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Studies on the Zeros of Bessel Functions and Methods for Their Computation: 3. Some New Works on Monotonicity, Convexity, and Other Properties

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Abstract—This paper continues the study of real zeros of Bessel functions begun in the previous parts of this work (see M. K. Kerimov, Comput. Math. Math. Phys. **54** (9), 1337–1388 (2014); **56** (7), 1175–1208 (2016)). Some new results regarding the monotonicity, convexity, concavity, and other properties of zeros are described. Additionally, the zeros of q-Bessel functions are investigated.

Keywords: Bessel functions, q-Bessel functions, real zeros, monotonicity, convexity, and concavity of zeros, overview.

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INTRODUCTION

In the first and second parts of this series (see M. K. Kerimov, Comput. Math. Math. Phys. **54** (9), 1337–1388 (2014); **56** (7), 1175–1208 (2016)), we analyzed a number of works concerning the monotonicity, convexity, and concavity of Bessel function zeros with respect to an index. In the present paper, other works on this subject are analyzed with the same degree of detail. In doing this, we will repeatedly refer to the previous parts of the work.

1. MONOTONICITY AND CONVEXITY PROPERTIES OF REAL POSITIVE ZEROS OF THE FIRST KIND BESSEL FUNCTIONS $J_\nu(x)$

We begin with Lewis and Muldoon's paper [37], which was motivated by the following physical problem.

In [48], Putterman, Kac, and Uhlenbeck proposed a purely quantum mechanical explanation of quantized vortex lines arising in rotating superfluid helium. This phenomenon is based on Blatt and Butler's results [3] (see also Lewis and Pule [38]), which showed that an ideal Bose gas rotating in a cylindrical bucket undergoes phase transitions. The total angular moment Ω increases as a linear function of the bucket's angular velocity ω between consecutive values ω_1, ω_2, ... at which there are jumps of size $N_0\hbar$, where N_0 is the number of condensed particles:

$$\Omega = \frac{1}{2}(N - N_0)mR^2\omega + N_0\hbar n, \quad \omega_n < \omega < \omega_{n+1}.$$

Here, N is the total number of particles, m is the mass of a particle, and R is the radius of the bucket. Assuming that the particle's wave function satisfies Dirichlet boundary conditions on the bucket walls, the critical velocity is given by the formula

$$\omega_n = \left(\frac{1}{2}\hbar m R^2\right)\left(j_{n,1}^2 - j_{n-1,1}^2\right),$$

where $j_{n,k}$ is the kth positive zero of the Bessel function $J_\nu(x)$.

From a physical point of view, it can be expected that $\omega_1, \omega_2 \ldots,$ is an increasing sequence; moreover, it has been hypothesized that

$$j_{n+2,1}^2 - 2j_{n+1,1}^2 + j_{n,1}^2 > 0, \quad n = 0, 1, \ldots. \tag{1}$$

The results of [37] were motivated by this physical problem.

The main result of the work is the proof that the derivative

$$dj_{\nu,k}^2/d\nu \quad \text{increases with} \quad \nu \quad \text{when} \quad 3 \le \nu < \infty. \tag{2}$$

Hypothesis (1) follows from this result if we use tabulated values of $j_{\nu,1}$ for $\nu = 1, 2, 3, 4$ from Watson's book [54, pp. 748–750]. To prove (2), the authors first show that $j_{\nu,1}/\nu$ decreases with increasing ν, $0 < \nu < \infty$, while $j_{\nu 1}^2/\nu$ increases with ν for $3 \le \nu < \infty$. It is also shown that all the results proved in the work are valid not only for $j_{\nu,1}$, but also for the zeros $j_{\nu,k}$, $k = 2, 3, \ldots$.

Let us describe the mathematical problem considered in the paper.

For $\nu \ge 0$, we consider the positive zeros $j_{\nu,k}$ and $j'_{\nu,k}$ of the functions $J_\nu(x)$ and $J'_\nu(x)$.

Lewis and Muldoon [37] examine the monotonicity and convexity of zeros of $J_\nu(x)$ with respect to the index ν. It is proved that $j_{\nu,k}/\nu$ is a decreasing function of ν in the interval $0 < \nu < \infty$ and that $j_{\nu,k}^2/\nu$ and $d^2 j_{\nu,k}^2/\nu$ are increasing functions of ν for sufficiently large ν. Specifically, it is proved that $j_{\nu,1}^2/\nu$ and $\dfrac{d}{d\nu} j_{\nu,1}^2$ increase for $3 \le \nu < \infty$. Some results of this type are also proved for zeros of the derivative $J'_\nu(x)$.

