, $u_0(x)$, $N(x)$, and $\mu(t)$ be given functions defined at $x \in \bar{\Omega}$, $t \in [0,T]$.

Inverse problem I. Find functions $u(x,t)$ and $q(t)$ related in the cylinder Q by the equation

$$u_t - \Delta u + q(t)u = f(x,t) \tag{1}$$

under the following conditions for the function $u(x,t)$:

$$u(x,0) = u_0(x), \quad x \in \Omega; \tag{2}$$

$$\frac{\partial u(x,t)}{\partial \nu_x} = 0 \quad \text{for} \quad (x,t) \in S; \tag{3}$$

$$\int_\Gamma N(x)u(x,t)ds_x = \mu(t), \quad 0 < t < T \tag{4}$$

(hereinafter, $\nu_x = (\nu_{x_1}, \dots, \nu_{x_n})$ is the inward normal to Γ at a current point x).

Inverse problem II. Find functions $u(x,t)$ and $p(t)$ related in the cylinder Q by the equation

$$p(t)u_t - \Delta u + c(x,t)u = f(x,t) \tag{5}$$

under conditions (2)–(4) for the function $u(x,t)$.

In inverse problems I and II, conditions (2) and (3) are conditions of an ordinary initial (second) boundary value problems for parabolic equations and condition (4) is an integral overdetermination boundary condition. The inverse problems of finding the solution together with the unknown time-dependent coefficient and with overdetermination condition (4) have not been studied earlier in the multidimensional case.

Inverse problems I and II have a simple physical interpretation (see [27]). The coefficients $q(t)$ and $p(t)$ in these problems are functions determining the loss (absorption) and specific heat capacity, respectively. If the medium is isotropic, it is natural to assume that these coefficients are functions of time. Condition (3) means that the medium is thermally isolated, and condition (4) gives information on the average surface temperature of the medium; such information can be easily obtained using of a system of sensors.

2. SOLVABILITY OF INVERSE PROBLEM I

To avoid cumbersome computations and formulations, we introduce the following notation. Throughout this subsection we shall assume that

$$|\mu(t)| \geq \mu_0 > 0 \quad \text{for} \quad t \in [0,T] \tag{6}$$

for the initial functions $f(x,t)$, $u_0(x)$, etc., all the derivatives and integrals defined below exist (the exact conditions for the initial data will be given below).

Define

$$f_0(t) = \int_\Gamma N(x)f(x,t)ds_x,$$

$$\alpha(t) = \frac{f_0(t) - \mu'(t)}{\mu(t)}, \quad \beta(t) = \frac{1}{\mu(t)},$$

$$\alpha_0 = \operatorname*{vrai\,min}_{0 \leq t \leq T} \alpha(t),$$

$$v_0(x) = \Delta u_0(x), \quad f_1(x,t) = \Delta f(x,t),$$

$$N_0 = \|N\|_{L_2(\Gamma)}, \quad R_0 = \left(\|f_1\|^2_{L_2(Q)} + \|v_0\|^2_{L_2(\Omega)}\right)e^T,$$

$$R_1 = \|f_1\|^2_{L_2(Q)} + \sum_{i=1}^n \|v_{0x_i}\|^2_{L_2(\Omega)}.$$

Let $w(x)$ be a function from the space $W_2^1(\Omega)$. By the embedding theorems (see, e.g., [28, Ch. II, Section 2]), we have the inequality

$$\int_\Gamma w^2 ds_x \leq k_0 \int_\Omega w^2 dx + k_1 \sum_{i=1}^n \int_\Omega w_{x_i}^2 dx, \tag{7}$$

in which the constants k_0 and k_1 are determined by the domain Ω only. Define the number M_0 as

$$M_0 = N_0 (k_0 R_0 + k_1 R_1)^{\frac{1}{2}}.$$

Theorem 1. *Let condition* (6) *and the memberships* $f(x,t) \in L_2(Q)$, $\Delta f(x,t) \in L_2(Q)$, $N(x) \in L_2(\Gamma)$, $u_0(x) \in W_2^3(\Omega)$, *and* $\mu(t) \in W_2^1([0,T])$ *be satisfied. Further on, suppose to be satisfied the matching conditions*

