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Entropy-Conservative Spatial Discretization of the Multidimensional Quasi-Gasdynamic System of Equations

A. A. Zlotnik

National Research University Higher School of Economics, Moscow, 101000 Russia
e-mail: azlotnik2007@mail.ru

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Abstract—The multidimensional quasi-gasdynamic system written in the form of mass, momentum, and total energy balance equations for a perfect polytropic gas with allowance for a body force and a heat source is considered. A new conservative symmetric spatial discretization of these equations on a nonuniform rectangular grid is constructed (with the basic unknown functions—density, velocity, and temperature—defined on a common grid and with fluxes and viscous stresses defined on staggered grids). Primary attention is given to the analysis of entropy behavior: the discretization is specially constructed so that the total entropy does not decrease. This is achieved via a substantial revision of the standard discretization and applying numerous original features. A simplification of the constructed discretization serves as a conservative discretization with nondecreasing total entropy for the simpler quasi-hydrodynamic system of equations. In the absence of regularizing terms, the results also hold for the Navier–Stokes equations of a viscous compressible heat-conducting gas.

Keywords: Navier–Stokes equations for viscous compressible heat-conducting gases, quasi-gasdynamic system of equations, spatial discretization, conservativeness, law of nondecreasing entropy.

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INTRODUCTION

A wide variety of numerical methods for gasdynamic simulations has been developed to date (see, e.g., [1, 2]). An original class of methods is associated with the construction of quasi-gasdynamic (QGD) systems of equations and their subsequent discretization. A detail description of this approach can be found in [3–5]. One of its advantages is the simplicity of parallel implementations of corresponding algorithms. Additionally, a number of important issues in the mathematical theory of QGD systems were investigated in [6–9].

The law of nondecreasing entropy plays a key role in both physical and mathematical theory of gas dynamics equations, namely, the Euler equations for an inviscid non-heat-conducting gas and the Navier–Stokes equations for a viscous heat-conducting gas (see, e.g., [10–14]). In numerical methods intended for gasdynamic simulations, the control of total entropy behavior is also an important issue of theory and practice. The interest in it has been growing in recent years (see [15–22] and earlier works [23–25] for various one-dimensional Navier–Stokes systems of equations written in Lagrangian mass coordinates).

At the same time, the discrete law of nondecreasing entropy is rather frequently overlooked in the development of numerical methods. This is caused both by the complexity of its derivation, which requires special nonlinear transformations of discrete equations (in contrast to other conservation laws) and by rather specific requirements imposed on the methods to be constructed.

Both these points are vividly manifested in the present paper, which continues to develop a rigorous mathematical theory of discretizations of QGD systems. Previously, this subject was addressed in the context of one-dimensional QGD systems [26–30] and a simplified barotropic QGD system in the multidimensional case [31]. Note that successful numerical experiments with new discretizations were presented in [29, 30].

In this paper, the discretization approaches used in [27, 31] are extended to the multidimensional QGD system written in the form of mass, momentum, and total energy balance equations for a perfect polytropic gas with allowance for a body force and a heat source. We construct a new conservative symmetric spatial discretization on a nonuniform rectangular grid with the basic unknown functions—density, velocity, and temperature—defined on a common grid and with fluxes and viscous stresses defined on staggered grids (following [4, 5]). Primary attention is given to the analysis of entropy behavior: the discretization is specially constructed so that finally the total entropy does not decrease. This is achieved by substantial revision of the standard discretization and applying numerous original features, including a new discretization of the Navier–Stokes viscous stress tensor, new representations for diagonal elements of the regularizing stress tensor and the heat flux, a new approach to averaging the regularization parameter τ as a multilplier, special "logarithmic" averages of the density and internal energy (in mass and total energy fluxes), a special role of temperature in discretizations of the viscosity coefficient and the regularizing stress tensor, the introduction of correcting terms/factors with small coefficients h^2 in the mass and total energy fluxes within the total energy equation and the body force approximation, a new discretization of the heat source, etc. Note that, in the one-dimensional case, the discretization is not identical to the one constructed in [27].

An important feature is that, in the absence of regularizing terms from the system, the results hold for the Navier–Stokes equations for a viscous compressible heat-conducting gas.

Along with the QGD system, the quasi-hydrodynamic (QHD) system of equations is used in practice [4, 5]. Mathematically, it is convenient to treat the latter formally as a simplification of the former. As a result, a conservative discretization with nondecreasing total entropy for the QHD system can easily be specified as a corresponding simplification of the discretization constructed for the QGD system, which is performed at the end of this paper.

Note that, in addition to its practical importance, the possibility of constructing a conservative discretization satisfying conservation laws, including the one of nondecreasing entropy, can be of significant theoretical importance. Relying on this possibility and the method developed in [32] (which can be referred to as the *method of reference difference schemes*), fairly strong results could be derived in the future concerning the properties of a large family of discretizations, including structurally simpler ones, for which conservation laws are satisfied only with an error.

This paper is organized as follows. In Section 1, we recall the QGD system of equations and the corresponding internal energy and entropy balance equations. Some of the terms are written in a form more convenient for the subsequent discretization. In Section 2, we first introduce the necessary notation and present auxiliary results from difference analysis. Then the discretization of the QGD equations is described in the early stages of construction. For this discretization, we derive total (over the domain) mass and energy conservation laws (Theorem 1) and obtain kinetic and internal energy balance equations (Lemma 2). In Section 3, we complete the construction of the discretization and analyze the behavior of the total entropy. Generally, the argument underlying the derivation of the law of nondecreasing total entropy follows the differential case, but is more complicated. It is divided into several steps, namely, Lemmas 3–7. Note that the discretization is elaborated to a higher degree of detail in parallel with the derivation procedure in order to demonstrate the role of its structural elements in the validity of the law. A direct consequence of these lemmas is the main result of the present work—Theorem 2 on a lower bound for the derivative of total entropy and the law of nondecreasing total entropy. In the two-dimensional case, part of the constructed discretization is written in expanded form. The discretization chosen for off-diagonal terms of the regularizing stress tensor is discussed at the end of Section 3. More specifically, we show how to modify the discretizations from [31] so as to reduce the stencil for such terms. Finally, Section 4 is briefly concerned with the QHD system of equations. Specifically, we consider a corresponding conservative discretization that is a simplification of the one constructed in Sections 2 and 3 and prove Theorem 3 (similar to Theorem 2) on the properties of total entropy.

1. QUASI-GASDYNAMIC SYSTEM OF EQUATIONS AND ENTROPY BALANCE

The QGD system of equations in the form of [4, 5] consists of the following mass, momentum, and total energy balance equations:

$$\partial_t\rho + \operatorname{div}\mathbf{J} = 0, \tag{1.1}$$

$$\partial_t(\rho u_l) + \operatorname{div}(\mathbf{J}u_l) + \partial_l p = \partial_i\Pi_{il} + \rho_* F_l, \tag{1.2}$$

$$\partial_t E + \operatorname{div}[(E + p)(\mathbf{u} - \mathbf{w})] = \partial_i(-q_i + \Pi_{ij}u_j) + J_i F_i + Q. \tag{1.3}$$

The unknown functions—the density $\rho > 0$, velocity $\mathbf{u} = (u_1, \ldots, u_n)$, and the total energy of the gas $E = 0.5\rho|\mathbf{u}|^2 + \rho\varepsilon$— depend on (x, t), where $x = (x_1, \ldots, x_n) \in \Omega$, Ω is a bounded domain in $\mathbb{R}^n$ ($n = 1, 2, 3$) with a boundary $\partial\Omega$, and $t \geq 0$. Here and below, summation from 1 to n is implied over repeated indices i and j

(and only over them), the indices l and k range from 1 to n, and the operators div and ∇ are taken with respect to x.

We use the equations of state of a perfect polytropic gas:

$$p = (\gamma - 1)\rho\varepsilon, \quad \varepsilon = c_V\theta \tag{1.4}$$

with constants $\gamma > 1$ and $c_V > 0$. The first of these equations is also applied in the form $p = K\rho\theta$, where $K = (\gamma - 1)c_V$. The functions p, ε, $\theta > 0$ are the pressure, internal energy, and absolute temperature. Let $\mathbf{F}(x, t) = (F_1, \ldots, F_n)$ denote the body force density and $Q(x, t) \geq 0$ be the heat source strength (both are given functions).

The regularized mass flux $\mathbf{J} = (J_1, \ldots, J_n)$ and the regularized density ρ_* are given by the formulas

$$\mathbf{J} = \rho(\mathbf{u} - \mathbf{w}), \quad \mathbf{w} = \frac{\tau}{\rho}[\operatorname{div}(\rho\mathbf{u})]\mathbf{u} + \hat{\mathbf{w}}, \quad \hat{w}_k = \tau\left[(\mathbf{u}\nabla)u_k + \frac{1}{\rho}\partial_k p - F_k\right], \tag{1.5}$$

$$\rho_* = \rho - \tau\operatorname{div}(\rho\mathbf{u}). \tag{1.6}$$

Here, $\mathbf{w} = (w_1, \ldots, w_n)$ and $\hat{\mathbf{w}} = (\hat{w}_1, \ldots, \hat{w}_n)$ are auxiliary regularizing vector functions, $\tau = \tau(\rho, \mathbf{u}, \theta) > 0$ is a regularization parameter, and $\mathbf{u}\nabla = u_i\partial_i$. Note a possible and sometimes useful formula

$$\mathbf{J} = \rho_*\mathbf{u} - \rho\hat{\mathbf{w}}. \tag{1.7}$$

The viscous stress tensor $\Pi = \Pi^{NS} + \Pi^{\tau}$ consists of the Navier–Stokes and regularizing terms

$$\Pi_{kk}^{NS} = 2\mu\partial_k u_k + \left(\lambda - \frac{2}{3}\mu\right)\operatorname{div}\mathbf{u}, \quad \Pi_{kl}^{NS} = \mu(\partial_k u_l + \partial_l u_k), \quad k \neq l, \tag{1.8}$$

$$\Pi_{kk}^{\tau} = \rho u_k\hat{w}_k + \tau[(\mathbf{u}\nabla)p + \gamma p\operatorname{div}\mathbf{u}] - \tau(\gamma - 1)Q, \quad \Pi_{kl}^{\tau} = \rho u_k\hat{w}_l, \quad k \neq l, \tag{1.9}$$

where $\mu = \mu(\rho, \mathbf{u}, \theta) \geq 0$ and $\lambda = \lambda(\rho, \mathbf{u}, \theta) \geq 0$ are the dynamic and bulk viscosity coefficients (in contrast to [4, 5], we consider the more general case $\lambda \not\equiv 0$). The second of the terms in Π_{kk}^{τ} is rewritten in a modified form suitable for the discretization constructed below:

$$\begin{aligned} \tau((\mathbf{u}\nabla)p + \gamma p\operatorname{div}\mathbf{u}) &= \tau[\operatorname{div}(p\mathbf{u}) + (\gamma - 1)p\operatorname{div}\mathbf{u}] \\ &= K\theta[\tau\operatorname{div}(\rho\mathbf{u}) + \tau\rho(\mathbf{u}\nabla)\ln\theta + (\gamma - 1)\tau\rho\operatorname{div}\mathbf{u}]. \end{aligned} \tag{1.10}$$

The heat flux $\mathbf{q} = (q_1, \ldots, q_n)$ is given by the formula

$$-\mathbf{q} = \varkappa\nabla\theta + \tau\left\{\rho\left[(\mathbf{u}\nabla)\varepsilon - \frac{p}{\rho^2}(\mathbf{u}\nabla)\rho\right] - Q\right\}\mathbf{u}$$

with the thermal conductivity coefficient $\varkappa = \varkappa(\rho, \mathbf{u}, \theta) \geq 0$. This formula is rewritten in another form used also for the discretization below:

$$-\mathbf{q} = \varkappa\nabla\theta + \tau\theta\left\{c_V\rho(\mathbf{u}\nabla)\ln\theta - K\rho(\mathbf{u}\nabla)\ln\rho - \frac{Q}{\theta}\right\}\mathbf{u}. \tag{1.11}$$

Note that, although μ, λ, and $\varkappa$ are usually independent of $\mathbf{u}$, such a dependence may arise when the QGD equations are used.

The indicated system of equations is a special regularization of the Navier–Stokes equations for a viscous compressible heat-conducting gas.