To prove their results, the authors consider (for $\nu > 0$) the boundary eigenvalue problem

$$-\frac{d}{dx}\left(\frac{xdy}{dx}\right) + \nu^2 x^{-1} y = \lambda x y, \quad \nu > 0, \tag{1.1}$$

$$y(0) = y(1) = 0, \tag{1.2}$$

and examine its eigenvalues $j_{\nu,k}^2$, $k = 1, 2, \ldots$, and eigenfunctions $J_\nu(j_{\nu,k}x)$. The normalized eigenfunctions have the form

$$\psi_\nu(x) = 2^{1/2}\left[J_{\nu+1}(j_{\nu,k})\right]^{-1} J_\nu(j_{\nu,k}x). \tag{1.3}$$

Additionally, they use the following well-known fact, which was proved by the Sturm method in Bôcher's work [4]: for fixed k, the function $j_{\nu,k}$ increases with ν for $0 < \nu < \infty$. Watson (see [54, pp. 507–508]) proved this result by applying Schläfli's formula

$$\frac{dj_{\nu,k}}{d\nu} = 2\nu\left[j_{\nu,k} J_{\nu+1}^2(j_{\nu,k})\right]^{-1} \int_0^{j_{\nu,k}} t^{-1} J_{\nu+1}^2(t)\,dt \tag{1.4}$$

and Watson's formula (see Watson [54, p. 508, formula (3)]), i.e., the integrodifferential equation

$$\frac{dj_{\nu,k}}{d\nu} = 2j_{\nu,k}\int_0^\infty K_0(2j_{\nu,k}\sinh t)e^{-2\nu t}\,dt. \tag{1.5}$$

First, some general results are presented regarding boundary value problems for ordinary differential equations with a parameter λ of the form

$$-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \nu^2 q(x) y = \lambda \varphi(x) y, \tag{1.6}$$

$$\lim_{x \to a+} p(x) y(x) y'(x) = p(b) y(b) y'(b). \tag{1.7}$$

It is assumed that $-\infty \le a < b < \infty$; $p(x) > 0$; and $p'(t)$, $q(x)$, and $\varphi(x)$ are continuous for $a < x \le b$.

Lemma 1.1. *Suppose that, for every* $\nu > 0$, *the boundary value problem* (1.6), (1.7) *has a discrete set of real eigenvalues. Let* λ_ν *be an eigenvalue of fixed rank and* $\psi_\nu(x)$ *be the corresponding eigenfunction satisfying the normalization condition*

$$\int_a^b \varphi(x)[\psi_\nu(x)]^2\,dx = 1. \tag{1.8}$$

Let, for every $\nu > 0$,

$$\lim_{\mu\to\nu}\int_a^b q(x)\,\psi_\mu(x)\,\psi_\nu(x)\,dx = \int_a^b q(x)[\psi_\nu(x)]^2\,dx \tag{1.9}$$

and

$$\lim_{\mu\to\nu}\int_a^b \varphi(x)\,\psi_\mu(x)\,\psi_\nu(x)\,dx = \int_a^b \varphi(x)[\psi_\nu(x)]^2\,dx = 1. \tag{1.10}$$

Additionally, suppose that, for any $\nu > 0$, *there exists a limit*

$$\lim_{x\to a+} p(x)\,\psi_\nu'(x) \tag{1.11}$$

such that $\psi_\nu(a+) = 0$ *and* $\psi_\nu(b) = 0$ *or* $\psi_\nu'(b) = 0$.

Then

$$\frac{d\lambda_\nu}{d\nu} = 2\nu\int_a^b q(x)[\psi_\nu(x)]^2\,dx, \tag{1.12}$$

$$\frac{d}{d\nu}\frac{\lambda_\nu}{\nu} = \int_a^b \left[2q(x) - \nu^{-2}\lambda_\nu\varphi(x)\right][\psi_\nu(x)]^2\,dx, \tag{1.13}$$

and

$$\frac{d}{d\nu}\frac{\lambda_\nu}{\nu} = -2\nu^{-3}\int_a^b p(x)[\psi_\nu'(x)]^2 dx. \tag{1.14}$$

Proof. To prove (1.12), we multiply the equations

$$-(p\psi_\mu')' + \mu^2 q\psi_\nu = \lambda_\mu\varphi\psi_\nu,$$

$$-(p\psi_\nu')' + \nu^2 q\psi_\nu = \lambda_\nu\varphi\psi_\nu$$

by ψ_0 and ψ_μ, respectively, subtract the second from the first, and integrate between the limits a and b to obtain

$$\left(\lambda_\nu - \lambda_\mu\right)\int_a^b \varphi(x)\,\psi_\mu(x)\,\psi_\nu(x)\,dx = \left(\nu^2 - \mu^2\right)\int_a^b \varphi(x)\,\psi_\mu(x)\,\psi_\nu(x)\,dx.$$

Dividing this result by $\nu - \mu$, sending μ to ν, and using formulas (1.8), (1.9), and (1.10), we derive (1.12).

Moreover,

$$\frac{d}{d\nu}\frac{\lambda_\nu}{\nu} = \nu^{-1}\frac{d\lambda_\nu}{d\nu} - \nu^{-2}\lambda_\nu = 2\int_a^b q(x)[\psi_\nu(x)]^2\,dx - \nu^{-2}\lambda_\nu\int_a^b \varphi(x)[\psi_\nu(x)]^2\,dx,$$

whence (1.13) follows.

We also have

$$\frac{d}{d\nu}\frac{\lambda_\nu}{\nu^2} = \nu^{-2}\frac{d\lambda_\nu}{d\nu} - 2\nu^{-3}\lambda_\nu.$$

Combining (1.12) and the relation

$$\lambda_\nu = -\int_a^b [p\psi'_\nu]'\psi_\nu dx + \nu^2\int_a^b q(x)[\psi_\nu(x)]^2\,dx,$$

which follows from (1.6) and (1.7), we obtain

$$\frac{d}{d\nu}\frac{\lambda_\nu}{\nu^2} = 2\nu^{-3}\int_a^b \frac{d}{dx}\{p(x)\psi'_\nu(x)\}\psi_\nu(x)\,dx.$$

Integrating by parts and using the boundary conditions yields (1.14). The lemma is proved.