$$\frac{\partial u_0(x)}{\partial \nu_x} = \frac{\partial v_0(x)}{\partial \nu_x} = 0 \quad \textit{for} \quad x \in \Gamma,$$

$$\int_\Gamma N(x) u_0(x) ds_x = \mu(0)$$

and the condition

$$\frac{\partial f(x,t)}{\partial \nu_x} = 0 \quad \textit{for} \quad (x,t) \in S,$$

$$\alpha_0 > 0, \quad M_0 \leq \alpha_0 \mu_0.$$

Then, inverse problem I *has a solution* $\{u(x,t), q(t)\}$ *such that* $u(x,t) \in W_2^{2,1}(Q) \cap L_\infty(0,T; W_2^1(\Omega))$, $\Delta u(x,t) \in W_2^{2,1}(Q) \cap L_\infty(0,T; W_2^1(\Omega))$, *and* $q(t) \in L_2([0,T])$.

Proof. For a fixed number M, define the function $G_M(\xi)$, $\xi \in \mathbb{R}$:

$$G_M(\xi) = \begin{cases} \xi, & |\xi| \leq M, \\ M, & \xi > M, \\ -M, & \xi < -M. \end{cases}$$

Further on, for an arbitrary function $w(x)$, define the function $\Phi(w)$:

$$\Phi(w) = \int_\Gamma N(x) w(x) ds_x.$$

Consider the following auxiliary boundary value problem: find a function $v(x,t)$ satisfying the equation

$$v_t - \Delta v + [\alpha(t) + \beta(t) G_{M_0}(\Phi(v(x,t)))] v = f_1(x,t) \tag{8}$$

in the cylinder Q and the additional conditions

$$\frac{\partial v(x,t)}{\partial \nu_x} = 0 \quad \text{for} \quad (x,t) \in S, \tag{9}$$

$$v(x,0) = v_0(x) \quad \text{for} \quad x \in \Omega. \tag{10}$$

For brevity, we will denote by V_0 the space $W_2^{2,1}(Q) \cap L_\infty(0,T; W_2^1(\Omega))$. Using the hypotheses of the theorem and applying the fixed-point method, we will show that problem (8)–(10) has a solution $v(x,t)$ belonging to the space V_0.

Let $w(x,t)$ be a fixed function from the space V_0. Consider the following problem: find a function $v(x,t)$ satisfying the equation

$$v_t - \Delta v + [\alpha(t) + \beta(t) G_{M_0}(\Phi(w(x,t)))] v = f_1(x,t) \tag{8_w}$$

in the cylinder Q and conditions (9) and (10). According to the theory of parabolic equations (see [29, Ch. III, Section 6]), this problem has a solution $v(x,t)$ belonging to the space V_0. In other words, boundary value problem (8_w), (9), and (10) generates an operator A mapping the space V_0 into itself: $A(w) = v$.

Let us show that the operator A satisfied all conditions of the Schauder fixed-point theorem (see [30, Ch. VIII, Section 35]).

Consider the equalities

$$\int_0^t\int_\Omega \{v_\tau - \Delta v + [\alpha(\tau) + \beta(\tau)G_{M_0}(\Phi(w(x,\tau)))]v\} v dx d\tau = \int_0^t\int_\Omega f_1 v dx d\tau,$$

$$-\int_0^t\int_\Omega \{v_\tau - \Delta v + [\alpha(\tau) + \beta(\tau)G_{M_0}(\Phi(w(x,\tau)))]v\} \Delta v dx d\tau = -\int_0^t\int_\Omega f_1 \Delta v dx d\tau,$$

which follow from Eq. (8_w). Integrating by parts, using the inequality $\alpha(\tau) + \beta(\tau)G_{M_0}(\Phi(w(x,\tau))) \geq 0$, and applying Young's inequality and Grönwall's lemma, we find that all solutions of boundary value problem (8_w), (9), and (10) satisfy the estimates