Equations (1.1)–(1.4) and the formula $\mathbf{J} = \rho(\mathbf{u} - \mathbf{w})$ imply the internal energy balance equation

$$\partial_t(\rho\varepsilon) + \operatorname{div}(\mathbf{J}\varepsilon) = -\operatorname{div}\mathbf{q} + \Pi_{ij}\partial_i u_j - p\operatorname{div}(\mathbf{u} - \mathbf{w}) + (\mathbf{w}\nabla)p - \rho\hat{w}_i F_i + Q. \tag{1.12}$$

It is well known that the entropy of a perfect polytropic gas is given by

$$S = S_0 - K\ln\rho + c_V\ln\theta, \tag{1.13}$$

where S_0 is a constant. The mass and internal energy balance equations imply the entropy balance equation (see [4, 5])

$$\begin{aligned}\partial_t(\rho S) + \mathrm{div}(\mathbf{J}S) &= \mathrm{div}\left(-\frac{\mathbf{q}}{\theta}\right) + \Xi^{NS} + \frac{\rho}{\tau\theta}|\hat{\mathbf{w}}|^2 + \frac{\tau K}{\rho}[\mathrm{div}(\rho\mathbf{u})]^2 \\ &+ \tau c_V\rho\left[(\gamma-1)\mathrm{div}\mathbf{u} + (\mathbf{u}\nabla)\ln\theta - \frac{(\gamma-1)Q}{2p}\right]^2 + \frac{Q}{\theta}\left(1 - \frac{\tau(\gamma-1)Q}{4p}\right)\end{aligned} \tag{1.14}$$

(see also [8] for $Q \not\equiv 0$), where

$$\Xi^{NS} = 2\frac{\mu}{\theta}\mathbb{D}_{ij}\mathbb{D}_{ij} + \left(\frac{\lambda}{\theta} - \frac{2}{3}\frac{\mu}{\theta}\right)(\mathrm{div}\,\mathbf{u})^2 + \varkappa\frac{|\nabla\theta|^2}{\theta^2} \geq 0, \quad n = 1,2,3, \quad \mathbb{D}_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i).$$

Combining Eq. (1.14) with the boundary conditions $\mathbf{J}\cdot\mathbf{n}|_{\partial\Omega} = 0$ and $\mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0$, where $\mathbf{n}$ is the outward normal to $\partial\Omega$, we immediately derive the total entropy balance equation

$$\begin{aligned}\partial_t\int_\Omega \rho S dx &= \int_\Omega\left\{\Xi^{NS} + \frac{\rho}{\tau\theta}|\hat{\mathbf{w}}|^2 dx + \frac{\tau K}{\rho}[\mathrm{div}(\rho\mathbf{u})]^2\right. \\ &\left.+ \tau c_V\rho\left[(\gamma-1)\mathrm{div}\mathbf{u} + (\mathbf{u}\nabla)\ln\theta - \frac{(\gamma-1)Q}{2p}\right]^2 + \frac{Q}{\theta}\left(1 - \frac{\tau(\gamma-1)Q}{4p}\right)\right\}dx, \quad t \geq 0.\end{aligned} \tag{1.15}$$

Its right-hand side represents total entropy production. More specifically, the integral of the sum $\Xi^{NS} + Q/\theta$ is the Navier–Stokes entropy production, and the other terms involve the multiplier τ and represent relaxation production. All terms under the integral sign are nonnegative, except for the last one, which contains Q twice and is nonnegative provided that $\tau(\gamma - 1)Q \leq 4p$. Thus, under the indicated condition, we have the *law of nondecreasing total entropy*

$$\partial_t\int_\Omega \rho S dx \geq 0.$$

Note that the penultimate terms on the right-hand sides of (1.14) and (1.15) are written in a somewhat modified form that is most suitable for deriving discrete analogues.

Note also that, under the standard boundary condition $\mathbf{u}|_{\partial\Omega} = 0$, the above-mentioned boundary conditions for $\mathbf{J}$ and $\mathbf{q}$ can be rewritten in a simpler equivalent form as $\left(\frac{1}{\rho}\nabla p - \mathbf{F}\right)\cdot\mathbf{n}\Big|_{\partial\Omega} = 0$ and $\varkappa\nabla\theta\cdot\mathbf{n}|_{\partial\Omega} = 0$.

2. NOTATION, EARLY STAGES OF SPATIAL DISCRETIZATION OF THE QGD SYSTEM, AND ITS FIRST PROPERTIES

2.1. Following [27, 31], we introduce the notation necessary for constructing a discretization and recall a few formulas. In what follows, let $\Omega = (0, X_1) \times \ldots \times (0, X_n)$. For $k = 1, \ldots, n$, on $[0, X_k]$ we introduce an arbitrary nonuniform grid $\bar{\omega}_{kh}$ in x_k with nodes $0 = x_{k0} < x_{k1} < \ldots < x_{kN_k} = X_k$. Let ω_{kh} consist of its interior nodes x_{km}, $1 \leq m \leq N_k - 1$. Additionally, let $x_{k(-1)} = -x_{k1}$ and $x_{k(N_k+1)} = x_{kN_k} + \left(x_{kN_k} - x_{k(N_k-1)}\right)$. The steps are $h_{km} = x_{km} - x_{k(m-1)}$.

We also introduce an auxiliary grid $\bar{\omega}^*_{kh}$ with nodes $x_{k(m+1/2)} = (x_{km} + x_{k(m+1)})/2$, $-1 \leq m \leq N_k$, and steps $\hat{h}_{km} = x_{k(m+1/2)} - x_{k(m-1/2)} = (h_{km} + h_{k(m+1)})/2$. Let ω^*_{kh} consist of its interior nodes $x_{k(m+1/2)}$, $0 \leq m \leq N_{k-1}$.

The set of functions defined on a grid ω is denoted by $H(\omega)$. For the subsequent analysis, it is natural to assume that $h_k \in H(\overline{\omega}^*_{kh})$ and $\hat{h}_k \in H(\overline{\omega}_{kh})$. For $v \in H(\overline{\omega}_{kh})$ and $y \in H(\overline{\omega}^*_{kh})$, we introduce the grid averages and difference quotients

$$(s_k v)_{m-1/2} = 0.5(v_{m-1} + v_m), \quad (\tilde{s}_k v)_{m-1/2} = \frac{h_{km}}{2\hat{h}_{k(m-1)}} v_{m-1} + \frac{h_{km}}{2\hat{h}_{km}} v_m, \quad \delta_k v_{m-1/2} = \frac{v_m - v_{m-1}}{h_{km}},$$

$$(s^*_k y)_m = \frac{h_{km}}{2\hat{h}_{km}} y_{m-1/2} + \frac{h_{k(m+1)}}{2\hat{h}_{km}} y_{m+1/2}, \quad \delta^*_k y_m = \frac{y_{m+1/2} - y_{m-1/2}}{\hat{h}_{km}}.$$

Clearly, s_k, $\tilde{s}_k$, δ_k: $H(\overline{\omega}_{kh}) \to H(\omega^*_{kh})$, while s^*_k, δ^*_k: $H(\overline{\omega}^*_{kh}) \to H(\overline{\omega}_{kh})$ and $H(\omega^*_{kh}) \to H(\omega_{kh})$. Let $(v_{k-})_{m-1/2} = v_{m-1}$ and $(v_{k+})_{m-1/2} = v_m$.

Let $\tilde{v} \in H(\overline{\omega}_{kh})$. The following difference analogues of the product rule and one more formula hold:

$$\delta^*_k(y s_k v) = (\delta^*_k y)v + s^*_k(y\delta_k v), \tag{2.1}$$

$$\delta^*_k[(s_k\tilde{v})s_k v - 0.25h_k^2(\delta_k\tilde{v})\delta_k v] = (\delta^*_k s_k\tilde{v})v + \tilde{v}\delta^*_k s_k v, \tag{2.2}$$

$$s_k(v\tilde{v}) - (s_k v)s_k\tilde{v} = 0.25h_k^2(\delta_k v)\delta_k\tilde{v}. \tag{2.3}$$

The last two formulas are used not only in the analysis, but also in the construction of the discretization.

The inner products in $H(\overline{\omega}_{kh})$, $H(\omega_{kh})$, and $H(\omega^*_{kh})$ are defined as

$$(v,\tilde{v})_{\overline{\omega}_{kh}} = 0.5v_0\tilde{v}_0 h_{k1} + (v,\tilde{v})_{\omega_{kh}} + 0.5v_{N_k}\tilde{v}_{N_k}h_{kN_k},$$

$$(v,\tilde{v})_{\omega_{kh}} = \sum_{m=1}^{N_k-1} v_m\tilde{v}_m\hat{h}_{km}, \quad (y,\tilde{y})_{\omega^*_{kh}} = \sum_{m=1}^{N_k} y_{m-1/2}\tilde{y}_{m-1/2}h_{km}$$

with the help of the composite trapezoidal and midpoint rules.

Below, we repeatedly use the summation-by-parts formula

$$(\delta^*_k y, v)_{\overline{\omega}_{kh}} = -(y, \delta_k v)_{\omega^*_{kh}} + (s^*_k y)_{N_k}v_{N_k} - (s^*_k y)_0 v_0, \tag{2.4}$$

where the substitutions are zero (for example) for $(s^*_k y)_0 = (s^*_k y)_{N_k} = 0$ or $v_0 = v_{N_k} = 0$, and the formula

$$(s^*_k y, v)_{\overline{\omega}_{kh}} = (y, s_k v)_{\omega^*_{kh}} + 0.25(\delta^*_k y)_{N_k}v_{N_k}h^2_{kN_k} - 0.25(\delta^*_k y)_0 v_0 h^2_{k1}, \tag{2.5}$$

where the substitutions are zero for $(\delta^*_k y)_0 = (\delta^*_k y)_{N_k} = 0$ or $v_0 = v_{N_k} = 0$. The indicated conditions on y are equivalent to the fact that y is an odd/even function, respectively, with respect to $x_k = 0$, X_k. For $v_0 = v_{N_k} = 0$, the inner products on left-hand sides of the formulas can be simplified to $(\cdot,\cdot)_{\omega_{kh}}$.

Lemma 1. *Let $a \in H(\overline{\omega}_{kh})$, where $a \ge 0$. Any $y \in H(\hat{\omega}^*_{kh})$ that is an even function of x_k with respect to $x_k = 0, X_k$ satisfies the inequality*

$$(\tilde{s}_k a, y^2)_{\omega^*_{kh}} \ge (a, (s^*_k y)^2)_{\overline{\omega}_{kh}}.$$

This result is a strengthening of Lemma 1 in [31]; its proof remains nearly the same. Recall that the derivation of this inequality dictates the form of the operator $\tilde{s}_k$, which is not standard. It is also shown in [31] that, under suitable conditions on the grid $\overline{\omega}_{kh}$, the coefficients of the operators $\tilde{s}_k$ and s_k are close (for a uniform grid $\overline{\omega}_{kh}$, they coincide). Note that $\tilde{s}_k(\hat{h}_k a) = h_k s_k a$ and, in principle, this formula can be effectively used when the parameter τ is chosen depending on the spatial grid.

We introduce the multidimensional grids $\overline{\omega}_h = \overline{\omega}_{1h} \times \ldots \times \overline{\omega}_{nh}$, $\omega_h = \omega_{1h} \times \ldots \times \omega_{nh}$, $\partial\omega_h = \overline{\omega}_h \backslash \omega_h$, and $\omega^*_h = \omega^*_{1h} \times \ldots \times \omega^*_{nh}$. Let the grids $\hat{\omega}_{i^*,h}$ and $\overline{\omega}_{i^*,h}$ be obtained from $\overline{\omega}_h$ by replacing the multiplier $\overline{\omega}_{ih}$ with

$\bar{\omega}_{ih}^*$ and ω_{ih}^*, respectively, and let $\omega_{i^*,h}$ be obtained from ω_h by replacing the multiplier ω_{ih} with ω_{ih}^*. The multidimensional averages are defined as

$$s = s_1 \dots s_n, \quad s_{\hat{i}} = \prod_{k\neq i} s_k, \quad s_{\hat{i}}^* = \prod_{k\neq i} s_k^*, \quad s_{\hat{i}\hat{j}} = \prod_{k\neq i,j} s_k, \quad s_{\hat{i}\hat{j}}^* = \prod_{k\neq i,j} s_k^*, \quad \text{where} \quad i \neq j.$$

The spaces $H(\bar{\omega}_h)$, $H(\omega_h)$, and $H(\omega_h^*)$ are equipped with the inner products

$$(v,\tilde{v})_{\bar{\omega}_h} = (\dots(v\tilde{v},1)_{\bar{\omega}_{1h}},\dots,1)_{\bar{\omega}_{nh}}, \quad (v,\tilde{v}) = (v,\tilde{v})_{\omega_h} \equiv (\dots(v\tilde{v},1)_{\omega_{1h}},\dots,1)_{\omega_{nh}},$$
$$(y,\tilde{y})_* = (y,\tilde{y})_{\omega_h^*} \equiv (\dots(y\tilde{y},1)_{\omega_{1h}^*},\dots,1)_{\omega_{nh}^*}.$$

Let the inner product $(v,\tilde{v})_{\bar{\omega}_{i^*,h}}$ be obtained from $(v,\tilde{v})_{\bar{\omega}_h}$ by replacing $(\cdot,1)_{\bar{\omega}_{ih}}$ with $(\cdot,1)_{\omega_{ih}^*}$, and $(v,\tilde{v})_{i^*}$ be obtained from $(v,\tilde{v})$ in a similar manner.

