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Comparison of Additional Second-Order Terms in Finite-Difference Euler Equations and Regularized Fluid Dynamics Equations

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Abstract—In recent years, an area of research in computational mathematics has emerged that is associated with the numerical solution of fluid flow problems based on regularized fluid dynamics equations involving additional terms with velocity, pressure, and body force. The inclusion of these functions in the additional terms has been physically substantiated only for pressure and body force. In this paper, the continuity equation obtained geometrically by Euler is shown to involve second-order terms in time that contain Jacobians of the velocity field and are consistent with some of the additional terms in the regularized fluid dynamics equations. The same Jacobians are contained in the inhomogeneous right-hand side of the wave equation and generate waves of pressure, density, and sound. Physical interpretations of the additional terms used in the regularized fluid dynamics equations are given.

Keywords: regularized fluid dynamics equations, Euler's finite-difference continuity equation, Jacobians, wave equation.

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INTRODUCTION

Studied in [1, 2], the finite-difference form of the continuity equation derived by Euler [3] contains second-order terms in time with Jacobians of the velocity field. In the transition to the wave equation, these terms lead to the generation of periodic pressure oscillations. To understand the nature of the arising oscillations and to find effective methods for their suppression in numerical computations, it is useful to compare the additional terms used in [4–6] with the second-order terms in time t calculated by Euler, which have the clear geometric interpretation of pairwise deformations (see [7]).

1. EULER'S FINITE-DIFFERENCE CONTINUITY EQUATION WITH SECOND-ORDER TERMS IN TIME

Recall the geometric procedure used by Euler in 1752 to derive the continuity equation and followed by N.E. Zhukovsky in 1876 to construct an ellipsoid of deformation. To simplify the presentation, in a plane two-dimensional laminar steady flow, we consider the deformation of a control volume whose shape is a unit square. The deformations are assumed to be only dilations along the x axis and contractions along the y axis. Expressing the dilation rate in terms of the derivative $\partial u/\partial x$ of the x-velocity u and the contraction rate in terms of the derivative $\partial v/\partial y$ of the y-velocity v and equating the areas before and after the deformation over the time Δt, we obtain

$$(1 + \Delta t\partial u/\partial x)(1 + \Delta t\partial v/\partial y) = 1$$

or

$$\partial u/\partial x + \partial v/\partial y + \Delta t(\partial u/\partial x)(\partial v/\partial y) = 0,$$

where the additional term reflects the pairwise nature of deformations, i.e., the presence of two deformations in mutually perpendicular directions.

Taking into account the shear strains, Euler [3] obtained a more complete additional term, which was expressed in [8] as the Jacobian $\partial(u, v)/\partial(x, y)$ multiplied by the time interval Δt:

$$\partial u/\partial x + \partial v/\partial y + \Delta t \partial(u,v)/\partial(x,y) = 0$$

or

$$\partial u/\partial x + \partial v/\partial y = -\Delta t \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}.$$

An analysis of the Jacobian shows that it reflects account of the deformations in mutually perpendicular directions. For a compressible gas, the continuity equation with high-order terms in time was written in [9] as

$$\partial \rho/\partial t + \partial(\rho u)/\partial x + \partial(\rho v)/\partial y + \Delta t \rho \partial(u,v)/\partial(x,y) = 0.$$

Periodic waves cannot emerge in an incompressible medium, while waves of density, pressure, or sound can develop in a compressible medium. When the equations are written in the wave equation approximation, the inhomogeneous right-hand side contains terms that lead to the generation of periodic waves by the flow.

Later studies [1, 10] showed that, when the wave equation is derived by the method of acoustic analogy, its inhomogeneous right-hand side involves the same second-order Jacobians, which pass from the convective terms of the momentum equations. They also lead to the generation of periodic pressure waves imposed on the steady laminar flow.

Thus, we have obtained that the classical system of gasdynamic equations (without additional terms) also generates periodic waves of pressure. Then the additional terms of the regularized fluid dynamics equations can be regarded as terms that suppress these natural sound oscillations of the flow and make it possible to obtain an averaged stationary solution at small Reynolds numbers for which usual methods fail to produce results.

2. REGULARIZED FLUID DYNAMICS EQUATIONS

In the history of fluid dynamics, there are examples when highly respected and merited experts found physical foundations for including additional terms in the classical system of fluid dynamics equations (see, e.g., [11–14]). To describe the domain of application and effectiveness of additional terms, one has to know their physical interpretation.