$$\int_\Omega v^2(x,t)dx \leq R_0, \tag{11}$$

$$\sum_{i=1}^n\int_\Omega v_{x_i}^2(x,t)dx \leq R_1, \tag{12}$$

$$\int_0^t\int_\Omega (\Delta v)^2(x,\tau)dx d\tau \leq R_1. \tag{13}$$

Estimates (11) and (12) and inequality (7) imply another estimate:

$$\int_\Gamma v^2(x,t)ds_x \leq k_0R_0 + k_1R_1. \tag{14}$$

Using estimates (11)–(14), it is easy to estimate the derivative v_t as

$$\int_0^t\int_\Omega v_\tau^2 dx d\tau \leq R, \tag{15}$$

where the number R is determined by the numbers R_0, R_1, k_0, and k_1 and the function $f(x,t)$ (the exact value of R_0 is not important).

Inequalities (11)–(15) imply that all solutions of boundary value problem (8_w), (9), and (10) satisfy the estimate

$$\|v\|_{V_0} \leq K_0, \tag{16}$$

in which the constant K_0 is determined only by the functions $f_1(x,t)$ and $v_0(x)$, the domain Ω, and the number T.

In turn, estimate (16) implies that the operator A maps a closed ball of the radius K_0 of the space V_0 into itself.

Let $\{w_m(x,t)\}_{m=1}^\infty$ be a sequence of functions from this ball, converging in the space V_0 to a function $w_0(x,t)$, and $v_m(x,t)$ and $v_0(x,t)$ be solutions to problems (8_{w_m}), (9), (10) and (8_{w_0}), (9), (10), respectively. Define $\overline{v}_m(x,t) = v_m(x,t) - v_0(x,t)$. We have the equalities

$$\overline{v}_{mt} - \Delta\overline{v}_m + [\alpha(t) + \beta(t)G_{M_0}(\Phi(w_m))]\overline{v}_m = \beta(t)[G(\Phi(w_m)) - G_{M_0}(\Phi(w_0))]v_0, \quad (x,t) \in Q,$$

$$\overline{v}_m(x,0) = 0, \quad x \in \Omega,$$

$$\frac{\partial\overline{v}_m(x,t)}{\partial\nu_x} = 0, \quad (x,t) \in S.$$

Repeating the proof of estimate (16), it is easy to obtain the inequality

$$\|\overline{v}_m\|_{V_0} \le K_1 \|\beta(t)[G(\Phi(w_m)) - G_{M_0}(\Phi(w_0))]\|_{L_2(Q)}. \tag{17}$$

The function $G_{M_0}(\xi)$ satisfies the inequality $|G_{M_0}(\xi_1) - G_{M_0}(\xi_2)| \le |\xi_1 - \xi_2|$. Using this estimate and applying the Hölder inequality and inequality (7), we replace (17) with

$$\|\overline{v}_m\|_{V_0} \le K_2 \left\{ \int_Q (w_m - w_0)^2 dxdt + \sum_{i=1}^n \int_Q [(w_m - w_0)_{x_i}]^2 dxdt \right\}.$$

Since the right-hand side in this inequality tends to zero as $m \to \infty$, we have $\overline{v}_m \to 0$ as $m \to \infty$ in the space V_0. But this means that the operator A is continuous.

Now let us prove that the operator A is compact on a closed ball of the radius K_0 of the space V_0.

Let $\{w_m(x,t)\}_{m=1}^{\infty}$ be an arbitrary sequence of functions from this ball. Since we have the embeddings $V_0 \subset W_2^1(Q) \subset L_2(S)$ and the second embedding is compact (see [28, Ch. II, Section 2; 31, Ch. I, Sections 8, 11], we can choose from $\{w_m(x,t)\}_{m=1}^{\infty}$ a subsequence $\{w_{m_k}(x,t)\}_{k=1}^{\infty}$ strongly converging in the space $L_2(S)$. Repeating the proof of the continuity for the sequence $\{w_{m_k}(x,t)\}_{k=1}^{\infty}$, we conclude that the sequence $\{A(w_{m_k})\}_{k=1}^{\infty}$ strongly converges in the space V_0. But this means that the operator A is compact.