2.2. The equations of QGD system (1.1)–(1.3) are discretized in space as follows:

$$\partial_t\rho + \delta_i^* J_i = 0 \quad \text{on} \quad \bar{\omega}_h, \tag{2.6}$$

$$\partial_t(\rho u_l) + \delta_i^*(J_i s_i u_l) + \delta_l^* s_l p = \delta_i^* \Pi_{il} + s_l^*(\rho_*^{(l)} F_l) \quad \text{on} \quad \omega_h, \tag{2.7}$$

$$\begin{gathered}\partial_t E + \delta_i^*[(E^{(i)} + s_i p)(s_i u_i - w_i) - 0.25h_i^2(\delta_i p)\delta_i u_i] = \delta_i^*(-q_i + \Pi_{ij} s_i u_j) \\ + s_i^*[(s_i\rho)(s_i u_i - w_i)F_i] + [s_i^*(R_{hi}F_i)]\theta + Q \quad \text{on} \quad \bar{\omega}_h,\end{gathered} \tag{2.8}$$

where, recall, $1 \le l \le n$. The boundary conditions are specified as

$$\mathbf{u}|_{\partial\omega_h} = 0, \quad s_k^* J_k|_{x_k=0,X_k} = 0, \quad s_k^* q_k|_{x_k=0,X_k} = 0, \quad 1 \le k \le n. \tag{2.9}$$

Here, the basic unknown functions $\rho(x,t) > 0$, $\mathbf{u}(x,t)$, and $\theta(x,t) > 0$ are defined for $x \in \bar{\omega}_h$ and $t \ge 0$. As before,

$$E = 0.5\rho|\mathbf{u}|^2 + \rho\varepsilon, \quad p = K\rho\theta, \quad \varepsilon = c_V\theta. \tag{2.10}$$

Additionally, u_k is sequentially extended as an even function of x_1 with respect to the points $0, X_1, \dots, X_n$ with respect to the points $0, X_n$ (except for x_k) and as odd function of x_k with respect to the points $x_k = 0$, X_k (for all $1 \le k \le n$), while the functions ρ, θ, and τ are extended evenly in all x_i, $1 \le i \le n$.

The functions F_k are given on $\bar{\omega}_{k^*,h}$ and are assumed to be odd in x_k with respect to the points $x_k = 0$, X_k and even in x_l with respect to the points $x_l = 0$, X_l for all $l \neq k$. The function Q is defined on $\bar{\omega}_h$.

By generalizing the formulas from [27], J_k and $E^{(k)}$ are defined on $\bar{\omega}_{k^*,h}$ as:

$$J_k = \rho^{(k)}(s_k u_k - w_k), \quad E^{(k)} = 0.5\rho^{(k)} u_{j,k-} u_{j,k+} + \rho^{(k)}\varepsilon^{\langle k\rangle}. \tag{2.11}$$

The averages $\rho^{(k)}$ and $\varepsilon^{\langle k\rangle}$ for ρ and ε on $\bar{\omega}_{k^*,h}$ are chosen in a special manner below. In the following theorem and lemma, they, together with $\rho_*^{(k)}$, w_k, Π_{kl}, q_k, and R_{hk}, are also defined on $\bar{\omega}_{k^*,h}$, and are as yet arbitrary. Here, the functions w_k (and, hence, J_k), R_{hk}, and q_k are assumed to have the same evenness/oddness properties as u_k. Assume that the functions $\rho_*^{(k)}$ and Π_{kk} are even, while Π_{kl} with $k \neq l$ are odd in x_k with respect to $x_k = 0$, X_k.

Note that the second and third boundary conditions in (2.9) are equivalent to the fact that J_k and q_k are odd functions of x_k with respect to $x_k = 0$, X_k.

It should be emphasized that Eqs. (2.6) and (2.8) are written not only on ω_h, but also on $\partial\omega_h$. For example, the first (simpler) of them at $x_k = 0$, X_k takes the form

$$\partial_t\rho|_{x_k=0} + \frac{2}{h_{k1}} J_k|_{x_k=0.5h_{k1}} = 0, \quad \partial_t\rho|_{x_k=X_k} - \frac{2}{h_{kN_k}} J_k|_{x_k-0.5h_{kN_k}} = 0,$$

since J_k is odd in x_k with respect to $x_k = 0$, X_k and in view of the property $J_l|_{x_k=0,X_k} = 0$ for all $l \neq k$; specifically, on grid edges, $\partial_t\rho|_{x_k=0,X_k;x_l=0,X_l} = 0$ for all $l \neq k$, $1 \le k \le n$.

Note that the subsequent analysis simplifies noticeably if boundary conditions (2.9) are replaced by the periodicity of the solution in x_1 with period X_1, ..., x_n with period X_n.

Theorem 1. *The total* (*over* Ω) *mass and energy conservation laws hold*:

$$\partial_t(\rho,1)_{\overline{\omega}_h} = 0,$$
$$\partial_t(E,1)_{\overline{\omega}_h} = ((s_i\rho)(s_iu_i - w_i), F_i)_{\overline{\omega}_{ih}} + (R_{hi}F_i, s_i\theta)_{\overline{\omega}_{ih}} + (Q,1)_{\overline{\omega}_h}.$$

Proof. Taking the scalar product of the mass and total energy balance equations (2.6) and (2.8) with 1 in $H(\overline{\omega}_h)$ and using formulas (2.4) and (2.5) and the fact that J_k, $(E^{(k)} + s_kp)(s_ku_k - w_k)$, $h_k^2(\delta_kp)\delta_ku_k$, q_k, and $\Pi_{kj}s_ku_j$ are odd functions, while $(s_k\rho)(s_ku_k - w_k)F_k$, and $R_{hk}F_k$ are even functions of x_k with respect to $x_k = 0$, X_k, we derive the required conservation laws.

Lemma 2. *For the indicated discretization, the following kinetic and internal energy balance equations on* $\overline{\omega}_h$ *are valid*:

$$\partial_t(0.5\rho|\mathbf{u}|^2 + \delta_i^*(0.5J_iu_{j,i-}u_{j,i+})) + (\delta_j^*s_jp)u_j = (\delta_i^*\Pi_{ij})u_j + [s_j^*(\rho_*^{(j)}F_j)]u_j, \tag{2.12}$$

$$\begin{aligned}\partial_t(\rho\varepsilon) + \delta_i^*(J_i\varepsilon^{\langle i\rangle}) &= -\delta_i^*q_i + s_i^*(\Pi_{ij}\delta_iu_j) - p\delta_i^*(s_iu_i - w_i) + s_i^*(w_i\delta_ip)\\ &+ s_i^*[(s_i\rho)(s_iu_i - w_i)F_i] - [s_i^*(\rho_*^{(i)}F_i)]u_i + [s_i^*(R_{hi}F_i)]\theta + Q.\end{aligned} \tag{2.13}$$

Proof. The momentum balance equation (2.7) is multiplied by u_l, and the second term on the left-hand side is transformed with the help of formulas (2.1):

$$0.5\partial_t(\rho u_l^2) + 0.5(\partial_t\rho)u_l^2 + \delta_i^*[J_i(s_iu_l)^2] - s_i^*[J_i(s_iu_l)\delta_iu_l] + (\delta_l^*s_lp)u_l = [\delta_l^*\Pi_{il} + s_l^*(\rho_*^{(l)}F_l)]u_l \tag{2.14}$$

on $\overline{\omega}_h$ (with the boundary condition $u_l|_{\partial\omega_h} = 0$ taken into account). The mass balance equation (2.6) is multiplied by u_l^2 and is also transformed with the help of formulas (2.1):

$$(\partial_t\rho)u_l^2 = -(\delta_i^*J_i)u_l^2 = -\delta_i^*[J_is_i(u_l^2)] + s_i^*[J_i\delta_i(u_l^2)].$$

Substituting this expression into (2.14), using the elementary formulas

$$(s_ku_l)^2 = 0.5s_k(u_l^2) + 0.5u_{l,k-}u_{l,k+}, \qquad \delta_k(u_l^2) = 2(s_ku_l)\delta_ku_l,$$

replacing the index l by j, and summing the result over j, we obtain Eq. (2.12).

According to formulas (2.11), we write

$$(E^{(k)} + s_kp)(s_ku_k - w_k) = (0.5u_{j,k-}u_{j,k+} + \varepsilon^{\langle k\rangle})J_k + (s_kp)s_ku_k - (s_kp)w_k,$$

and, with the help of (2.1) and (2.2), Eq. (2.8) can be rewritten as

$$\begin{aligned}&\partial_tE + \delta_i^*[J_i(0.5u_{j,i-}u_{j,i+} + \varepsilon^{\langle i\rangle})] + (\delta_i^*s_ip)u_i + p\delta_i^*s_iu_i - s_i^*[(\delta_ip)w_i] - p\delta_i^*w_i\\ &= -\delta_i^*q_i + (\delta_i^*\Pi_{ij})u_j + s_i^*(\Pi_{ij}\delta_iu_j) + s_i^*[(s_ip)(s_iu_i - w_i)F_i] + [s_i^*(R_{hi}F_i)]\theta + Q.\end{aligned}$$

Subtracting (2.12) from this equation yields Eq. (2.13).

Note that Eq. (2.13) is a natural discretization of differential equation (1.12) (anyway, for $\mathbf{F} = 0$), which is not only important in itself, but is also used heavily below to derive the law of nondecreasing total entropy. This is achieved by using the nonstandard discretization of $|\mathbf{u}|^2$ in the form of $u_{j,k-}u_{j,k+}$ in (2.11) and by introducing the correcting divergent terms $\delta_i^*[-0.25h_i^2(\delta_ip)\delta_iu_i]$ (containing the small multipliers h_i^2) into the total energy balance equation (2.8). Note that Eq. (2.8) in system (2.6)–(2.8) can be equivalently replaced by Eq. (2.13).

3. COMPLETING THE CONSTRUCTION OF THE DISCRETIZATION AND ANALYSIS OF THE TOTAL ENTROPY BEHAVIOR

3.1. Applying the formulas from [27], for ρ and ε, we introduce nonstandard "logarithmic" averages over x_k:

$$\rho^{(k)} := \frac{1}{\ln(\rho_{k-};\rho_{k+})}, \quad \varepsilon^{\langle k\rangle} := \varepsilon_{k-}\varepsilon_{k+}\ln(\varepsilon_{k-};\varepsilon_{k+}) \quad \text{on} \quad \overline{\omega}_{k^*,h}. \tag{3.1}$$

Note that

$$\varepsilon^{\langle k\rangle} = c_V\theta_{k-}\theta_{k+}\ln(\theta_{k-};\theta_{k+}) = c_V \ln\left(\frac{1}{\theta_{k-}};\frac{1}{\theta_{k+}}\right).$$

Here, $\ln(\alpha;\beta)$ is the divided difference for the logarithmic function:

$$\ln(\alpha;\beta) = \frac{\ln\beta - \ln\alpha}{\beta-\alpha} \quad \text{for} \quad \alpha \neq \beta, \quad \ln(\alpha;\alpha) = \frac{1}{\alpha}, \quad \alpha > 0, \quad \beta > 0.$$

The functions $\rho^{(k)}$ and $\varepsilon^{\langle k\rangle}$ are assumed to be even in x_k with respect to $x_k = 0, X_k$, $1 \le k \le n$. Recall that, in order to avoid the loss of numerical stability for small $|(\beta/\alpha) - 1|$, it is recommended that $\ln(\alpha;\beta)$ be calculated using an approximation of its integral representation [29].

Lemma 3. *The following equality holds on* $\overline{\omega}_h$:

$$\partial_t(\rho S) + \delta_i^*(J_i s_i S) = [\partial_t(\rho\varepsilon) + \delta_i^*(J_i\varepsilon^{\langle i\rangle})]\frac{1}{\theta} - s_i^*[K(s_iu_i - w_i)\delta_i\rho] + \delta_i^*\left[J_i\left(\gamma c_V - \varepsilon^{\langle i\rangle}s_i\frac{1}{\theta}\right)\right]. \tag{3.2}$$

Proof. By entropy definition (1.13) and the mass balance equation (2.6), we have

$$\partial_t(\rho S) = (\partial_t\rho)S + \rho\left(-\frac{K}{\rho}\partial_t\rho + \frac{c_V}{\theta}\partial_t\theta\right) = (\partial_t\rho)S - (K + c_V)\partial_t\rho + [\partial_t(\rho\varepsilon)]\frac{1}{\theta}$$
$$= -(\delta_i^*J_i)S + \gamma c_V\delta_i^*J_i + [\partial_t(\rho\varepsilon)]\frac{1}{\theta}.$$

Therefore, with the help of formula (2.1),

$$\partial_t(\rho S) + \delta_i^*(J_is_iS) = \gamma c_V\delta_i^*J_i + [\partial_t(\rho\varepsilon)]\frac{1}{\theta} + s_i^*(J_i\delta_iS).$$

Furthermore, applying formula (2.1) with $y = J_k\varepsilon^{\langle k\rangle}$ and $v = \frac{1}{\theta}$, we can write

$$\partial_t(\rho S) + \delta_i^*(J_is_iS) = [\partial_t(\rho\varepsilon) + \delta_i^*(J_i\varepsilon^{\langle i\rangle})]\frac{1}{\theta} + s_i^*\left[J_i\left(\delta_iS + \varepsilon^{\langle i\rangle}\delta_i\frac{1}{\theta}\right)\right] + \delta_i^*\left[J_i\left(\gamma c_V - \varepsilon^{\langle i\rangle}s_i\frac{1}{\theta}\right)\right]. \tag{3.3}$$

According to entropy definition (1.13),

$$\delta_kS = -K\delta_k\ln\rho + c_V\delta_k\ln\varepsilon = -K\ln(\rho_{k-};\rho_{k+})\delta_k\rho + c_V\ln(\varepsilon_{k-};\varepsilon_{k+})\delta_k\varepsilon;$$

additionally, we have the elementary formulas

$$\delta_k\frac{1}{\theta} = -\frac{\delta_k\theta}{\theta_{k-}\theta_{k+}}, \quad s_k\frac{1}{\theta} = \frac{s_k\theta}{\theta_{k-}\theta_{k+}}, \tag{3.4}$$

which imply that

$$J_k\left(\delta_kS + \varepsilon^{\langle k\rangle}\delta_k\frac{1}{\theta}\right) = -\rho^{(k)}(s_ku_k - w_k)K\ln(\rho_{k-};\rho_{k+})\delta_k\rho$$
$$+ J_k\left(c_V^2\ln(\varepsilon_{k-};\varepsilon_{k+}) - \varepsilon^{\langle k\rangle}\frac{1}{\theta_{k-}\theta_{k+}}\right)\delta_k\theta = -K(s_ku_k - w_k)\delta_k\rho,$$

where the fulfillment of the second equality is ensured namely by the choice of averages (3.1). Combining this equality with (3.3) yields (3.2).