Note that results of type (1.12) are well-known in perturbation theory. In the given work, this result is obtained by a simpler, direct method.

Let us return to problem (1.1)–(1.3). Since $J_\nu(x) = O[x^\nu]$, $x \to 0+$, the assumptions of Lemma 1.1 are satisfied. Therefore, (1.12) implies (1.4), i.e., a result proved in a different fashion in Watson's book [57, pp. 507–508].

Below is the main result of [37].

Theorem 1.1. *The following assertions hold*:

(i) *For every fixed k, the function* $j_{\nu,k}/\nu$ *decreases in* ν, $0 < \nu < \infty$.

(ii) *For every fixed k, the function* $j^2_{\nu,k}/\nu$ *increases as* ν *grows to sufficiently large values; specifically,* $j^2_{\nu,1}/\nu$ *increases with* ν *on the interval* $3 \le \nu \le \infty$.

(iii) *For every fixed k, the function* $\frac{d}{d\nu}j^2_{\nu,k}$ *increases as* ν *grows to sufficiently large values*; *specifically,* $\frac{d}{d\nu}j^2_{\nu,1}$ *increases on the interval* $3 \le \nu < \infty$.

Proof. Item (i) is a consequence of (1.14) in Lemma 1.1. It follows from (1.13) that

$$\frac{d}{d\nu}\frac{j^2_{\nu,k}}{\nu} = \int_0^1 \left[2 - j^2_{\nu,k}\nu^{-2}x^2\right]\psi^2_\nu(x)\,x^{-1}dx,$$

where $\psi_\nu(x)$ is given by (1.3). Here, the integrand is positive if $j^2_{\nu,k}/\nu^2 < 2$. This holds for sufficiently large ν, since $j_{\nu,k}/\nu \to 1$ as $\nu \to \infty$ (see Olver [46, p. 408]). Thus, the first part of (ii) is proved. However, in this work, an alternative proof is given, which yields a sharper result in (ii).

Using formula (1.5), we obtain

$$\frac{d}{d\nu}\frac{j^2_{\nu,k}}{\nu} = 4\nu^{-1}j^2_{\nu,k}\int_0^\infty K_0(2j_{\nu,k}\sinh)\,e^{-2\nu t}dt - \nu^{-2}j^2_{\nu,k}.$$

This expression is positive if $I(\nu, k) > 1$, where

$$I(\nu,k) = 4\nu\int_0^\infty K_0(2j_{\nu,k}\sinh t)\,e^{-2\nu t}dt. \tag{1.15}$$

We have

$$I(\nu,k) = 2\int_0^\infty K_0(2j_{\nu,k}\sinh(u/(2\nu)))\,e^{-u}du.$$

Part (i) of the theorem implies that $j_{\nu,k}/\nu$ is a decreasing function of ν, $0 < \nu < \infty$; therefore, for every $u > 0$, this also holds for $2\nu\sinh\{u/(2\nu)\}$ and for the product $2j_{\nu,k}\sinh\{u/(2u)\}$. Since $K_0(\cdot)$ is a decreasing function of its argument and a decreasing function of a decreasing function is an increasing one, $I(\nu, k)$ increases with ν, $0 < \nu < \infty$.

Furthermore,

$$\lim_{\nu\to\infty} I(\nu,k) = 2\int_0^\infty K_0(u)e^{-u}du = 2$$

(see Watson [54, p. 388]). Therefore, for every k, there exists a unique function $\nu(k)$ such that $I(\nu(k), k) = 1$ and $I(\nu, k) > 1$ for $\nu > \nu(k)$.

Thus, we again see that $j_{\nu,k}^2/\nu$ increases with ν for sufficiently large ν.

Now we estimate $\nu(1)$. Using the formula

$$-4/\pi\int_0^\infty K_0(z\sinh t)e^{-2\nu t}dt = J_\nu(z)\frac{\partial Y_\nu(z)}{\partial\nu} - Y_\nu(z)\frac{\partial J_\nu(z)}{\partial\nu}$$

(see Watson [54, p. 444, formula (2)]), the formula

$$\left[\frac{\partial J_\nu(z)}{\partial\nu}\right]_{\nu=n} = \frac{\pi}{2}Y_n(z) + \frac{1}{2}n!\left(\frac{z}{2}\right)^{-n}\sum_{s=0}^{n-1}\left(\frac{z}{2}\right)^s [s!(n-s)]^{-1}J_s(z)$$

(see Olver [47, p. 244]), and tabulated values of $J_\nu(z)$ and $Y_\nu(z)$ (see Watson [54, pp. 666–733] and Abramowitz and Stegun [1, Chapter 9]), we obtain $I(2, 1) < 1$ and $I(3, 1) > 1$. Therefore, $2 < \nu(1) < 3$, $I(\nu, 1) > 1$ for $3 \le \nu < \infty$, and $j_{\nu,1}^2/\nu$ increases for these ν. This completes the proof of (ii).

To prove (iii), it follows from (1.5) that

$$\frac{d}{d\nu}j_{\nu,k}^2 = 4j_{\nu,k}^2\int_0^\infty K_0(2j_{\nu,k}\sinh t)e^{-2\nu t}dt = \frac{j_{\nu,k}^2}{\nu}I(\nu,k),$$

where $I(\nu, k)$ is defined by (1.15). Since $I(\nu, k)$ increases with ν, $0 < \nu < \infty$, we see that $\frac{d}{d\nu}j_{\nu,k}^2$ increases for those ν at which $j_{\nu,k}^2/\nu$ increases, and part (iii) follows from (ii).