Thus, the operator A maps a closed ball of the radius K_0 in the space V_0 into itself and is compact on it. By the Schauder theorem, the operator A on this ball has at least one fixed point: $A(v) = v$.

At a fixed point $v(x,t)$ of the operator A, Eq. (8) and conditions (9) and (10) are satisfied. Further on, the function $v(x,t)$ satisfies the inequality

$$|\Phi(v)| \le N_0(k_0 R_0 + k_1 R_1)^{\frac{1}{2}}.$$

Therefore, we have $G_{M_0}(\Phi(v)) = \Phi(v)$. In other words, the solution $v(x,t)$ of problem (8)–(10) is a solution of the equation

$$v_t - \Delta v + [\alpha(t) + \beta(t)\Phi(v)]v = f_1(x,t).$$

It should be noted that the function $v(x,t)$ at all t from the interval $(0,T)$ satisfies the equality

$$\int_\Omega v(x,t)dx = 0.$$

Hence, we can determine the function $u(x,t)$ as a solution to the problem

$$\Delta u(x,t) = v(x,t), \quad (x,t) \in Q,$$

$$\frac{\partial u(x,t)}{\partial \nu_x} = 0, \quad (x,t) \in S$$

(here, the variable t is a parameter).

Define

$$w(x,t) = u_t(x,t) - \Delta u(x,t) + [\alpha(t) + \beta(t)\Phi(v)]v(x,t) - f(x,t).$$

We have the equalities

$$\Delta w(x,t) = 0, \quad (x,t) \in Q,$$

$$\frac{\partial w(x,t)}{\partial \nu_x} = 0, \quad (x,t) \in S,$$

$$\int_\Omega w(x,t)dx = 0, \quad t \in (0,T).$$

Hence, $w(x,t)$ is identically zero in Q.

Define

$$q(t) = \alpha(t) + \beta(t)\Phi(v(x,t)).$$

It follows from the aforesaid that the functions $u(x,t)$ and $q(t)$ are related in the cylinder Q by Eq. (1).

The function $u(x,t)$ obviously satisfies conditions (2) and (3). Let us show that $u(x,t)$ also satisfies condition (4).

Multiply Eq. (1) by the function $N(x)$ and integrate over the boundary Γ of the domain Ω. As a result, we obtain the equality

$$\frac{\partial}{\partial t}\left(\int_\Gamma N(x)u(x,t)ds_x\right) - \int_\Gamma N(x)\Delta u(x,t)ds_x + q(t)\int_\Gamma N(x)u(x,t)ds_x = f_0(t).$$

On the other hand, we have the equality

$$q(t)\mu(t) = f_0(t) - \mu'(t) + \int_\Gamma N(x)\Delta u(x,t)ds_x.$$

These two equalities imply the relationship

$$\frac{\partial}{\partial t}\left[\int_\Gamma N(x)u(x,t)ds_x - \mu(t)\right] + q(t)\left[\int_\Gamma N(x)u(x,t)ds_x - \mu(t)\right] = 0. \tag{18}$$

Since we have the condition

$$\int_\Gamma N(x)u(x,0)ds_x - \mu(0) = 0 \tag{19}$$

and the function $q(t)$ is nonnegative, equalities (18) and (19) obviously imply

$$\int_\Gamma N(x)u(x,t)ds_x = \mu(t).$$

In other words, in addition to conditions (2) and (3), the function $u(x,t)$ also satisfies condition (4). The membership of the functions $u(x,t)$ and $q(t)$ in the required classes is obvious too. The constructed functions $u(x,t)$ and $q(t)$ give the sought-for solution of inverse problem I. The theorem is proven.

Let us present another variant of the solvability theorem for inverse problem I.