Lemmas 2 and 3 are proved by generalizing the corresponding arguments in [27] to the multidimensional case.

For the subsequent analysis, we note that, by virtue of the second formula in (3.4), it holds that

$$\frac{1}{(s_k\theta)s_k\dfrac{1}{\theta}}-1=\frac{\theta_{k-}\theta_{k+}-(s_k\theta)^2}{(s_k\theta)^2}=-\frac{h_k^2(\delta_k\theta)^2}{4(s_k\theta)^2}\le 0,\quad (s_k\theta)s_k\frac{1}{\theta}=\frac{(s_k\theta)^2}{\theta_{k-}\theta_{k+}}\ge 1. \tag{3.5}$$

The following lemma gives a preliminary expression for total entropy production, so it begins the derivation of the law of nondecreasing total entropy. Recall that this derivation is based on the mass and internal energy balance equations.

Lemma 4. *The following formula holds for the derivative of the total entropy:*

$$\begin{aligned}\partial_t(\rho,S)_{\bar\omega_h}&=\left(q_i,\delta_i\frac{1}{\theta}\right)_{\bar\omega_{i*,h}}+\left(\Pi_{ij}\delta_iu_j+w_i\delta_ip,s_i\frac{1}{\theta}\right)_{\bar\omega_{i*,h}}\\&+\left((s_i\rho)(s_iu_i-w_i)F_i,s_i\frac{1}{\theta}\right)_{\bar\omega_{i*,h}}-\left(\rho_*^{(i)}F_i,s_i\frac{u_i}{\theta}\right)_{\bar\omega_{i*,h}}+(R_{hi},F_i)_{\bar\omega_{i*,h}}+\left(Q,\frac{1}{\theta}\right)_{\bar\omega_h}.\end{aligned} \tag{3.6}$$

Proof. Taking the scalar product of either side of Eq. (2.13) with $1/\theta$ in $H(\bar\omega_h)$ and of Eq. (3.2) with 1 and applying formulas (2.4) and (2.5), we derive

$$\begin{aligned}\left(\partial_t(\rho\varepsilon)+\delta_i^*(J_i\varepsilon^{\langle i\rangle}),\frac{1}{\theta}\right)_{\bar\omega_h}&=\left(q_i,\delta_i\frac{1}{\theta}\right)_{\bar\omega_{i*,h}}+\left(\Pi_{ij}\delta_iu_j+w_i\delta_ip,s_i\frac{1}{\theta}\right)_{\bar\omega_{i*,h}}+\left(s_iu_i-w_i,\delta_i\frac{p}{\theta}\right)_{\bar\omega_{i*,h}}\\&+\left((s_i\rho)(s_iu_i-w_i)F_i,s_i\frac{1}{\theta}\right)_{\bar\omega_{i*,h}}-\left(\rho_*^{(i)}F_i,s_i\frac{u_i}{\theta}\right)_{\bar\omega_{i*,h}}+(R_{hi},F_i)_{\bar\omega_{i*,h}}+\left(Q,\frac{1}{\theta}\right)_{\bar\omega_h},\\\partial_t(\rho,S)_{\bar\omega_h}&=\left(\partial_t(\rho\varepsilon)+\delta_i^*(J_i\varepsilon^{\langle i\rangle}),\frac{1}{\theta}\right)_{\bar\omega_h}-(K(s_iu_i-w_i),\delta_i\rho)_{\bar\omega_{i*,h}}.\end{aligned}$$

Here, we took into account that q_k, $s_ku_k-w_k$, and $J_k\left(\gamma c_V-\varepsilon^{\langle k\rangle}s_k\dfrac{1}{\theta}\right)$ are odd functions of x_k, while $\Pi_{kl}\delta_ku_l+w_k\delta_kp$, $(s_k\rho)(s_ku_k-w_k)F_i$, and $R_{hk}F_k$ are even functions of x_k with respect to $x_k=0$, X_k, and $\left.\dfrac{u_k}{\theta}\right|_{x_k=0,X_k}=0$ for all $1\le k\le n$.

Substituting the first of these equalities into the second one and combining like terms with the help of the formula $p=K\rho\theta$, we obtain formula (3.6).

Let us construct special discretizations of Π, $\mathbf{w}$, and $\mathbf{q}$ ensuring that the total entropy production (i.e., the right-hand side of (3.6)) is nonnegative. First, we discretize the viscous stress tensor Π, see (1.8)–(1.10). Let $\tilde\mu=\mu/\theta$ and $\tilde\lambda=\lambda/\theta$. Note that, in solving the Euler equations with the help of the QGD equations, one usually sets $\mu=\alpha_{Sc}\tau p$ (see [4, 5]), where $\alpha_{Sc}>0$ is the Schmidt number; then simply $\tilde\mu=K\alpha_{Sc}\tau\rho$. Let

$$\Pi_{kl}=(s_k^{(-1)}\theta)\tilde\Pi_{kl}^{NS}+\Pi_{kl}^{\tau}\quad\text{with}\quad s_k^{(-1)}\theta:=\left(s_k\frac{1}{\theta}\right)^{-1}=\frac{\theta_{k-}\theta_{k+}}{s_k\theta}. \tag{3.7}$$

Define $\tilde\Pi_{kl}^{NS}$ and Π_{kl}^{τ} on $\bar\omega_{k*,h}$ by the formulas

$$\tilde\Pi_{kk}^{NS}=\left[s_{\hat k}^*\left(\frac{4}{3}\tilde\mu+\tilde\lambda\right)\right]\delta_ku_k+(1-\delta^{(kj)})s_j^*\left[\left(s_{\widehat{kj}}^*\left(\tilde\lambda-\frac{2}{3}\tilde\mu\right)\right)\delta_js_ku_j\right], \tag{3.8}$$

$$\tilde\Pi_{kl}^{NS}=(s_{\hat k}^*\tilde\mu)\delta_ku_l+s_l^*[(s_{\widehat{kl}}^*\tilde\mu)\delta_ls_ku_k],\quad k\ne l, \tag{3.9}$$

$$\begin{aligned}\Pi_{kk}^{\tau}&=(s_ku_k)(s_k\rho)\hat w_k\\&+(s_k^{(-1)}\theta)\left[K(\tau\operatorname{div})^{(k)}(\rho\mathbf{u})+K(\tau\rho\mathbf{u}\nabla)^{(k)}\ln\theta+K(\gamma-1)(\tau\rho\operatorname{div})^{(k)}\mathbf{u}-(\gamma-1)s_k\left(\frac{\tau Q}{\theta}\right)\right],\end{aligned} \tag{3.10}$$

$$\Pi_{kl}^{\tau}=(s_ku_k)(s_k\rho)\hat w_l^{(k)},\quad k\ne l, \tag{3.11}$$

where $\tilde{\mu} = \tilde{\mu}(s\rho, s\mathbf{u}, s\theta)$ and $\tilde{\lambda} = \tilde{\lambda}(s\rho, s\mathbf{u}, s\theta)$ (or $\tilde{\mu} = s\tilde{\mu}(\rho, \mathbf{u}, \theta)$ and $\tilde{\lambda} = s\tilde{\lambda}(\rho, \mathbf{u}, \theta)$) are defined on ω_h^*, while $\tau = \tau(\rho, \mathbf{u}, \theta)$ is defined on $\bar{\omega}_h$. Here, $\delta^{(kj)}$ is the Kronecker delta. It is assumed that $\tilde{\Pi}_{kk}^{NS}$ and Π_{kk}^{τ} are even, while $\tilde{\Pi}_{kl}^{NS}$ and Π_{kl}^{τ} with $k \neq l$ are odd in x_k with respect to $x_k = 0, X_k$.

An essentially new element is the extraction (in (3.7)) of the multiplier $s_k^{(-1)}\theta$, which approximates θ. Discretizations (3.8) and (3.9) are identical to those successfully used recently in [31] (with μ and λ in the role of $\tilde{\mu}$ and $\tilde{\lambda}$).

Formula (3.10) also involves nonstandard approximations of θ and τQ, namely, $s_k^{(-1)}\theta$ and $(s_k^{(-1)}\theta)s_k\left(\frac{\tau Q}{\theta}\right)$, respectively.

Let us also construct such discretizations for the components of the vectors $\mathbf{w}$ and $\hat{\mathbf{w}}$ (see (1.5)):

$$w_k = \frac{s_k^{(-1)}\theta}{(s_k\theta)s_k\rho}(\tau\mathrm{div})^{(k)}(\rho\mathbf{u})s_k u_k + \hat{w}_k, \tag{3.12}$$

$$\hat{w}_k = \frac{s_k^{(-1)}\theta}{s_k\rho}\left\{\left(\tilde{s}_k\frac{\tau\rho}{\theta}\right)\left[(s_k u_k)\delta_k u_k + \frac{1}{s_k\rho}\delta_k p - F_k\right] + (1-\delta^{(kj)})s_k\left[\frac{\tau\rho}{\theta}s_j^*[(s_j u_j)\delta_j u_k]\right]\right\}, \tag{3.13}$$

$$\begin{aligned}\hat{w}_k^{(l)} = \frac{s_l^{(-1)}\theta}{s_l\rho}\Bigg\{ & s_l\left\{\frac{\tau\rho}{\theta}s_k^*\left[(s_k u_k)\delta_k u_k + \frac{1}{s_k\rho}\delta_k p - F_k\right]\right\} + \left(\tilde{s}_l\frac{\tau\rho}{\theta}\right)(s_l u_l)\delta_l u_k \\ & + s_l\left\{\frac{\tau\rho}{\theta}s_{\widehat{kl}}^*[(s_{\widehat{kl}}u_{\widehat{kl}})\delta_{\widehat{kl}}u_k]\right\}\Bigg\} \quad \text{on} \quad \bar{\omega}_{l*,h}, \quad k \neq l.\end{aligned} \tag{3.14}$$

The first two of these quantities are defined on $\bar{\omega}_{k*,h}$. The last term in (3.14) is nonzero only for $n = 3$; moreover, in this term, $u_{\widehat{kl}} = u_m$ and $\delta_{\widehat{kl}} = \delta_m$, where (k, l, m) is a permutation of (1, 2, 3). All these formulas contain correcting multipliers depending on θ, including $\frac{s_k^{(-1)}\theta}{s_k\theta} \approx 1$; moreover, the corresponding difference involves the small multiplier h_k^2 (see the first formula in (3.5)). The difference expressions on $\bar{\omega}_{k*,h}$ applied here and in (3.10) are given by

$$(\tau\mathrm{div})^{(k)}(\rho\mathbf{u}) = \tilde{s}_k(\tau\rho)\frac{1}{s_k\rho}\delta_k(\rho u_k) + (1-\delta^{(kj)})s_k\left\{\tau\rho s_j^*\left[\frac{1}{s_j\rho}\delta_j(\rho u_j)\right]\right\}, \tag{3.15}$$

$$(\tau\rho\mathrm{div})^{(k)}\mathbf{u} = [\tilde{s}_k(\tau\rho)]\delta_k u_k + (1-\delta^{(kj)})s_k(\tau\rho s_j^*\delta_j u_j). \tag{3.16}$$

Clearly, these formulas approximate (in the same manner) the respective quantities

$$\tau\mathrm{div}(\rho\mathbf{u}) = (\tau\rho)\left[\frac{1}{\rho}\partial_k(\rho u_k)\right] + (1-\delta^{(kj)})(\tau\rho)\left[\frac{1}{\rho}\partial_j(\rho u_j)\right], \quad \tau\rho\mathrm{div}\mathbf{u} = \tau\rho\partial_k u_k + (1-\delta^{(kj)})(\tau\rho)\partial_j u_j,$$

which are independent of k, on n different grids $\bar{\omega}_{k*,h}$, $1 \le k \le n$. Note that, in (3.13)–(3.16) and below, the averages of τ are not extracted as multipliers (in contrast to more traditional discretizations [4, 5]); instead, τ is averaged in complexes with other unknowns.