Remark 1.1. Computations show that (iii) holds for all $\nu > 0$ at least in the case $k = 1$. Item (ii) does not hold for small positive ν, since $j_{\nu,k}^2/\nu \to \infty$ as $\nu \to 0+$.

Remark 1.2. The authors hypothesize, but fail to prove, that $dj_{\nu,k}/d\nu$ decreases in ν, $0 < \nu < \infty$. The weak result that $d(\log j_{\nu,k})/d\nu$ decreases for $0 < \nu < \infty$ follows from (1.5). Indeed, for every fixed k, we have $d^2(\log c_{\nu,k})/d\nu^2 < 0$ for $0 < \nu < \infty$, where $c_{\nu,k}$ is a zero of fixed rank of any solution $AJ_\nu(x) + BY_\nu(x)$ to the Bessel equation. In implicit form, this result is presented in [41, p. 389]), but it is proved there only for the first positive zero $y_{\nu,1}$ of $Y_\nu(x)$.

Remark 1.3. Lorch and Szego [42] investigated the monotonicity, with respect to the order, of differences (and higher order differences) of consecutive Bessel function zeros. For fixed ν, high order monotonicity with respect to the rank k was also investigated earlier by Lorch, Muldoon, and Szego [41]. The following results were proved in these works.

Let $c_{\nu,m}$ and $\gamma_{\nu,k}$ be the mth and kth positive zeros, respectively, of any pair of (distinct or not) real zeros of the Bessel function $C_\nu(x)$ of order ν located so that $c_{\nu,m} > \gamma_{\nu,k}$, where m and k (corresponding ranks of the zeros) are fixed positive integers. Either zero increases with ν (see Watson [54, p. 508]).

In particular, we can set $m = k + 1$, where l takes any positive integer values. When $l = 1$, we obtain usual finite differences given by

$$\Delta c_{\nu,m} = c_{\nu,k+1} - c_{\nu,k}, \quad \Delta^{n+1}c_{\nu,k} = \Delta\left(\Delta^n c_{\nu,k}\right), \quad n = 1,2,\ldots.$$

For all n, varying k, and fixed ν, where $\nu > 1/2$, these differences were investigated by Lorch and Szego [42, 43]. Specifically, it was shown that

$$(-1)^{n-1}\Delta^n c_{\nu,k} > 0, \quad n,k = 1,2,\ldots, \quad \nu > \frac{1}{2}.$$

The following theorems were proved in the work under study.

Theorem 1.2. *For* $\nu > 1/2$, *the expression* $(-1)^{n-1}\Delta^n_{\nu,k}$ (*which is positive*) *increases with* ν *for any pair of fixed positive integers* k, $n = 1, 2, \ldots$.

Theorem 1.3. *For every fixed pair of positive integers m and n, the difference* $c_{\nu,m} - \gamma_{\nu,k}$ *increases with* ν *for* $\nu \geq 0$ *and*

$$D_\nu(c_{\nu,m} - \gamma_{\nu,k}) < 0 \quad \textit{for} \quad \nu \leq -1/2,$$

where D_ν *denotes the differentiation operator with respect to* ν.

Corollary 1.1. For every fixed pair of positive integers k and l, the difference $c_{\nu,k+1} - c_{\nu,k}$ increases with ν for $\nu \geq 0$ and

$$D_\nu(c_{\nu,k+1} - c_{\nu,k}) < 0 \quad \text{for} \quad \nu \leq -1/2.$$

Corollary 1.2. If $c_{\nu,k}$ and $\gamma_{\nu,k}$ are the kth positive zeros of linearly independent Bessel functions of order ν and if $c_{\nu,1} > \gamma_{\nu,1}$, then the difference $c_{\nu,k} - \gamma_{\nu,k}$ increases with ν for $\nu \geq 0$ and

$$D_\nu(c_{\nu,k} - \gamma_{\nu,k}) < 0 \quad \text{for} \quad \nu \leq -1/2$$

for any fixed $k = 1, 2, \ldots$; specifically, the difference $j_{\nu,k} - y_{\nu,k}$ increases with ν for $\nu \geq 0$ and

$$D_\nu(j_{\nu,k} - y_{\nu,k}) < 0 \quad \textit{for} \quad \nu \leq -1/2$$

for every fixed $k = 1, 2, \ldots$.

Corollary 1.1 follows from Theorem 1.2. To derive Corollary 1.2 to this theorem, it suffices to note that the zeros $c_{\nu,k}$ and $\gamma_{\nu,k}$ interlace (see Watson [54, p. 481]) and $j_{\nu,1} > y_{\nu,1}$ (see Watson [54, p. 487, formula (10)]).

Remark 1.4. The fact that $j_{\nu,k}$ increases with ν for $\nu > 0$ was proved by Bôcher [4] with the use of the Sturm theorem. Watson's result based on formula (1.5) is important in the sense that it proves an increase in $j_{\nu,k}$ for $\nu > -1$ and, in fact, for all real ν when a suitably rank of the zero is chosen. Note also that, for $\nu > -1$, this also follows from the fractional integral

$$x^{(\nu+\varepsilon)/2}J_{\nu+\varepsilon}\left(x^{1/2}\right) = \left[2^\varepsilon\Gamma(\varepsilon)\right]^{-1}\int_0^\infty (x-t)^{\varepsilon-1}t^{\nu/2}J_\nu\left(t^{1/2}\right)dt, \quad \nu > -1, \quad \varepsilon > 0,$$

which is similar to the first Sonin integral (see Watson [54, p. 373]).