Let δ_0 be a fixed number from the interval $(0,1)$. Define

$$R_0' = \frac{1}{\delta_0\alpha_0}\|f_1\|^2_{L_2(Q)} + \|v_0\|^2_{L_2(\Omega)},$$

$$R_1' = \|f_1\|^2_{L_2(Q)} + \sum_{i=1}^{n}\|v_{0x_i}\|^2_{L_2(\Omega)}.$$

Theorem 2. *Let condition* (6) *and the memberships* $f(x,t) \in L_2(Q)$, $\Delta f(x,t) \in L_2(Q)$, $N(x) \in L_2(\Gamma)$, $u_0(x) \in W_2^3(\Omega)$, *and* $\mu(t) \in W_2^1([0,T])$ *be satisfied. Further on, suppose to be satisfied the matching conditions*

$$\frac{\partial u_0(x)}{\partial \nu_x} = \frac{\partial v_0(x)}{\partial \nu_x} = 0, \quad x \in \Gamma,$$

$$\int_\Gamma N(x)u_0(x)ds_x = \mu(0)$$

and the conditions

$$\frac{\partial f(x,t)}{\partial \nu_x} = 0, \quad (x,t) \in S,$$

$$\alpha_0 > 0, \quad N_0(k_0R_0' + k_1R_1')^{\frac{1}{2}} \le (1-\delta_0)\alpha_0\mu_0.$$

Then, inverse problem I *has a solution* $\{u(x,t), q(t)\}$ *such that* $u(x,t) \in W_2^{2,1}(Q) \cap L_\infty(0,T;W_2^1(\Omega))$, $\Delta u(x,t) \in W_2^{2,1}(Q) \cap L_\infty(0,T;W_2^1(\Omega))$, *and* $q(t) \in L_2([0,T])$.

Proof. Theorem 2 is proven by analogy with Theorem 1; the only difference is that the equality

$$\int_0^t\int_\Omega \{v_\tau - \Delta v + [\alpha(\tau) + \beta(\tau)G_{M_0'}(\Phi(w(x,\tau)))]v\}v\,dxd\tau = \int_0^t\int_\Omega f_1 v\,dxd\tau$$

($M_0' = N_0(k_0R_0' + k_1R_1')^{\frac{1}{2}}$) is analyzed not with Grönwall's lemma but with Young's inequality and the inequality $\alpha(t) + \beta(t)G_{M_0'}(\Phi(w(x,\tau))) \ge \delta_0\alpha_0$.

Now let us discuss the uniqueness of the solution to inverse problem I.

Theorem 3. *Suppose that all hypotheses of Theorem* 1 *are satisfied. Then, for any two solutions* $\{u_1(x,t), q_1(t)\}$ *and* $\{u_2(x,t), q_2(t)\}$ *of inverse problem* I, *we have* $u_1(x,t) \equiv u_2(x,t)$ *for* $(x,t) \in \bar{Q}$ *and* $q_1(t) = q_2(t)$ *for* $t \in [0,T]$.

Proof. Define $w(x,t) = u_1(x,t) - u_2(x,t)$. We have the equalities

$$w_t - \Delta w + q_1(t)w = -\beta(t)\int_\Gamma N(y)w(y,t)ds_y u_2(x,t), \quad (x,t) \in Q,$$

$$w(x,0) = 0, \quad x \in \Omega,$$

$$\frac{\partial w(x,t)}{\partial \nu_x} = 0, \quad (x,t) \in S.$$

Successively considering the equalities

$$\int_0^t\int_\Omega [w_\tau - \Delta w + q_1(\tau)w]w\,dxd\tau = -\int_0^t\int_\Omega \beta\int_\Gamma N(y)w(y,\tau)ds_y u_2 w\,dxd\tau,$$

$$\int_0^t\int_\Omega [w_\tau - \Delta w + q_1(\tau)w]\Delta w\,dxd\tau = \int_0^t\int_\Omega \beta\int_\Gamma N(y)w(y,\tau)ds_y u_2 \Delta w\,dxd\tau,$$

integrating by parts, and applying Young's and Hölder's inequalities and inequality (7), we obtain the estimate

$$\int_\Omega \left[w^2(x,t) + \sum_{i=1}^n w_{x_i}^2(x,t)\right]dx \le C\int_0^t\int_\Omega \left[w^2(x,\tau) + \sum_{i=1}^n w_{x_i}^2(x,\tau)\right]dxd\tau,$$

in which the constant C is determined only by the input data of the problem. This estimate and Grönwall's lemma imply that $w(x,t)$ is identically zero in $\bar{Q}$.