Since $\hat{w}_k$ is discretized not only on the grid $\bar{\omega}_{k*,h}$, but also on $\bar{\omega}_{l*,h}$ with $l \neq k$ according to (3.13) and (3.14), the stencil of formula (3.11) does not expand (see the discussion below). Such a technique was used in [4, 5], but the corresponding discretizations were substantially different.

The constructed functions w_k and $\hat{w}_k$ are assumed to have the same evenness/oddness properties with respect to all variables as u_k does (see above).

The expression $(\tau\rho\mathbf{u}\nabla)^{(k)}\ln\theta$ in (3.10) is specified below as a term of $\mathbf{q}$ (see (3.28) below). Its form, like the form of $(\tau\rho\,\mathrm{div})^{(k)}\mathbf{u}$, is of no importance up to Lemma 7.

Recalling definition (1.6) and generalizing the formulas from [27] with the θ-dependent multiplier in (3.12) taken into account, we define

$$\rho_*^{(k)} = s_k\rho - \frac{s_k^{(-1)}\theta}{s_k\theta}(\tau\mathrm{div})^{(k)}(\rho\mathbf{u}), \quad R_{hk} = 0.25h_k^2\rho_*^{(k)}(\delta_k u_k)\delta_k\frac{1}{\theta} \quad \text{on} \quad \bar\omega_{k*,h}. \tag{3.17}$$

These quantities are assumed to have the above-mentioned evenness/oddness properties. Once again, $\rho_*^{(k)}$ contains the correcting multiplier $\frac{s_k^{(-1)}\theta}{s_k\theta} \approx 1$, while R_{hk} is itself a correcting term with a small multiplier h_k^2.

Lemma 5. *The Navier–Stokes terms in the total entropy production satisfy the lower bound*

$$\begin{gathered}\left(\Pi_{ij}^{NS}\delta_i u_j, s_i\frac{1}{\theta}\right)_{\bar\omega_{i*,h}} = (\tilde\Pi_{ij}^{NS}, \delta_i u_j)_{i*} \\ \geq (2\tilde\mu, s_{\hat i}[(\delta_i u_i)^2] + (1-\delta^{(ij)})s_{\hat{ij}}(\tilde{\mathbb{D}}_{hij}^2))_* + \left(\left(\tilde\lambda - \frac{2}{3}\tilde\mu\right), (\mathrm{div}_h^*\mathbf{u})^2\right)_* \geq 0,\end{gathered} \tag{3.18}$$

where $\tilde{\mathbb{D}}_{hkl}(\mathbf{u}) = 0.5(\delta_k s_l u_l + \delta_l s_k u_k)$ *for* $k \neq l$ *and* $\mathrm{div}_h^*\mathbf{u} = \delta_i s_{\hat i} u_i$ *on* ω_h^*.

The equality in (3.18) follows from the property $\delta_k u_j|_{x_l=0,X_l} = 0$ for all $k \neq l$ and j. The inequalities follow directly from [31, Lemma 3]. Note that they can be slightly strengthened, but we do not go into more detail about this issue.

To write the total entropy production, for $\hat w_k$ and $\mathrm{div}(\rho\mathbf{u})$, we introduce the discretizations

$$\hat w_k^{(0)} = \tau s_k^*\left[(s_k u_k)\delta_k u_k + \frac{1}{s_k\rho}\delta_k p - F_k\right] + (1-\delta^{(kj)})\tau s_j^*[(s_j u_j)\delta_j u_k] \quad \text{on} \quad \bar\omega_h, \tag{3.19}$$

$$\mathrm{div}_h(\rho\mathbf{u}) = \rho s_i^*\left[\frac{1}{s_i\rho}\delta_i(\rho u_i)\right] \quad \text{on} \quad \bar\omega_h; \tag{3.20}$$

they are not used for the discretization of the QGD equations. In (3.20), ρ is factored out of the averaging operators, so this formula is also not standard.

Lemma 6. *The regularizing terms and terms with* **F** *in the total entropy production satisfy the lower bound*

$$\begin{gathered}\left(\Pi_{ij}^{\tau}\delta_i u_j + w_i\delta_i p, s_i\frac{1}{\theta}\right)_{\bar\omega_{i*,h}} + \left((s_i\rho)(s_i u_i - w_i)F_i, s_i\frac{1}{\theta}\right)_{\bar\omega_{i*,h}} - \left(\rho_*^{(i)}F_i, s_i\frac{u_i}{\theta}\right)_{\bar\omega_{i*,h}} + (R_{hi}, F_i)_{\bar\omega_{i*,h}} \\ \geq \left(\frac{\rho}{\tau\theta}, |\hat{\mathbf{w}}^{(0)}|^2\right)_{\bar\omega_h} + \left(\frac{K\tau}{\rho}, [\mathrm{div}_h(\rho\mathbf{u})]^2\right)_{\bar\omega_h} + KA^{\tau,\theta} - A^{\tau,Q},\end{gathered} \tag{3.21}$$

where $\hat{\mathbf{w}}^{(0)} = (\hat w_1^{(0)}, \ldots, \hat w_n^{(0)})$ *and*

$$A^{\tau,\theta} = (\gamma-1)((\tau\rho\mathrm{div})^{(i)}\mathbf{u}, \delta_i u_i)_{i*} + ((\tau\rho\mathbf{u}\nabla)^{(i)}\ln\theta, \delta_i u_i)_{i*} + \left((\tau\mathrm{div})^{(i)}(\rho\mathbf{u}), (s_i u_i)\frac{1}{s_i\theta}\delta_i\theta\right)_{i*}, \tag{3.22}$$

$$A^{\tau,Q} = (\gamma-1)\left(s_i\left(\frac{\tau Q}{\theta}\right), \delta_i u_i\right)_{i*}. \tag{3.23}$$

Proof. To transform the first term on the left-hand side of (3.21), we use formulas (3.10)–(3.12) and extract terms with components of $\hat{\mathbf{w}}$ and **F**:

$$\left(\Pi_{ij}^{\tau}\delta_i u_j + w_i\delta_i p, s_i\frac{1}{\theta}\right)_{\bar\omega_{i*,h}} = \delta^{(ii)}\widehat{W}_i^{\tau} + \left((s_i\rho)\left(s_i\frac{1}{\theta}\right)\widehat{w}_i, F_i\right)_{\bar\omega_{i*,h}} + A^{\tau}, \tag{3.24}$$

where (combining the terms with discretizations of $\hat w_k$)

$$\begin{gathered}\widehat{W}_k^{\tau} := \left((s_k u_k)(s_k\rho)\widehat{w}_k(\delta_k u_k) + \widehat{w}_k[\delta_k p - (s_k\rho)F], s_k\frac{1}{\theta}\right)_{\bar\omega_{k*,h}} \\ + (1-\delta^{(kj)})\left((s_j u_j)(s_j\rho)\widehat{w}_k^{(j)}\delta_j u_k, s_j\frac{1}{\theta}\right)_{\bar\omega_{j*,h}},\end{gathered} \tag{3.25}$$

$$A^\tau := \left(K(\tau\mathrm{div})^{(i)}(\rho\mathbf{u}) + K(\tau\rho\mathbf{u}\nabla)^{(i)}\ln\theta + K(\gamma-1)(\tau\rho\mathrm{div})^{(i)}\mathbf{u} - (\gamma-1)s_i\left(\frac{\tau Q}{\theta}\right), \delta_i u_i\right)_{i*}$$
$$+\left(\frac{s_i^{(-1)}\theta}{(s_i\theta)s_i\rho}(\tau\mathrm{div})^{(i)}(\rho\mathbf{u})(s_iu_i)\delta_i p, s_i\frac{1}{\theta}\right)_{i*}.$$

Here, we used the fact that $s_ku_k|_{x_l=0,X_l} = \delta_k u_k|_{x_l=0,X_l} = 0$ for all $k \neq l$.

Rearranging the multipliers in the terms of $\widehat{W}_k^\tau$, introducing shortened notation, and using formulas (3.13) and (3.14), we obtain

$$\widehat{W}_k^\tau = \left(\left(s_k\frac{1}{\theta}\right)(s_k\rho)\widehat{w}_k, (s_ku_k)\delta_k u_k + \frac{1}{s_k\rho}\delta_k p - F_k\right)_{\overline{\omega}_{k*,h}}$$
$$+ (1-\delta^{(kj)})\left(\left(s_j\frac{1}{\theta}\right)(s_j\rho)\widehat{w}_k^{(j)}, (s_ju_j)\delta_j u_k\right)_{\overline{\omega}_{j*,h}}$$
$$= \left(\left(s_k\frac{1}{\theta}\right)(s_k\rho)\widehat{w}_k, \alpha_{kk}+\beta_k\right)_{\overline{\omega}_{k*,h}} + (1-\delta^{(kj)})\left(\left(s_j\frac{1}{\theta}\right)(s_j\rho)\widehat{w}_k^{(j)}, \alpha_{jk}\right)_{\overline{\omega}_{j*,h}}$$
$$= \left(\tilde{s}_k\frac{\tau\rho}{\theta}, (\alpha_{kk}+\beta_k)^2\right)_{\overline{\omega}_{k*,h}} + (1-\delta^{(kj)})\left(s_k\left(\frac{\tau\rho}{\theta}s_j^*\alpha_{jk}\right), \alpha_{kk}+\beta_k\right)_{\overline{\omega}_{k*,h}}$$
$$+ (1-\delta^{(kj)})\left\{\left(s_j\left[\frac{\tau\rho}{\theta}s_k^*(\alpha_{kk}+\beta_k)\right], \alpha_{jk}\right)_{\overline{\omega}_{j*,h}} + \left(\tilde{s}_j\frac{\tau\rho}{\theta}, \alpha_{jk}^2\right)_{\overline{\omega}_{j*,h}} + \left(s_j\left(\frac{\tau\rho}{\theta}s_{\widehat{kj}}^*\alpha_{\widehat{kj}k}\right), \alpha_{jk}\right)_{\overline{\omega}_{j*,h}}\right\}.$$

Applying Lemma 1 and formula (2.5) several times yields the inequality

$$\widehat{W}_k^\tau \geq \left(\frac{\tau\rho}{\theta}, [s_k^*(\alpha_{kk}+\beta_k)]^2\right.$$
$$\left.+(1-\delta^{(kj)})\{(s_j^*\alpha_{jk})s_k^*(\alpha_{kk}+\beta_k) + [s_k^*(\alpha_{kk}+\beta_k)]s_j^*\alpha_{jk} + (s_j^*\alpha_{jk})^2 + (s_{\widehat{kj}}^*\alpha_{\widehat{kj}k})s_j^*\alpha_{jk}\}\right)_{\overline{\omega}_h}$$
$$= \left(\frac{\tau\rho}{\theta}, [s_k^*(\alpha_{kk}+\beta_k) + (1-\delta^{(kj)})s_j^*\alpha_{jk}]^2\right)_{\overline{\omega}_h} = \left(\frac{\rho}{\tau\theta}, (\widehat{w}_k^{(0)})^2\right)_{\overline{\omega}_h}.$$

It should be emphasized that the sum over j in the penultimate expression is taken inside square brackets. Here, we used the property $\alpha_{lk}|_{x_k=0,X_k} = 0$ for $k \neq l$ and the fact that α_{jk} are even functions of x_j with respect to $x_j = 0$, X_j for $k \neq j$.

Now formula (3.24) implies that

$$\left(\Pi_{ij}^\tau\delta_i u_j + w_i\delta_i p, s_i\frac{1}{\theta}\right)_{\overline{\omega}_{i*,h}} = \left(\frac{\rho}{\tau\theta}, |\hat{\mathbf{w}}^{(0)}|^2\right)_{\overline{\omega}_h} + \left((s_i\rho)\left(s_i\frac{1}{\theta}\right)\widehat{w}_i, F_i\right)_{\overline{\omega}_{i*,h}} + A^\tau. \tag{3.26}$$

In view of definitions (3.12) and (3.17) and formula (2.3), the other terms on the left-hand side of (3.21) are transformed as

$$\left((s_i\rho)(s_iu_i - w_i)F_i, s_i\frac{1}{\theta}\right)_{\overline{\omega}_{i*,h}} - \left(\rho_*^{(i)}F_i, s_i\frac{u_i}{\theta}\right)_{\overline{\omega}_{i*,h}} + (R_{hi}, F_i)_{\overline{\omega}_{i*,h}}$$
$$= -\left((s_i\rho)\widehat{w}_iF_i, s_i\frac{1}{\theta}\right)_{\overline{\omega}_{i*,h}} - \left(\rho_*^{(i)}F_i, s_i\frac{u_i}{\theta} - (s_iu_i)s_i\frac{1}{\theta}\right)_{\overline{\omega}_{i*,h}} + (R_{hi}, F_i)_{\overline{\omega}_{i*,h}}$$
$$= -\left((s_i\rho)\left(s_i\frac{1}{\theta}\right)\widehat{w}_i, F_i\right)_{\overline{\omega}_{i*,h}};$$

here, we took into account the formula $(s_k\rho)(s_ku_k - w_k) = \rho_*^{(k)}s_ku_k - (s_k\rho)\widehat{w}_k$ (cf. (1.7)). The terms with R_{hi} appearing in Eq. (2.8) and their form (3.17) are explained by the necessity of satisfying the last equality. When it is added to (3.41), the term with F_i on the right-hand side cancels out.