In all the above works, the additional terms take into account self-diffusion, i.e., they involve the pressure gradient, density, and external forces. They have been physically substantiated. Terms with velocity are contained only in quasi-hydrodynamic models. There is no physical substantiation for them. However, computations show that terms with velocities are important and cannot be dropped. To the author's knowledge, there have been no attempts to offer a physical interpretation of the velocity field taken into account in the additional terms. Accordingly, the identification of the Jacobian-containing additional terms calculated by Euler with terms of the regularized fluid dynamics equations seems useful.

The modern foundations of additional terms used in the numerical solution of fluid dynamics problems were laid by Chetverushkin [4].

The regularized fluid dynamics equations studied by Elizarova and Sheretov [5, 6] were found to be highly effective in enhancing the stability of numerical solutions of fluid dynamics problems. For a weakly compressible fluid, they have the form

$$\operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{w},$$
$$\rho \partial \mathbf{u}/\partial t + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \rho f + \rho(\mathbf{w} \cdot \nabla)\mathbf{u} + \rho(\mathbf{u} \cdot \nabla)\mathbf{w},$$
$$\mathbf{w} = \tau((\mathbf{u} \cdot \nabla)\mathbf{u} + \rho^{-1}\nabla p - \mathbf{f}),$$

where the parameter τ in the vector $\mathbf{w}$ has the dimension of time:

$$\tau = \nu/c_0^2.$$

Here, ν corresponds to the kinematic viscosity for viscous flows and c_0 is the speed of sound. However, the time parameter τ is not regarded as the ratio of the kinematic viscosity to the squared sound speed. With use of this additional term, a number of complicated gasdynamic problems at low and high Reynolds numbers were simulated in [5, 6].

Below, the additional terms in the continuity equation used in [5, 6], i.e., ones containing velocity gradients are compared with the second-order terms in time t calculated by Euler, which have the clear geometric interpretation of pairwise deformations (see [7]). For this purpose, we transform the first parenthesized term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ on the right-hand side of the formula for the vector $\mathbf{w}$. The terms containing the pressure gradient and the body force are ignored. For illustrative purposes, the analysis performed for a plane two-dimensional flow with velocity components u and v along the x and y axes. According to the vector analysis formula

$$(\mathbf{G} \cdot \nabla)\mathbf{F} = G_x\partial\mathbf{F}/\partial x + G_y\partial\mathbf{F}/\partial y$$

(see [15, p. 167]), the first parenthesized term in the formula for $\mathbf{w}$ can be expanded as

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = u\partial\mathbf{u}/\partial x + v\partial\mathbf{u}/\partial y = u\begin{vmatrix}\partial u/\partial x\\ \partial v/\partial x\end{vmatrix} + v\begin{vmatrix}\partial u/\partial y\\ \partial v/\partial y\end{vmatrix} = \begin{vmatrix}u\partial u/\partial x + v\partial u/\partial y\\ u\partial v/\partial x + v\partial v/\partial y\end{vmatrix}.$$

Thus, the vector $\mathbf{w}$ has two components along the x and y axes:

$$w_x = (u\partial u/\partial x + v\partial u/\partial y)\tau,$$

$$w_y = (u\partial v/\partial x + v\partial v/\partial y)\tau.$$

Assuming that all derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, and $\partial v/\partial y$ in the control volume and its neighborhood have constant values, we compute the divergence of $\mathbf{w}$ on the right-hand side of the continuity equation from [6] divided by τ:

$$\begin{gathered}(\operatorname{div}\mathbf{w})/\tau = \partial(u\partial u/\partial x)/\partial x + \partial(v\partial u/\partial y)/\partial x\\ + \partial(u\partial v/\partial x)/\partial y + \partial(v\partial v/\partial y)/\partial y = (\partial u/\partial x)(\partial u/\partial x) + (\partial v/\partial x)(\partial u/\partial y)\\ + (\partial v/\partial x)(\partial u/\partial y) + (\partial v/\partial y)(\partial v/\partial y) = -(\partial u/\partial x)(\partial v/\partial y)\\ + (\partial v/\partial x)(\partial u/\partial y) + (\partial v/\partial x)(\partial u/\partial y) - (\partial u/\partial x)(\partial v/\partial y)\\ = -2\begin{vmatrix}\partial u/\partial x & \partial u/\partial y\\ \partial v/\partial x & \partial v/\partial y\end{vmatrix} = -2\partial(u,v)/\partial(x,y).\end{gathered}$$