To study the zeros of the derivative of a Bessel function, we again consider Eq. (1.1) with boundary conditions $y(0) = y'(1) = 0$. Then the following assertion holds.

Theorem 1.4. *Let* $j'_{\nu,k}$ *be the kth positive zero of* $J'_\nu(x)$. *Then the formula*

$$\frac{dj'_{\nu,k}}{dx} = 2\nu\left[j'_{\nu,k}\left\{(j'_{\nu,k}/\nu)^2 - 1\right\}J^2_{\nu+1}(j'_{\nu,k})\right]^{-1}\int_0^{j_{\nu,k}} x^{-1}J^2_\nu(x)\,dx \tag{1.16}$$

holds, the function $j'_{\nu,k}/\nu$ *decreases to* 1 *with increasing* ν, $0 < \nu < \infty$, *and* $(j'_{\nu,k})^2/\nu$ *increases for at least those* ν *for which* $j'_{\nu,k} < 2^{1/2\nu}$.

Formula (1.16) is not present in Watson's book; possibly, it is new. A consequence of this formula is that $j'_{\nu,k}$ increases with ν. This result was proved in a different way in Watson's book [55, p. 510]. The proof of Theorem 1.4 is similar to that of Theorem 1.1, but it is not presented in the work. The results obtained in the work are extended to the zeros of the skew product of Bessel functions:

$$J_\nu(ax)Y_\nu(x) - Y_\nu(ax)J_\nu(x),$$

and also to the zeros of the modified Bessel function of the second kind

$$K_{i\nu}(x) = \int_0^\infty e^{-x\cosh t}\cos(\nu t)\,dt.$$

These results will be described elsewhere.

2. ON THE CONVEXITY OF SQUARED ZEROS OF THE BESSEL FUNCTION $J_\nu(x)$

Elbert and Laforgia [9] proved that the squared zero of $J_\nu(x)$ (i.e., $j^2_{\nu,\chi}$ in χ-notation) is a convex function of ν for $\nu \geq 0$ and $\chi \geq \chi_0$, where χ_0 is defined as

$$\begin{gathered}\chi_0 = \inf\left\{\chi,\ \chi > 0,\ j'_{\nu,\chi} > 1 \quad \text{for all} \quad \nu \geq 0\right\},\\ 0 \leq \chi_0 < \chi\left(\frac{1}{4}\right), \quad 0 < \chi_0 < 1, \quad \frac{d}{d\nu} j_{\nu,\chi} = j'_{\nu,\chi}.\end{gathered} \tag{2.1}$$

First, we prove the following result.

Lemma 2.1. *If* $0 \leq \nu < \infty$ *and* $j_{\nu,\chi} > \nu + 1/4$, *then*

$$j' = j'_{\nu,\chi} = \frac{d}{d\nu} j_{\nu,\chi} > 1. \tag{2.2}$$

Proof. Consider the domain $D = \{(\nu, j);\ 0 \leq \nu < \infty, j \geq \nu + 1/4\}$ and Watson's integrodifferential equation

$$\frac{d}{d\nu} j = 2j\int_0^\infty K_0(2j\sinh t)e^{-2\nu t}dt, \quad \chi > 0, \quad j = j_{\nu,\chi} \tag{2.3}$$

with the boundary condition

$$\lim_{\nu\to-\chi+0} j_{\nu,\chi} = 0. \tag{2.4}$$

Let us prove that

$$I = I(\nu, j) = 2j\int_0^\infty K_0(2j\sinh t)e^{-2\nu t}dt > 1, \quad (\nu, j) \in D. \tag{2.5}$$

Making the substitution $u = 2j\sinh t$ in (2.5) yields

$$I = \int_0^\infty K_0(u)\frac{e^{-2\nu\,\mathrm{arccosh}(u/2j)}}{\sqrt{1 + u^2/4j^2}}du. \tag{2.6}$$

On the other hand, it is well known (see Watson [50, p. 388]) that

$$\int_0^\infty K_0(u)e^{-u}du = 1.$$

Therefore, to prove (2.5), it suffices to show that

$$\frac{e^{-2\nu\,\mathrm{arcsinh}(u/2j)}}{\sqrt{1 + u^2/4j^2}} > e^{-u}$$

for $u > 0$ and $(\nu, j) \in D$. This inequality is equivalent to

$$\nu\frac{\log\left(x + \sqrt{1 + x^2}\right)}{x} + \frac{1}{4}\frac{\log\left(1 + x^2\right)}{x} < j,$$

where $x = u/2j$. Since

$$\frac{\log\left(x+\sqrt{1+x^2}\right)}{x} < 1, \qquad \frac{\log\left(1+x^2\right)}{x} < 1$$

for $x > 0$, the proof of Lemma 2.1 is complete.