Applying the formula

$$\frac{\delta_k p}{s_k\rho} = K\frac{\delta_k\rho}{s_k\rho}s_k\theta + K\delta_k\theta$$

to the last term A^τ and rearranging terms, we obtain the decomposition

$$A^\tau = KA^{\tau,\mathbf{u}} + KA^{\tau,\theta} - A^{\tau,Q},$$

where

$$A^{\tau,\mathbf{u}} := \left((\tau\mathrm{div})^{(i)}(\rho\mathbf{u}), \frac{\delta_i\rho}{s_i\rho}s_iu_i + \delta_iu_i\right)_{i*},$$

while $A^{\tau,\theta}$ and $A^{\tau,Q}$ are given by formulas (3.22) and (3.23). Setting $d_k = \frac{1}{s_k\rho}\delta_k(\rho u_k)$ and applying definition (3.15), Lemma 1, and formulas (2.5) and (3.20), we write the chain of relations

$$A^{\tau,\mathbf{u}} = ([\tilde{s}_i(\tau\rho)]d_i, d_i)_{i*} + (1-\delta^{(ij)})(s_i(\tau\rho s_j^* d_j), d_i)_{i*}$$
$$\geq (\tau\rho, s_i^*(d_i^2))_{\overline{\omega}_h} + (1-\delta^{(ij)})(\tau\rho s_j^* d_j, s_i^* d_i)_{\overline{\omega}_h} = \left(\frac{\tau}{\rho}, [\mathrm{div}_h(\rho\mathbf{u})]^2\right)_{\overline{\omega}_h}.$$

Here, we took into account that d_k are even functions of x_k with respect to $x_k = 0, X_k$ and used the equality $d_l|_{x_k=0,X_k} = 0$ for all $l \neq k$. The lemma is proved.

Note that the discretizations of the terms with components of $\mathbf{F}$ in Eqs. (2.7) and (2.8) and the manipulations with them performed in the analysis of total entropy are direct generalizations of the corresponding ones presented in [27].

It remains to discretize the heat flux $\mathbf{q}$ written in form (1.11). Specifically, for $1 \leq k \leq n$, let

$$-q_k = (s_k\varkappa)\delta_k\theta - q_k^\tau, \quad -q_k^\tau = (s_k^{(-1)}\theta)\left[c_V(\tau\rho\mathbf{u}\nabla)^{(k)}\ln\theta - K(\tau\rho\mathbf{u}\nabla)^{(k)}\ln\rho - s_k\left(\frac{\tau Q}{\theta}\right)\right]s_ku_k \quad (3.27)$$

on $\overline{\omega}_{k*,h}$, where the coefficient $\varkappa = \varkappa(\rho,\mathbf{u},\theta)$ is defined on $\overline{\omega}_h$, and let

$$(\tau\rho\mathbf{u}\nabla)^{(k)}\ln v = [\tilde{s}_k(\tau\rho)](s_ku_k)\frac{1}{s_kv}\delta_kv + (1-\delta^{(kj)})s_k\left\{\tau\rho s_j^*\left[(s_ju_j)\frac{1}{s_jv}\delta_jv\right]\right\} \quad (3.28)$$

for $v \in H(\overline{\omega}_h)$. In these formulas, we used the same nonstandard approximations for θ and τQ as before, while the derivatives $\partial_k\ln\rho$ and $\partial_k\ln\theta$ were approximated as $\frac{1}{s_k\rho}\delta_k\rho$ and $\frac{1}{s_k\theta}\delta_k\theta$.

The functions q_k and q_k^τ are assumed to be odd in x_k with respect to $x_k = 0, X_k$.

To write the total entropy production, we introduce the following discretizations for $\mathrm{div}\,\mathbf{u}$ and $(\mathbf{u}\nabla)\ln\theta$ (they are not used for the discretization of the QGD equations):

$$\mathrm{div}_h\mathbf{u} = s_i^*\delta_iu_i, \quad (\mathbf{u}\nabla)_h\ln\theta = s_i^*\left[(s_iu_i)\frac{1}{s_i\theta}\delta_i\theta\right] \quad \text{on} \quad \overline{\omega}_h. \quad (3.29)$$

Clearly, $\mathrm{div}_h\mathbf{u} = \mathrm{div}_h(\rho\mathbf{u})|_{\rho\equiv1}$.

Note that the discretization of the QGD equations constructed for $n = 1$ is somewhat different from the one proposed in [27].

Lemma 7. *The total entropy production terms containing* θ *and* Q *satisfy the lower bound*

$$\left(q_i, \delta_i\frac{1}{\theta}\right)_{\overline{\omega}_{i*,h}} + KA^{\tau,\theta} - A^{\tau,Q} + \left(Q, \frac{1}{\theta}\right)_{\overline{\omega}_h} \geq \left(s_i\varkappa, \frac{(\delta_i\theta)^2}{\theta_{i-}\theta_{i+}}\right)_{\overline{\omega}_{i*,h}}$$
$$+ c_V\left(\tau\rho, \left[(\gamma-1)\mathrm{div}_h\mathbf{u} + (\mathbf{u}\nabla)_h\ln\theta - \frac{(\gamma-1)Q}{2p}\right]^2\right)_{\overline{\omega}_h} + \left(\frac{Q}{\theta}, 1 - \frac{\tau(\gamma-1)Q}{4p}\right)_{\overline{\omega}_h}. \quad (3.30)$$

Proof. According to definition (3.27) and the first formula in (3.4), we write

$$\left(q_i, \delta_i \frac{1}{\theta}\right)_{\bar{\omega}_{i^*,h}} = \left(s_i \varkappa, \frac{(\delta_i \theta)^2}{\theta_{i-}\theta_{i+}}\right)_{\bar{\omega}_{i^*,h}} + \left(q_i^\tau, \delta_i \frac{1}{\theta}\right)_{\bar{\omega}_{i^*,h}}, \tag{3.31}$$

where, in view of the relation

$$-(s_k^{(-1)}\theta)\delta_k \frac{1}{\theta} = \frac{1}{s_k\theta}\delta_k\theta \tag{3.32}$$

(following from (3.4)) and the boundary condition $\mathbf{u}|_{\partial\Omega} = 0$, we have

$$\begin{gathered}\left(q_i^\tau, \delta_i \frac{1}{\theta}\right)_{\bar{\omega}_{i^*,h}} = c_V\left((\tau\rho\mathbf{u}\nabla)^{(i)} \ln\theta, (s_i u_i)\frac{1}{s_i\theta}\delta_i\theta\right)_{i^*} - K\left((\tau\rho\mathbf{u}\nabla)^{(i)} \ln\rho, (s_i u_i)\frac{1}{s_i\theta}\delta_i\theta\right)_{i^*} \\ -\left(s_i\left(\frac{\tau Q}{\theta}\right), (s_i u_i)\frac{1}{s_i\theta}\delta_i\theta\right)_{i^*}.\end{gathered} \tag{3.33}$$

Formula (3.32) explains the choice of the approximations used for θ and $\partial_k \ln\theta$ in (3.27).

For convenience, let

$$a_k = (\gamma - 1)\delta_k u_k, \quad b_k = (s_k u_k)\frac{1}{s_k\theta}\delta_k\theta.$$

Then $(\tau\rho\mathbf{u}\nabla)^{(k)}\ln\theta = [\tilde{s}_k(\tau\rho)]b_k + (1 - \delta^{(kj)})s_k(\tau\rho s_j^* b_j)$ according to definition (3.28), so

$$\begin{gathered}\left(q_i^\tau, \delta_i \frac{1}{\theta}\right)_{i^*} = c_V([\tilde{s}_i(\tau\rho)]b_i, b_i)_{i^*} + c_V(1 - \delta^{(ij)})(s_i(\tau\rho s_j^* b_j), b_i)_{i^*} \\ -(K(\tau\rho\mathbf{u}\nabla)^{(i)}\ln\rho, b_i)_{i^*} - \left(s_i\left(\frac{\tau Q}{\theta}\right), b_i\right)_{i^*}.\end{gathered} \tag{3.34}$$

Since $(\gamma - 1)(\tau\rho\,\mathrm{div})^{(k)}\mathbf{u} = [\tilde{s}_k(\tau\rho)]a_k + (1 - \delta^{(kj)})s_k(\tau\rho s_j^* a_j)$ according to definition (3.16) and $K = c_V(\gamma - 1)$, we can write

$$\begin{gathered}KA^{\tau,\theta} = c_V[([\tilde{s}_i(\tau\rho)]a_i, a_i)_{i^*} + (1 - \delta^{(ij)})(s_i(\tau\rho s_j^* a_j), a_i)_{i^*} \\ + ([\tilde{s}_i(\tau\rho)]b_i, a_i)_{i^*} + (1 - \delta^{(ij)})(s_i(\tau\rho s_j^* b_j), a_i)_{i^*} + (\gamma - 1)((\tau\mathrm{div})^{(i)}(\rho\mathbf{u}), b_i)_{i^*}].\end{gathered} \tag{3.35}$$

By definitions (3.15) and (3.28) and the formula

$$\frac{1}{s_k\rho}\delta_k(\rho u_k) - (s_k u_k)\frac{1}{s_k\rho}\delta_k\rho = \delta_k u_k,$$

we derive

$$\begin{gathered}(\gamma - 1)((\tau\mathrm{div})^{(i)}(\rho\mathbf{u}), b_i)_{i^*} - (\gamma - 1)((\tau\rho\mathbf{u}\nabla)^{(i)}\ln\rho, b_i)_{i^*} \\ = ([\tilde{s}_i(\tau\rho)]a_i, b_i)_{i^*} + (1 - \delta^{(ij)})(s_i(\tau\rho s_j^* a_j), b_j)_{i^*}.\end{gathered}$$

Therefore, it follows from (3.34) and (3.35) that

$$\begin{gathered}\left(q_i^\tau, \delta_i \frac{1}{\theta}\right)_{i^*} + KA^{\tau,\theta} - A^{\tau,Q} = c_V[(\tilde{s}_i(\tau\rho), a_i^2 + 2a_i b_i + b_i^2)_{i^*} \\ + (1 - \delta^{(ij)})(s_i(\tau\rho s_j^* a_j), a_i + b_i)_{i^*} + (1 - \delta^{(ij)})(s_i(\tau\rho s_j^* b_j), a_i + b_i)_{i^*}] \\ -\left(s_i\left(\frac{\tau Q}{\theta}\right), a_i + b_i\right)_{i^*}.\end{gathered}$$

By virtue of formula (2.5),

$$\left(q_i^\tau, \delta_i \frac{1}{\theta}\right)_{i*} + KA^{\tau,\theta} - A^{\tau,Q} = c_V\Big[(\tilde{s}_i(\tau\rho), (a_i + b_i)^2)_{\bar{\omega}_{i*,h}}$$
$$+ (1 - \delta^{(ij)})(\tau\rho s_j^*(a_j + b_j), s_i^*(a_i + b_i))_{\bar{\omega}_h}\Big] - \left(\frac{\tau Q}{\theta}, s_i^*(a_i + b_i)\right)_{\bar{\omega}_h}.$$

Here, we took into account that $a_k + b_k$ are even functions of x_k with respect to $x_k = 0$, X_k and $(a_k + b_k)|_{x_l = 0, X_l} = 0$ for all $l \neq k$. Applying Lemma 1, rearranging the terms twice, and recalling definitions (3.29), we obtain

$$\left(q_i^\tau, \delta_i \frac{1}{\theta}\right)_{i*} + KA^{\tau,\theta} - A^{\tau,Q} \geq c_V\Big[(\tau\rho, s_i^*[(a_i + b_i)]^2)_{\bar{\omega}_h}$$
$$+ (1 - \delta^{(ij)})(\tau\rho s_j^*(a_j + b_j), s_i^*(a_i + b_i))_{\bar{\omega}_h}\Big] - \left(\frac{\tau Q}{\theta}, s_i^*(a_i + b_i)\right)_{\bar{\omega}_h}$$
$$= c_V(\tau\rho, [\delta^{(ii)} s_i^*(a_i + b_i)]^2)_{\bar{\omega}_h} - 2c_V\left(\tau\rho, [s_i^*(a_i + b_i)]\frac{(\gamma - 1)Q}{2p}\right)_{\bar{\omega}_h}$$
$$= c_V\left(\tau\rho, \left[\delta^{(ii)} s_i^*(a_i + b_i) - \frac{(\gamma - 1)Q}{2p}\right]^2\right)_{\bar{\omega}_h} - c_V\left(\tau\rho, \left[\frac{(\gamma - 1)Q}{2p}\right]^2\right)_{\bar{\omega}_h}$$
$$= c_V\left(\tau\rho, \left[(\gamma - 1)\mathrm{div}_h\mathbf{u} + (\mathbf{u}\nabla)_h \ln\theta - \frac{(\gamma - 1)Q}{2p}\right]^2\right)_{\bar{\omega}_h} - \left(\frac{Q}{\theta}, \frac{\tau(\gamma - 1)Q}{4p}\right)_{\bar{\omega}_h}.$$

It should be stressed that the sum over i in both terms with $\delta^{(ii)}$ is taken inside square brackets. Estimate (3.30) is proved.