Thus, the functions $u_1(x,t)$ and $u_2(x,t)$ coincide in $\bar{Q}$. Therefore, the functions $q_1(t)$ and $q_2(t)$ also coincide for $t \in [0,T]$.

The theorem is proven.

3. SOLVABILITY OF INVERSE PROBLEM II

Analysis of the solvability of inverse problem II is performed by analogy with the analysis of the solvability of inverse problem I.

Suppose to be satisfied the following conditions:

$$|\mu'(t)| \ge \mu_1 > 0, \quad t \in [0,T]; \tag{20}$$

$$c(x,t) \ge c_0 > 0, \quad (x,t) \in \bar{Q},$$
$$c(x,t) = c_1(x) + c_2(t), \quad (x,t) \in \bar{Q}. \tag{21}$$

Define

$$a(t) = \frac{f_0(t) - c_2(t)\mu(t)}{\mu'(t)}, \quad b(t) = \frac{1}{\mu'(t)}, \quad a_0 = \underset{0\le t\le T}{\operatorname{vrai\,min}}\, a(t),$$

$$v_1(x) = \Delta u_0(x) - c_1(x)u_0(x), \quad f_2(x,t) = \Delta f(x,t) - c_1(x)f(x,t).$$

Hereinafter, we assume that a_0 is a positive number. Let δ_0 be a fixed number from the interval $(0,1)$. Define the numbers R_2, R_3, and M_1 as follows:

$$R_2 = \frac{1}{\delta_0 a_0}\|f_2\|^2_{L_2(Q)} + \sum_{i=1}^n \|v_{1x_i}\|^2_{L_2(\Omega)} + \left\|\sqrt{c(x,0)}v_1\right\|^2_{L_2(\Omega)},$$

$$R_3 = \left(k_1 + \frac{k_0}{c_0}\right)R_2,$$

$$M_1 = N_0 R_3^{\frac{1}{2}}.$$

Theorem 4. *Let condition* (20) *and* (21) *and the memberships* $f(x,t)\in L_2(Q)$, $\Delta f(x,t)\in L_2(Q)$, $f_0(t)\in L_\infty([0,T])$, $c_1(x)\in C^2(\bar{\Omega})$, $c_2(t)\in C^1([0,T])$, $N(x)\in L_2(\Gamma)$, $u_0(x)\in W_2^3(\Omega)$, *and* $\mu(t)\in W_2^1([0,T])$ *be satisfied. Further on, suppose to be satisfied the matching conditions*

$$\frac{\partial u_0(x)}{\partial \nu_x} = \frac{\partial v_1(x)}{\partial \nu_x} = 0, \quad x\in\Gamma,$$

$$\int_\Gamma N(x)u_0(x)ds_x = \mu(0)$$

and the conditions

$$\frac{\partial f(x,t)}{\partial \nu_x} = 0, \quad (x,t)\in S,$$

$$c_2'(t) \le 0, \quad t\in[0,T],$$

$$a_0 > 0,$$

$$\exists\delta_0\in(0,1): \quad M_1 \le (1-\delta_0)a_0\mu_1.$$

Then, inverse problem II *has a solution* $\{u(x,t), p(t)\}$ *such that* $u(x,t)\in V_0$, $\Delta u(x,t)\in V_0$, *and* $p(t)\in L_\infty([0,T])$.