Computations show that a larger domain with $j' > 1$ can be obtained. For this purpose, we choose an initial point $(0, j_0) \in D$. Then the solution $j = j(\nu; 0, j_0)$ of problem (2.3), (2.4) stays in D for all $\nu > 0$, and this solution can be continued on the left to the value $\nu = -\chi(j_0)$, where

$$\lim_{\nu\to-\chi(j_0)+0} j(\nu; 0, j_0) = 0.$$

Therefore, $j(\nu; 0, j_0) = j_{\nu,\chi}(j_0)$.

Since the solution of Eq. (2.3) is unique, we conclude that $\chi = \chi(j_0)$ is an increasing function of j_0. By Lemma 2.1, $(\nu, j_{\nu,\chi(1/4)}) \in D$ for all $\nu \geq 0$. Therefore, $j'_{\nu,\chi} > 1$ for all $\chi \geq \chi(1/4)$. Since $j_{0,1} = 2.40... > 1/4$, we have $0 < \chi(1/4) < 1$. By the definition of χ_0, it is true that $0 \leq \chi_0 < \chi(1/4)$. In view of (2.6), we have

$$\lim_{j\to+0} I(0, j) = 0.$$

Therefore, formula (2.3) yields $j'(0, j_0) < 1$ if j_0 is sufficiently small. Hence, $\chi_0 > 0$.

Remark 2.1. A consequence of the definition (2.1) of χ_0 is

$$j_{\nu,\chi} > j_{0,\chi} + \nu, \quad \nu > 0, \quad \chi \geq \chi_0. \tag{2.7}$$

This inequality generalizes a similar one proved by Laforgia and Muldoon only for $k = 1, 2, \ldots$ (see [35, formula (2.4)]).

Remark 2.2. Concerning the role played by χ_0, we have the limiting relation

$$\lim_{\nu\to\infty} \frac{j_{\nu,\chi}}{\nu} = 1 \quad \text{for} \quad \chi \geq \chi_0. \tag{2.8}$$

Indeed, in view of (2.7) and the inequality $j_{\nu,\chi'} < j_{\nu,\chi''}$ for $\nu > -\chi'$ and $0 < \chi' < \chi''$, the function $j_{\nu,\chi}$ satisfies

$$j_{0,\chi} + \nu < j_{\nu,\chi} < j_{\nu,[\chi]+1},$$

where $[\chi]$ denotes the largest integer smaller than or equal to x.

By applying Tricomi's asymptotic formula (see Tricomi [52])

$$j_{\nu,k} = \nu + a_k \nu^{1/3} + O\left(\nu^{-1/3}\right), \quad \nu \to \infty, \quad \kappa = 1, 2, \ldots,$$

where a_k are constants independent of ν, we obtain formula (2.8). On the other hand, in view of (2.5),

$$I(\nu, \nu) = 2\nu \int_0^\infty K_0(2\nu \sinh t) e^{-2\nu t} dt.$$

Recalling that $K_0(u)$ is a decreasing function of u, we obtain

$$I(\nu, \nu) < 2\nu \int_0^\infty K_0(2\nu t) e^{-2\nu t} dt = \int_0^\infty K_0(u) e^{-u} du = 1.$$

Therefore, the solution $j(\nu; \nu_0, \nu_0)$ cannot intersect the line $j = \nu$; moreover, $j(\nu; \nu_0, \nu_0) < \nu$ for $\nu > \nu_0$ and $j(\nu; \nu_0, \nu_0) = j_{\nu,\chi}$ with some $\tilde{\chi} = \tilde{\chi}(\nu_0) < \chi_0$.

Lewis and Muldoon [37, Theorem 3.1] proved that $j^2_{\nu,1}$ is a convex function for $\nu \geq 3$. Here, a more general is proved.

Theorem 2.1. *The function* $j^2_{\nu,\chi}$ *is convex in* ν *for* $\nu \geq 0$ *and any* $\chi \geq \chi_0$.

Proof. To prove the theorem, it is sufficient to show that

$$\left(\frac{j^2}{2}\right)'' = j'^2 + jj'' > 0. \tag{2.9}$$

Differentiating Eq. (2.3) yields

$$j'' = 2j'\int_0^\infty K_0(2j\sinh t)e^{-2\nu t}dt + 2j\int_0^\infty K_0'(2j\sinh t)2j'\sinh t e^{-2\nu t}dt - 4j\int_0^\infty K_0(2j\sinh t)e^{-2\nu t}dt. \quad (2.10)$$

In view of (2.3), three terms on the right-hand side of (2.10) can be written as

$$j'' = \frac{j'^2}{j} + I_1 - I_2. \quad (2.11)$$

By making the substitution $u = 2j\sinh t$, the integral I_1 is brought to the form

$$I_1 = 2j'\int_0^\infty K_0'(u)\phi\left(\frac{u}{2j}\right)du,$$

where

$$\phi(x) = \frac{xe^{-2\nu \operatorname{arcsinh} x}}{\sqrt{1+x^2}}.$$

Integration by parts gives

$$I_1 = 2j'\left[K_0(u)\phi\left(\frac{u}{2j}\right)\right]_0^\infty - \frac{j'}{j}\int_0^\infty K_0(u)\phi'\left(\frac{u}{2j}\right)du, \quad (2.12)$$

where

$$\phi'(x) = \frac{1-2\nu x\sqrt{1+x^2}}{\left(1+x^2\right)^{3/2}}e^{-2\nu \operatorname{arcsinh} x}.$$

It follows that

$$\phi'(x) < \frac{e^{-2\nu \operatorname{arcsinh} x}}{\left(1+x^2\right)^{3/2}} < 1, \quad x > 0, \quad \nu \geq 0.$$