Combining Lemmas 4–7, we obtain the main result of this paper.

Theorem 2. *The spatial discretization of the QGD equations consists of Eqs.* (2.6)–(2.8), *together with formulas* (2.10), (2.11), *and* (3.1), *where* Π, $\mathbf{w}$, *and* $\mathbf{q}$ *are given by formulas* (3.7)–(3.16), (3.27), *and* (3.28) *and* ρ_* *and* R_h *are given by formulas* (3.17).

For this discretization under boundary conditions (2.9), *the derivative of the total entropy satisfies the lower bound*

$$\partial_t(\rho, S)_{\bar{\omega}_h} \geq (\tilde{\Pi}_{ij}^{NS}, \delta_i u_j)_{i*} + \left(s_i\varkappa, \frac{(\delta_i\theta)^2}{\theta_{i-}\theta_{i+}}\right)_{\bar{\omega}_{i*,h}} + \left(\frac{\rho}{\tau\theta}, |\hat{\mathbf{w}}^{(0)}|^2\right)_{\bar{\omega}_h} + \left(\frac{K\tau}{\rho}, [\mathrm{div}_h(\rho\mathbf{u})]^2\right)_{\bar{\omega}_h}$$
$$+ c_V\left(\tau\rho, \left[(\gamma - 1)\mathrm{div}_h\mathbf{u} + (\mathbf{u}\nabla)_h \ln\theta - \frac{(\gamma - 1)Q}{2p}\right]^2\right)_{\bar{\omega}_h} + \left(\frac{Q}{\theta}, 1 - \frac{\tau(\gamma - 1)Q}{4p}\right)_{\bar{\omega}_h}, \quad (3.36)$$

where the quantities $\hat{\mathbf{w}}^{(0)}$, $\mathrm{div}_h(\rho\mathbf{u})$, $\mathrm{div}_h\mathbf{u}$, *and* $(\mathbf{u}\nabla)_h\ln\theta$ *are defined in* (3.19), (3.20), *and* (3.29). *The first four terms on the right-hand side of this bound are always nonnegative* (*in view of Lemma* 5), *while the last term is nonnegative provided that* $\tau(\gamma - 1)Q \leq 4p$ *on* $\bar{\omega}_h$.

Under the indicated condition, this bound implies the law of nondecreasing total entropy

$$\partial_t(\rho, S)_{\bar{\omega}_h} \geq 0.$$

Inequality (3.36) is a discrete analogue of the total entropy balance equation (1.15) (the transition from the equality to the inequality was used for illustrative purposes with several specific nonnegative difference terms dropped).

The properties indicated in the theorem remain valid for $\tau \geq 0$. To show this, the third term on the right-hand side of (3.36) has to be rewritten as

$$\left(\frac{\tau\rho}{\theta}, \delta^{(ii)}\left\{s_i^*\left[(s_i u_i)\delta_i u_i + \frac{1}{s_i\rho}\delta_i p - F_i\right] + (1 - \delta^{(ij)})\tau s_j^*[(s_j u_j)\delta_j u_i]\right\}^2\right)$$

(the sum over j is taken inside curly brackets). Specifically, for $\tau = 0$, the result holds for the Navier–Stokes equations for a viscous compressible heat-conducting gas (in which case only the first and second terms, together with $\left(\frac{Q}{\theta},1\right)_{\bar{\omega}_h}$, are retained on the right-hand side of (3.36)); the second boundary condition disappears from (2.9) (it holds automatically by virtue of the above-mentioned extension of the components of $\mathbf{u}$), while the third boundary condition becomes $s_k^*[(s_k\varkappa)\delta_k\theta]\|_{x_k=0,X_k} = 0$.

3.2. For illustrative purposes, the part of the constructed discretization involving summation over repeated indices (or containing several indices) is written in expanded form for the two-dimensional case ($n = 2$). Equations (2.6)–(2.8) become

$$\partial_t\rho + \delta_1^*J_1 + \delta_2^*J_2 = 0,$$

$$\partial_t(\rho u_1) + \delta_1^*(J_1s_1u_1 + s_1p - \Pi_{11}) + \delta_2^*(J_2s_2u_2 - \Pi_{21}) = s_1^*(\rho_*^{(1)}F_1),$$
$$\partial_t(\rho u_2) + \delta_1^*(J_1s_1u_2 - \Pi_{12}) + \delta_2^*(J_2s_2u_2 + s_2p - \Pi_{22}) = s_2^*(\rho_*^{(2)}F_2),$$

$$\partial_tE + \delta_1^*[(E^{(1)} + s_1p)(s_1u_1 - w_1) - 0.25h_1^2(\delta_1p)\delta_1u_1 + q_1 - \Pi_{11}s_1u_1 - \Pi_{12}s_1u_2]$$
$$+ \delta_2^*[(E^{(2)} + s_2p)(s_2u_2 - w_2) - 0.25h_2^2(\delta_2p)\delta_2u_2 + q_2 - \Pi_{21}s_2u_1 - \Pi_{22}s_2u_2]$$
$$= s_1^*[(s_1\rho)(s_1u_1 - w_1)F_1] + s_2^*[(s_2\rho)(s_2u_2 - w_2)F_2] + [s_1^*(R_{h1}F_1) + s_2^*(R_{h2}F_2)]\theta + Q.$$

Formula (2.11) for $E^{(k)}$ and formulas (3.13), (3.14) for $\hat{\mathbf{w}}$ are

$$E^{(1)} = \rho^{(1)}[0.5(u_{1,1-}u_{1,1+} + u_{2,1-}u_{2,1+}) + \varepsilon^{\langle 1\rangle}], \quad E^{(2)} = \rho^{(2)}[0.5(u_{1,2-}u_{1,2+} + u_{2,2-}u_{2,2+}) + \varepsilon^{\langle 2\rangle}],$$

$$\hat{w}_1 = \frac{s_1^{(-1)}\theta}{s_1\rho}\left\{\left(\tilde{s}_1\frac{\tau\rho}{\theta}\right)\left[(s_1u_1)\delta_1u_1 + \frac{1}{s_1\rho}\delta_1p - F_1\right] + s_1\left[\frac{\tau\rho}{\theta}s_2^*[(s_2u_2)\delta_2u_1]\right]\right\},$$
$$\hat{w}_2 = \frac{s_2^{(-1)}\theta}{s_2\rho}\left\{\left(\tilde{s}_2\frac{\tau\rho}{\theta}\right)\left[(s_2u_2)\delta_2u_2 + \frac{1}{s_2\rho}\delta_2p - F_2\right] + s_2\left[\frac{\tau\rho}{\theta}s_1^*[(s_1u_1)\delta_1u_2]\right]\right\},$$

$$\hat{w}_1^{(2)} = \frac{s_2^{(-1)}\theta}{s_2\rho}\left\{s_2\left\{\frac{\tau\rho}{\theta}s_1^*\left[(s_1u_1)\delta_1u_1 + \frac{1}{s_1\rho}\delta_1p - F_1\right]\right\} + \left(\tilde{s}_2\frac{\tau\rho}{\theta}\right)(s_2u_2)\delta_2u_1\right\},$$
$$\hat{w}_2^{(1)} = \frac{s_1^{(-1)}\theta}{s_1\rho}\left\{s_1\left\{\frac{\tau\rho}{\theta}s_2^*\left[(s_2u_2)\delta_2u_2 + \frac{1}{s_2\rho}\delta_2p - F_2\right]\right\} + \left(\tilde{s}_1\frac{\tau\rho}{\theta}\right)(s_1u_1)\delta_1u_2\right\}.$$

Formulas (3.15), (3.16), and (3.28) become

$$(\tau\mathrm{div})^{(1)}(\rho\mathbf{u}) = \tilde{s}_1(\tau\rho)\frac{1}{s_1\rho}\delta_1(\rho u_1) + s_1\left\{\tau\rho s_2^*\left[\frac{1}{s_2\rho}\delta_2(\rho u_2)\right]\right\}, \quad (\tau\rho\mathrm{div})^{(1)}\mathbf{u} = [\tilde{s}_1(\tau\rho)]\delta_1u_1 + s_1(\tau\rho s_2^*\delta_2u_2),$$
$$(\tau\rho\mathbf{u}\nabla)^{(1)}\ln v = [\tilde{s}_1(\tau\rho)](s_1u_1)\frac{1}{s_1v}\delta_1v + s_1\left\{\tau\rho s_2^*\left[(s_2u_2)\frac{1}{s_2v}\delta_2v\right]\right\},$$

$$(\tau\mathrm{div})^{(2)}(\rho\mathbf{u}) = s\left\{\tau\rho s_1^*\left[\frac{1}{s_1\rho}\delta_1(\rho u_1)\right]\right\} + \tilde{s}_2(\tau\rho)\frac{1}{s_2\rho}\delta_2(\rho u_2), \quad (\tau\rho\mathrm{div})^{(2)}\mathbf{u} = s_2(\tau\rho s_1^*\delta_1u_1) + [\tilde{s}_2(\tau\rho)]\delta_2u_2,$$
$$(\tau\rho\nabla)^{(2)}\ln v = s_2\left\{\tau\rho s_1^*\left[(s_1u_1)\frac{1}{s_1v}\delta_1v\right]\right\} + [\tilde{s}_2(\tau\rho)](s_2u_2)\frac{1}{s_2v}\delta_2v.$$

Here, the quantities with upper or lower indices 1 correspond to the grid $\bar{\omega}_{1*,h}$, while the quantities with indices 2 correspond to $\bar{\omega}_{2*,h}$ (if there are two indices, priority is given to the upper one).

Formulas (3.8) and (3.9) for components of stress tensor (3.7) are written as

$$\tilde{\Pi}_{11}^{NS} = \left[s_2^*\left(\frac{4}{3}\tilde{\mu} + \tilde{\lambda}\right)\right]\delta_1 u_1 + s_2^*\left[\left(\tilde{\lambda} - \frac{2}{3}\tilde{\mu}\right)\delta_2 s_1 u_2\right], \quad \tilde{\Pi}_{21}^{NS} = (s_1^*\tilde{\mu})\delta_2 u_1 + s_1^*(\tilde{\mu}\delta_1 s_2 u_2),$$
$$\tilde{\Pi}_{22}^{NS} = s_1^*\left[\left(\tilde{\lambda} - \frac{2}{3}\tilde{\mu}\right)\delta_1 s_2 u_1\right] + \left[s_1^*\left(\frac{4}{3}\tilde{\mu} + \tilde{\lambda}\right)\right]\delta_2 u_2, \quad \tilde{\Pi}_{12}^{NS} = s_2^*(\tilde{\mu}\delta_2 s_1 u_1) + (s_2^*\tilde{\mu})\delta_1 u_2.$$

The first and fourth of these formulas correspond to the grid $\overline{\omega}_{1*,h}$, while the second and third, to $\overline{\omega}_{2*,h}$.

3.3. Let us discuss the discretization of Π_{kl}^{τ} and $\hat{w}_k$ (see (3.11), (3.13), (3.14)). Instead of discretizing $\hat{w}_k$ on n grids, we can use the single grid $\overline{\omega}_{k*,h}$ and apply the simpler formulas

$$\hat{w}_k = (s_k\tau)\left[(\mathbf{u}\nabla)^{(k)}u_k + \frac{1}{s_k\rho}\delta_k p - F_k\right], \tag{3.37}$$

$$(\mathbf{u}\nabla)^{(k)}v = (s_k u_k)\delta_k v + (1 - \delta^{(kj)})s_k s_j^*[(s_j u_j)\delta_j v] \quad \text{for} \quad v \in H(\overline{\omega}_h). \tag{3.38}$$

The quantity $(\mathbf{u}\nabla)^{(k)}v$ discretizes

$$(\mathbf{u}\nabla)v = u_k\partial_k v + (1 - \delta^{(kj)})u_j\partial_j v$$

on n grids $\overline{\omega}_{k*,h}$, $1 \le k \le n$; there is an obvious analogy between formulas (3.38) and (3.28). However, here we need to use the formula

$$\Pi_{kl}^{\tau} = (s_k^{(-1)}\theta)(s_k u_k)s_k s_l^*\left[(s_l\rho)\left(s_l\frac{1}{\theta}\right)\hat{w}_l\right], \quad k \ne l, \tag{3.39}$$

where the operator $s_k s_l^*$ maps a function from the grid $\overline{\omega}_{l*,h}$ to $\overline{\omega}_{k*,h}$. Unfortunately, this leads to a considerable expansion of the stencil used.