Proof. Consider the following auxiliary boundary value problem: find a function $v(x,t)$ satisfying the equation

$$[a(t) + b(t)G_{M_1}(\Phi(v))]v_t - \Delta v + c(x,t)v = f_2(x,t) \tag{22}$$

in the cylinder Q and the conditions

$$\frac{\partial v(x,t)}{\partial \nu_x} = 0, \quad (x,t)\in S, \tag{23}$$

$$v(x,0) = v_1(x), \quad x\in\Omega. \tag{24}$$

The solvability of this problem is proven by the fixed-point method.

Let $w(x,t)$ be a function from the space V_0. Consider the boundary value problem: find a function $v(x,t)$ satisfying in the cylinder Q the equation

$$[a(t) + b(t)G_{M_1}(\Phi(w))]v_t - \Delta v + c(x,t)v = f_2(x,t) \tag{22_w}$$

and conditions (23) and (24). This problem has a solution $v(x,t)$ belonging to the space V_0 (see [28, Ch. III, Section 6]). Therefore, it generates an operator A mapping the space V_0 into itself: $A(w) = v$. Let us show that the operator A has fixed point in the space V_0.

Consider the equality

$$\int_0^t\int_\Omega\{[a(\tau)+b(\tau)G_{M_1}(\Phi(w))]v_\tau-\Delta v+cv\}v_\tau dxd\tau=\int_0^t\int_\Omega f_2v_\tau dxd\tau.$$

Integrating by parts, applying Young's inequality, and using the hypotheses of the theorem, we find that the solutions $v(x,t)$ of boundary value problem (22_w), (23), and (24) satisfies the estimate

$$\delta_0a_0\int_0^t\int_\Omega v_\tau^2dxd\tau+\sum_{i=1}^n\int_\Omega v_{x_i}^2(x,t)dx+c_0\int_\Omega v^2(x,t)dx\le R_2. \tag{25}$$

Using (25), is it easy to estimate the second derivatives of the function $v(x,t)$ and obtain the total estimate

$$\|v\|_{V_0}\le K_2 \tag{26}$$

of all solutions of boundary value problem (22_w), (23), and (24), in which the constant K_2 is determined by the functions $c(x,t)$, $f_2(x,t)$, $v_1(x)$, the numbers δ_0 and a_0, and the domain Ω.

Estimate (26) implies that the operator A maps a closed ball of the radius K_2 of the space V_0 into itself. Using estimate (26) and the compactness of the embedding $W_2^1(Q)\subset L_2(S)$, it is easy to prove that the operator A is continuous and compact on a closed ball of the radius K_2 in the space V_0. Therefore, the operator A in the space V_0 has fixed points. These fixed points give the solution $v(x,t)$ of boundary value problem (22)–(24).

Solutions $v(x,t)$ of problem (22)–(24) have the same estimates (25) and (26). In particular, (25) implies that $v(x,t)$ satisfies the inequality

$$\int_\Gamma v^2(x,t)dx\le R_3.$$

Hence, $G_{M_1}(\Phi(v))=\Phi(v)$ and the fixed point $v(x,t)$ of the operator A is a solution of the equation

$$[a(t)+b(t)\Phi(v)]v_t-\Delta v+c(x,t)v=f_2(x,t). \tag{27}$$

Set

$$p(t)=a(t)+b(t)\Phi(v)$$

and define the function $u(x,t)$ as a solution of the problem

$$\Delta u(x,t)-c_1(x)u(x,t)=0,\quad \frac{\partial u(x,t)}{\partial \nu_x}=0.$$

It is easy to check that the functions $u(x,t)$ and $p(t)$ are related on the cylinder Q by Eq. (5) and the function $u(x,t)$ satisfies overdetermination condition (4) (in fact, one has only to repeat the reasoning used to prove Theorem 1).

The fact that the function $u(x,t)$ satisfied conditions (2) and (3) and the membership of the functions $u(x,t)$ and $p(t)$ in the required classes are obvious.

The theorem is proven.