Recalling the asymptotic formula for function $K_0(x)$,

$$K_0(x) = \begin{cases} O(\log(1/x)), & x > 0, \quad x \sim 0, \\ O\left(e^{-x}\right), & x \gg 1, \end{cases} \quad (*)$$

we conclude that the first term on the right-hand side of (2.12) is zero. Then

$$I_1 = -\frac{j'}{j}\int_0^\infty K_0(u)\phi'\left(\frac{u}{2j}\right)du. \quad (2.13)$$

Similarly, for I_2, we obtain

$$I_2 = 2\int_0^\infty K_0(u)\frac{\operatorname{arcsinh}(u/2j)}{\sqrt{1+u^2/uj^2}}e^{-2\nu \operatorname{arcsinh}(u/2j)}du = \frac{1}{j}\int_0^\infty K_0(u)u\psi\left(\frac{u}{2j}\right)du, \quad (2.14)$$

where

$$\psi(x) = \frac{1}{\sqrt{1+x^2}}\frac{\operatorname{arcsinh} x}{x}e^{-2\nu \operatorname{arcsinh} x}.$$

Since

$$\psi(x) < \psi(0) = 1, \quad x > 0,$$

we have

$$I_2 < \frac{1}{j}\int_0^{\infty} K_0(u)\,u\,du = \frac{1}{j} \tag{2.15}$$

(see Watson [54, p. 388]).

Combining (2.11) with (2.14), (2.15), and the inequality $j' > 1$, we obtain

$$\left(\frac{j^2}{2}\right)'' > j'^2 - j'\int_0^{\infty} K_0(u)\,u\,\psi'\left(\frac{u}{2j}\right)du.$$

Therefore, it suffices to show that

$$I_3 = j' - \int_0^{\infty} K_0(u)\,\psi'\left(\frac{u}{2j}\right)du > 0.$$

In view of (2.6) and (2.12),

$$I_3 = \int_0^{\infty} K_0(u)\frac{e^{-2\nu\operatorname{arcsinh}(u/2j)}}{\sqrt{1+u^2/uj^2}}\left[1 + \frac{1 - 2\nu(u/2j)\sqrt{1+u^2/4j^2}}{1+u^2/4j^2}\right]du,$$

where the expression in square brackets is positive for $u > 0$. Thus, the theorem is completely proved, so the function $j_{\nu,\chi}^2$ is convex for $\chi \ge \chi_0$ and $\nu \ge 0$.

Corollary 2.1. In the special case $\chi \equiv k = 1, 2, \ldots$, the function $j_{\nu,\chi}^2$ is convex for $\nu \ge 0$.

Several remarks have to be made regarding what was said above. Since $j_{\nu,\chi}^2$ is convex, the graph of $j_{\nu,\chi}^2$ lies below the chord joining the points $(0,\, j_{\nu,\chi}^2)$ and $(\nu^*, j_{\nu^*,\chi}^2)$.It follows that

$$\frac{j_{\nu,\chi}^2 - j_{0,\chi}^2}{\nu} < \frac{j_{\nu^*,\chi}^2 - j_{0,\chi}^2}{\nu^*}, \quad 0 < \nu < \nu^*,$$

i.e., $\left(j_{\nu,\chi}^2 - j_{0,\chi}^2\right)/\nu$ increases with ν for $\nu > 0$. Consider the chord joining the points $(0,\, j_{0,\chi}^2)$ and $\left(\frac{1}{2},\, j_{1/2,\chi}^2\right)$ in the graph of $j_{\nu,\chi}^2$ plotted as a function of ν. The convexity of the graph implies that

$$j_{\nu,\chi}^2 < j_{0,\chi}^2 + 2\nu\left[\chi^2\pi^2 - j_{0,\chi}^2\right], \quad 0 < \nu < 1/2,$$

where the inequality holds as an equality only for $\nu = 0$ and $\nu = 1/2$. Similarly, the convexity of $j_{\nu,\chi}^2$ implies that

$$j_{\nu,\chi}^2 > j_{0,\chi}^2 + 2j_{0,\chi}\nu\left[\frac{dj_{\nu,\chi}}{d\nu}\right]_{\nu=0}, \quad \nu > 0, \quad \chi > \chi_0,$$

and, since $j'_{\nu,\chi} > 1$ for $\chi \ge \chi_0$, we have

$$j_{\nu,\chi}^2 > j_{0,\chi}^2 + 2j_{0,\chi}\nu, \quad \nu > 0.$$

The convexity property can be used to obtain some other inequalities.

The question arises as to whether the convexity property holds in the entire domain of $j_{\nu,\chi}$, i.e., on the interval $(-\chi, \infty)$. Consider the case $\chi \equiv k = 1, 2, \ldots$. Then the equation for determining the zeros $j_{\nu,k}$ of $J_\nu(x)$ yields the equation (see Watson [55, p. 15])

$$0 = (\nu + k)\,\Gamma(\nu+1)\left(\frac{j_{\nu,k}}{2}\right)^{-\nu} J_\nu(j_{\nu,k}).$$

$$0 = (\nu + k)\Gamma(\nu + 1)\left(\frac{j'_{\nu,k}}{2}\right)^{-\nu} J_\nu(j'_{\nu,k}).$$