For the indicated discretizations, differences in the derivation of Theorem 2 arise only in the proof of Lemma 6. Transforming the first term on the left-hand side of (3.21) with the help of formulas (3.10), (3.39), and (3.37), extracting terms with components of $\hat{\mathbf{w}}$, and applying formula (2.5) twice (with respect to x_j and x_i), we obtain

$$\begin{aligned}\left(\Pi_{ij}^{\tau}\delta_i u_j + w_i\delta_i p, s_i\frac{1}{\theta}\right)_{\overline{\omega}_{i*,h}} &= \left((s_i u_i)(s_i\rho)\hat{w}_i(\delta_i u_i) + \hat{w}_i\delta_i p, s_i\frac{1}{\theta}\right)_{\overline{\omega}_{i*,h}} + A^{\tau} \\ &+ (1 - \delta^{(ij)})\left((s_i u_i)s_i s_j^*\left[(s_j\rho)\left(s_j\frac{1}{\theta}\right)\hat{w}_j\right], \delta_i u_j\right)_{\overline{\omega}_{i*,h}} \\ = \left((s_i\rho)\left(s_i\frac{1}{\theta}\right)\hat{w}_i, (s_i u_i)\delta_i u_i + \frac{1}{s_i\rho}\delta_i p\right)_{\overline{\omega}_{i*,h}} &+ (1 - \delta^{(ij)})\left((s_j\rho)\left(s_j\frac{1}{\theta}\right)\hat{w}_j, s_j^* s_j[(s_i u_i)\delta_i u_j]\right)_{\overline{\omega}_{j*,h}} + A^{\tau}\end{aligned} \tag{3.40}$$

with the same term A^{τ}. Here, we again take into account that $s_k u_k|_{x_l=0,X_l} = \delta_k u_k|_{x_l=0,X_l} = 0$ and use the fact that $(s_k u_k)\delta_k u_l$ is even in x_k with respect to $x_k = 0$, X_k (both properties hold for all $k \ne l$).

In formula (3.40), interchanging the indices i and j in the last term with multiplier $(1 - \delta^{(ij)})$ and applying formula (3.38), we derive

$$\left(\Pi_{ij}^{\tau}\delta_i u_j + w_i\delta_i p, s_i\frac{1}{\theta}\right)_{\overline{\omega}_{i*,h}} = \left((s_i\rho)\left(s_i\frac{1}{\theta}\right)\hat{w}_i, \frac{\hat{w}_i}{s_i\tau}\right)_{\overline{\omega}_{i*,h}} + \left((s_i\rho)\left(s_i\frac{1}{\theta}\right)\hat{w}_i, F_i\right)_{\overline{\omega}_{i*,h}} + A^{\tau}. \tag{3.41}$$

Therefore, Lemma 6 and Theorem 2 both remain valid when the term

$$\left(\frac{\rho}{\tau\theta}, |\hat{\mathbf{w}}^{(0)}|^2\right)_{\overline{\omega}_h} \quad \text{is replaced by} \quad \left(\frac{s_i p}{s_i\tau}s_i\frac{1}{\theta}, \hat{w}_i^2\right)_{\overline{\omega}_{i*,h}}.$$

3.4. For the multidimensional QGD system with a potential body force, discretizations of type (3.39), (3.37), (3.38) were recently used in [31]. Let us describe how they can be replaced by discretizations of type (3.11), (3.13), and (3.14) in order to reduce the stencil for Π^{τ}_{kl}, $k \neq l$.

For the first of the spatial discretizations proposed in [31], we use the formulas

$$\Pi^{\tau}_{kl} = (s_k u_k)(s_k\rho)\widehat{w}_l^{(k)} \quad \text{on} \quad \overline{\omega}_{k^*,h}, \quad k \neq l,$$

$$\widehat{w}_k = \frac{1}{s_k\rho}\{[\tilde{s}_k(\tau\rho)][(s_k u_k)\delta_k u_k + \delta_k(P_0'(\rho) - \Phi)]\} + (1 - \delta^{(kj)})s_k[\tau\rho s_j^*[(s_j u_j)\delta_j u_k]]\} \quad \text{on} \quad \overline{\omega}_{k^*,h},$$

$$\widehat{w}_k^{(l)} = \frac{1}{s_l\rho}\{s_l\{\tau\rho s_k^*[(s_k u_k)\delta_k u_k + \delta_k(P_0'(\rho) - \Phi)]\} + [\tilde{s}_l(\tau\rho)](s_l u_l)\delta_l u_k$$
$$+ s_l\{\tau\rho s^*_{\widehat{kl}}[(s_{\widehat{kl}} u_{\widehat{kl}})\delta_{\widehat{kl}} u_k]\}\} \quad \text{on} \quad \overline{\omega}_{l^*,h}, \quad k \neq l.$$

The first of these formulas is identical to (3.11), while the other two can be viewed as simplifications of (3.13) and (3.14). Then in [31] there appears a sum $\delta^{(ii)}\widehat{W}^{\tau}_i$, where $\widehat{W}^{\tau}_k$ (see (3.25)) is given by

$$\widehat{W}^{\tau}_k := ((s_k\rho)\hat{w}_k, \delta_k(P_0'(\rho) - \Phi))_{\overline{\omega}_{k^*,h}} + ((s_k u_k)(s_k\rho)\hat{w}_k, \delta_k u_k)_{\overline{\omega}_{k^*,h}}$$
$$+ (1 - \delta^{(kj)})((s_j u_j)(s_j\rho)\hat{w}_k^{(j)}, \delta_j u_k)_{\overline{\omega}_{j^*,h}}.$$

In a similar manner to the proof of Lemma 6, we have the chain of transformations

$$\widehat{W}^{\tau}_k := ((s_k\rho)\widehat{w}_k, (s_k u_k)(s_k\rho)\delta_k u_k + \delta_k(P_0'(\rho) - \Phi))_{\overline{\omega}_{k^*,h}} + (1 - \delta^{(kj)})((s_j\rho)\widehat{w}_k^{(j)}, (s_j u_j)\delta_j u_k)_{\overline{\omega}_{j^*,h}}$$
$$= (\tilde{s}_k(\tau\rho), (\alpha_{kk} + \beta_k)^2)_{\overline{\omega}_{k^*,h}} + (1 - \delta^{(kj)})(s_k(\tau\rho s_j^*\alpha_{jk}), \alpha_{kk} + \beta_k)_{\overline{\omega}_{k^*,h}}$$
$$+ (1 - \delta^{(kj)})\{(s_j[\tau\rho s_k^*(\alpha_{kk} + \beta_k)], \alpha_{jk})_{\overline{\omega}_{j^*,h}} + (\tilde{s}_j(\tau\rho), \alpha_{jk}^2)_{\overline{\omega}_{j^*,h}} + (s_j(\tau\rho s^*_{\widehat{kj}}\alpha_{\widehat{kj}k}), \alpha_{jk})_{\overline{\omega}_{j^*,h}}\}$$
$$\geq \left(\tau\rho, [s_k^*(\alpha_{kk} + \beta_k) + (1 - \delta^{(kj)})s_j^*\alpha_{jk}]^2\right)_{\overline{\omega}_h} = \left(\frac{\rho}{\tau}, (\widehat{w}_k^{(0)})^2\right)_{\overline{\omega}_h},$$

where the sum over j in the penultimate expression is taken inside square brackets. Finally, Theorem 2 from [31] remains valid when the term

$$\delta^{(jj)}\left\|\sqrt{\frac{s_j\rho}{s_j\tau}}\,\hat{w}_j\right\|^2_{\overline{\omega}_{j^*,h}} \quad \text{is replaced by} \quad \left\|\sqrt{\frac{\rho}{\tau}}\,|\hat{\mathbf{w}}^{(0)}|\right\|^2_{\overline{\omega}_h}, \tag{3.42}$$

where now

$$\hat{w}_k^{(0)} = \tau s_k^*[(s_k u_k)\delta_k u_k + \delta_k(P_0'(\rho) - \Phi)] + (1 - \delta^{(kj)})\tau s_j^*[(s_j u_j)\delta_j u_k] \quad \text{on} \quad \overline{\omega}_h.$$

For the second of the spatial discretizations proposed in [31], we can use the formulas

$$\Pi^{\tau}_{kl} = (s_k u_k)(s_{kp}\rho)\widehat{w}_l^{(k)} \quad \text{on} \quad \overline{\omega}_{k^*,h}, \quad k \neq l,$$

$$\widehat{w}_k = \frac{1}{s_{kp}\rho}\left\{[\tilde{s}_k(\tau\rho)]\left[(s_k u_k)\delta_k u_k + \frac{1}{s_{kp}\rho}\delta_k p(\rho) - \delta_k\Phi\right] + (1 - \delta^{(kj)})s_k[\tau\rho s_j^*[(s_j u_j)\delta_j u_k]]\right\} \quad \text{on} \quad \overline{\omega}_{k^*,h},$$

$$\widehat{w}_k^{(l)} = \frac{1}{s_{lp}\rho}\left\{s_l\left\{\tau\rho s_k^*\left[(s_k u_k)\delta_k u_k + \frac{1}{s_{kp}\rho}\delta_k p(\rho) - \delta_k\Phi\right]\right\} + [\tilde{s}_l(\tau\rho)](s_l u_l)\delta_l u_k\right.$$
$$\left. + s_l\{\tau\rho s^*_{\widehat{kl}}[(s_{\widehat{kl}} u_{\widehat{kl}})\delta_{\widehat{kl}} u_k]\}\right\} \quad \text{on} \quad \overline{\omega}_{l^*,h}, \quad k \neq l,$$

where $s_{lp}\rho$ is a special average of ρ. Similar arguments lead to the validity of Theorem 3 from [31] with substitution like (3.42), where $s_j\rho$ is replaced by $s_{jp}\rho$.

4. ENTROPY CONSERVATIVE DISCRETIZATION OF THE QUASI-HYDRODYNAMIC SYSTEM OF EQUATIONS

As before, the QHD system of equations (see [4, 5]) consists of mass, momentum, and total energy balance equations of form (1.1)–(1.3), but with a number of substantial simplifications. Specifically, the regularizing terms $\tau\mathrm{div}(\rho\mathbf{u})$ are omitted from the expressions for $\mathbf{w}$ and ρ_* (see (1.5) and (1.6)) and only the first terms are retained in the expressions for Π^{τ}_{kk} and $\mathbf{q}$ (see (1.9) and (1.11)), so now

$$\mathbf{w} = \hat{\mathbf{w}}, \quad \rho_* = \rho, \quad \Pi^{\tau}_{kl} = \rho u_k \hat{w}_l \quad \text{for all} \quad k, l, \quad -\mathbf{q} = \varkappa\nabla\theta,$$

while $\hat{w}_k$ and Π^{NS} are given as before by formulas (1.5) and (1.8). The QHD system can be regarded as another regularization of the Navier–Stokes equations for a viscous compressible heat-conducting gas. Some of its basic mathematical properties were investigated in [33].

For the QHD system, the total entropy balance equation is much simpler than (1.15):

$$\partial_t \int_\Omega \rho S dx = \int_\Omega \left(\Xi^{NS} + \frac{\rho}{\tau\theta}|\hat{\mathbf{w}}|^2 dx + \frac{Q}{\theta}\right) dx \geq 0 \quad \text{for} \quad t \geq 0. \tag{4.1}$$

To obtain a spatial discretization of the QHD system, as before, we use Eqs. (2.6)–(2.8), together with formulas (2.10) for E, p, and ε, (2.11) for J_k and $E^{(k)}$, and (3.1) for $\rho^{(k)}$ and $\varepsilon^{\langle k\rangle}$. However, now we set $\mathbf{w} = \hat{\mathbf{w}}$ and

$$\rho_*^{(k)} = s_k\rho, \quad \Pi^{\tau}_{kk} = (s_k u_k)(s_k\rho)\hat{w}_k, \quad q_k = -(s_k\varkappa)\delta_k\theta \quad \text{on} \quad \overline{\omega}_{k*,h}.$$

Formulas (3.7)–(3.9) and (3.11) for Π, $\tilde{\Pi}^{NS}$, and $\tilde{\Pi}^{\tau}_{kl}$; (3.13) and (3.14) for $\hat{w}_k$; and (3.17) for R_{hk} remain valid.

Theorem 3. *For the indicated spatial discretization of the QHD equations with boundary conditions* (2.9), *the derivative of the total entropy satisfies the lower bound*

$$\partial_t(\rho, S)_{\overline{\omega}_h} \geq (\tilde{\Pi}^{NS}_{ij}, \delta_i u_j)_{i*} + \left(s_i\varkappa, \frac{(\delta_i\theta)^2}{\theta_{i-}\theta_{i+}}\right)_{\overline{\omega}_{i*,h}} + \left(\frac{\rho}{\tau\theta}, |\hat{\mathbf{w}}^{(0)}|^2\right)_{\overline{\omega}_h} + \left(\frac{Q}{\theta}, 1\right)_{\overline{\omega}_h} \geq 0. \tag{4.2}$$

Proof. Indeed, neither the formulations nor the proofs of Lemmas 2–5 change in the case of the QHD equations. In Lemma 6, we now have $A^{\tau,\theta} = A^{\tau,Q} = 0$, and its proof simplifies considerably, since, additionally, $A^{\tau} = A^{\tau,\mathbf{u}} = 0$. Therefore, the right-hand side of bound (3.30) in Lemma 7 contains only the first term plus $\left(\frac{Q}{\theta}, 1\right)_{\overline{\omega}_h}$. As a result, its proof is reduced to formula (3.31) with $q_i^{\tau} = 0$. The theorem is proved.

Inequality (4.2) is a discrete analogue of the total entropy balance equation (4.1).

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