Theorem 5. *Suppose that all hypotheses of Theorem* 3 *are satisfied. Then, for any two solutions* $\{u_1(x,t),q_1(t)\}$ *and* $\{u_2(x,t),q_2(t)\}$ *such that* $u_i(x,t)\in V_0$ *and* $p_i(t)\in L_\infty([0,T])$, *we have* $u_1(x,t)\equiv u_2(x,t)$ *and* $p_1(t)\equiv p_2(t)$.

This theorem is proven by analogy with Theorem 3.

4. COMMENTS AND ADDENDA

1. In general, similar results on the solvability of inverse problems I and II can be obtained in more general situations than studied here: (a) for equations of form (1) or (5) in which the Laplace operator is replaced with an arbitrary second-order linear elliptic operator and conditions (3) is replaced with the condition of the second or third boundary value problem with the corresponding conormal derivative;

(b) in the case of a function N depending on the variables $x_1, \ldots, x_n$ and t. The corresponding solvability conditions can be derived rather easily using the technique proposed above.

2. In the general case, the numbers k_0 and k_1 in the inequality (7) depend in a very complicated manner on the geometrical characteristics of the boundary of the domain Ω. However, these numbers are easily computed, for example, for star domains with respect to some interior point of the domain Ω.

3. Let us present some examples demonstrating that the set of initial data in inverse problems I and II for which the conditions of the existence theorems are satisfied is not empty.

A simplest example for inverse problems I is constructed as follows. Let $u_0(x) \equiv \bar{u}_0 = \text{const}$ for $x \in \bar{\Omega}$, $f(x,t) \equiv \bar{f}(t)$ for $(x,t) \in \bar{Q}$, $N(x) \equiv 1$ for $x \in \bar{\Omega}$, $\mu(t) \equiv \mu_0 > 0$ for $t \in [0,T]$, and

$$\mu_0 = \bar{u}_0 \operatorname{mes}\Gamma, \quad \bar{f}(t) \geq \bar{f}_0 > 0, \quad t \in [0,T].$$

It is obvious that all hypotheses of Theorem 1 are satisfied and the solution of inverse problem I is defined by the equalities

$$u(x,t) = \bar{u}_0 \operatorname{mes}\Gamma, \quad q(t) = \frac{\bar{f}(t)\operatorname{mes}\Gamma}{\mu_0}.$$

Now replace $f(x,t)$ with a function $A - \tilde{f}(x)$, where A is a positive number and $\tilde{f}(x)$ is a function with a support lying strictly inside the domain Ω. Let $N(x) \equiv 1$ for $x \in \bar{\Omega}$, $u_0(x) \equiv \bar{u}_0 > 0$ for $x \in \bar{\Omega}$, and $\mu(t) = \bar{u}_0 \operatorname{mes}\Gamma$. We have the equalities $\alpha(t) = \frac{A}{\bar{u}_0}$, $N_0 = \operatorname{mes}^{\frac{1}{2}}\Gamma$, and $M_0 = \operatorname{mes}^{\frac{1}{2}}\Gamma \cdot (k_0 e^T + k_1)^{\frac{1}{2}} \|\Delta\tilde{f}\|_{L_2(\Omega)} T^{\frac{1}{2}}$. The hypothesis $M_0 \leq \alpha_0\mu_0$ of Theorem 1 is obviously satisfied for sufficiently large numbers A.

In a similar manner, if $f(x,t) = A - \tilde{f}(x)$, $c_1(x) \equiv 0$, $c_2(t) \equiv c_0 > 0$, $N(x) \equiv 1$, $u_0(x) \equiv 0$, $\mu(t) = \mu_1 t$, and $\mu_1 > 0$, the hypothesis $M_1 \leq (1 - \delta_0)a_0\mu_1$ of Theorem 4 will be satisfied for sufficiently large numbers A.

These examples demonstrate that the set of initial data in inverse problems I and II satisfying the conditions of the existence theorems is not empty.

4. The condition $(\partial f(x,t))/(\partial \nu_x) = 0$ for $(x,t) \in S$ is purely technical and was introduced to simplify the computations.

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