The continuity equation

$$\operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{w}$$

in the system of regularized fluid dynamics equations (1.1) from [6, p. 15] becomes

$$\partial u/\partial x + \partial v/\partial y = -2\tau\partial(u,v)/\partial(x,y)$$

or

$$\partial u/\partial x + \partial v/\partial y + 2\tau\partial(u,v)/\partial(x,y) = 0.$$

The time parameter τ is not regarded as the ratio of the kinematic viscosity to the squared oscillation velocity: ν/c_0^2. Instead, τ is treated as a unified variable. The only difference from Euler's finite-difference continuity equation

$$\Delta u/\Delta x + \Delta v/\Delta y + \Delta t\begin{vmatrix}\Delta u/\Delta x & \Delta u/\Delta y\\ \Delta v/\Delta x & \Delta v/\Delta y\end{vmatrix} = 0$$

(see, e.g., [1]) is that the Jacobian is preceded by a coefficient of 2. The cause of occurring this coefficient will be explained below.

We have found that the divergence of the first parenthesized term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ on the right-hand side of the formula for $\mathbf{w}$ reflects the area of the second deformation of the control volume.

3. MECHANISM OF ENHANCING THE STABILITY OF THE NUMERICAL SOLUTION

The analysis presented in [1, pp. 125–128] has shown that the instability of the numerical solution to the gasdynamic equations can be caused by both convective terms in the momentum equations and by the second-order terms in the finite-difference continuity equation, which lead to the generation of periodic pressure oscillations in the transition to the wave equation. The following situation arises. If the inhomogeneous wave equation is set up using the classical system of gasdynamic equations and only terms associated with the velocity field u are retained on the inhomogeneous right-hand side with the help of the weakly compressible fluid model (see [6, p. 15]), then we obtain the wave equation

$$\partial^2 p/\partial x^2 + \partial^2 p/\partial y^2 - c_0^{-2}\partial^2 p/\partial t^2 = 2\rho\partial(u,v)/\partial(x,y)$$

(see [1, p. 128]). Moreover, Jacobians arise only from the convective terms of the momentum equations along the x and y axes. However, if the wave equation is set up using Euler's finite-difference continuity equation

$$\Delta u/\Delta x + \Delta v/\Delta y + \Delta t\begin{vmatrix}\Delta u/\Delta x & \Delta u/\Delta y\\ \Delta v/\Delta x & \Delta v/\Delta y\end{vmatrix} = 0$$

or its form for a compressible gas,

$$\Delta\rho/\Delta t + \Delta(\rho u)/\Delta x + \Delta(\rho v)/\Delta y + \rho\Delta t\begin{vmatrix}\Delta u/\Delta x & \Delta u/\Delta y\\ \Delta v/\Delta x & \Delta v/\Delta y\end{vmatrix} = 0,$$

then the wave equation receives a coefficient of 3 preceding the Jacobian on the inhomogeneous right-hand side.

By using additional terms with a tuned coefficient τ, the terms leading to the generation of oscillations cancel out. Due to the additional terms, the inhomogeneous terms of the wave equation cancel out partially or completely.

The case of $\mathbf{w}$ involving body forces and the pressure drop is beyond the scope of this paper.

CONCLUSIONS

The paper offers a physical interpretation of the additional terms that have been used in the regularized fluid dynamics equations over the last few decades. Methods for solving these equations have been successfully tested numerically by Chetverushkin, Elizarova, Sheretov, and their students as applied to a large class of problems. More than five monographs have been published in this area of research in our country and abroad. Numerical experiments have shown that this new direction has great potential and promises advances in the numerical simulation of new hard-to-solve problems.

To conclude, we note that the requirement that the Euler continuity equation with high-order terms in time obey the Galilean transformations, which correspond to uniform linear motion, is not valid, since the additional terms generate periodic oscillations (e.g., sound waves) accompanied by a sinusoidal variation in the acceleration with time (see [10]). The same is true of the regularized fluid dynamics equations, in which the terms responsible for sinusoidal oscillations represent the basic tool in the method for obtaining results.

The fact that the additional terms in the regularized fluid dynamics equations agree with Euler's exact geometric constructions suggests that these terms are valid and outlines possible ways of their improvement for deriving stable numerical results.

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