In view of the series for $J_\nu(j_{\nu,k})$, this equation can be written in expanded form as

$$0 = (\nu + k)\Gamma(\nu + 1)\left(\frac{j_{\nu,k}}{2}\right)^{-\nu} J_\nu(j_{\nu,k})$$

$$= (\nu + k)\left[1 - \frac{(j_{\nu,k}/2)^2}{1!(\nu + 1)} + \ldots + (-1)^{k-1}\frac{(j_{\nu,k}/2)^{2(k-1)}}{(k-1)!(\nu+1)\ldots(\nu+k-1)}\right]$$

$$+ (-1)^k \frac{(j_{\nu,k}/2)^{2k}}{k!(\nu+1)\ldots(\nu+k-1)}\left[1 - \frac{(j_{\nu,k}/2)^2}{(k+1)(\nu+k+1)} + \ldots\right].$$

In a right neighborhood of $\nu = -k$, we have

$$\frac{(j_{\nu,k}/2)^{2k}}{\nu + k} = k!(k-1)(1-\varepsilon)\left(1 - \frac{\varepsilon}{2}\right)\ldots\left(1 - \frac{\varepsilon}{k-1}\right)k\,\frac{1 + \dfrac{(j_{\nu,k}/2)^2}{k-1-\varepsilon} + \ldots\dfrac{(j_{\nu,k}/2)^{2(k-1)}}{(k-1)!(k-1-\varepsilon)\ldots(1-\varepsilon)}}{1 - \dfrac{(j_{\nu,k}/2)^2}{(k+1)(1+\varepsilon)} + \ldots},$$

where $\varepsilon = \nu + k$.

Setting $\varepsilon \to 0$ and $j_{\nu,k} \to 0$ yields

$$\lim_{\nu\to -k+0}\frac{(j_{\nu,k}/2)^{2k}}{\nu + k} = k!(k-1)!.$$

This relation can be written as

$$\left(\frac{j_{\nu,k}}{2}\right)^2 = [k!(k-1)!(\nu + k)]^{1/k}[1 + o(1)], \quad \nu \to -\infty.$$

For $k = 2, 3, \ldots$, we obtain more accurate approximations

$$\frac{(j_{\nu,k}/2)^{2k}}{\nu + k} = k!(k-1)!\left\{1 + \frac{2k}{k^2 - 1}[k!(k-1)!\varepsilon]^{1/k}[1 + o(1)]\right\},$$

where $\varepsilon = \nu + k$. Therefore,

$$\left(\frac{j_{\nu,k}}{2}\right)^2 = [k!(k-1)!(\nu+k)]^{1/k} + \frac{2}{k^2-1}[k!(k-1)!(\nu+k)]^{2/k}[1 + o(1)], \quad \nu \to -k, \quad k = 2, 3, \ldots.$$

For $k = 1$, we have

$$\left(\frac{j_{\nu,1}}{2}\right)^2 = \nu + 1 + \frac{1}{2}(\nu+1)^2[1 + o(1)], \quad \nu \to -1.$$

These approximations show that $j^2_{\nu,k}$ cannot be convex on the entire interval $(-k, \infty)$ for $k = 2, 3, \ldots$. The authors do not know whether $j^2_{\nu,1}$ is convex on $(-1, 0)$, but they believe it is.

In [18] Elbert and Siafarikas prove that $j^2_{\nu,1}$ is a convex function of ν on the interval $-2 < \nu < 0$. Earlier, this result was proved by Elbert and Laforgia [19] for $\nu > 0$ only. The monotonicity of the functions

$$\frac{j^2_{\nu,1}}{4(\nu+1)}, \quad \frac{j^2_{\nu,1}}{4(\nu+1)\sqrt{\nu+2}}$$

is also investigated. The research method is based on expanding $J_\nu(z)$ in a power series and is especially effective on the interval $-2 < \nu < -1$. Let us recall some well-known facts.

Consider a power series for $J_\nu(z)$:

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\left(\frac{z}{2}\right)^{2n+\nu}}{\Gamma(n+\nu+1)}, \quad z > 0. \tag{2.16}$$

It is well known (see Watson [54, p. 483], Hurwitz [22], Kerimov [33]) that $J_\nu(z)$ has infinitely many positive zeros $j_{\nu,k}$, $k = 1, 2, \ldots$:

$$0 < j_{\nu,1} < j_{\nu,2} < \ldots,$$

which tend to infinity as $\nu \to \infty$.

The first zero $j_{\nu,1}$ can be analytically continued to $\nu = -1$, where it vanishes. Continuing $j_{\nu,1}$ analytically to the interval $(-2, -1)$, we find that, by the Hurwitz theorem [22] (see also Watson [54, p. 483], Kerimov [33]), the zero $j_{\nu,1}$ becomes purely imaginary. At the point $\nu = -2$, the function $j_{\nu,1}$ again vanishes.

Concerning the behavior of $j_{\nu,1}$, it is well known (see Piessens [47]) that $j_{\nu,1}$ can be represented by the power series

$$j_{\nu,1} = 2(\nu+1)^{1/2}\left[1 + \frac{\nu+1}{4} - \frac{7}{96}(\nu+1)^2 + \ldots\right] \tag{2.17}$$

about the point $\nu = -1$.

Let is analyze the behavior of $j_{\nu,1}^2$ for $\nu > -2$, where it is real. Consider the function

$$l(\nu) = j_{\nu,1}^2/4(\nu+1), \tag{2.18}$$

which can be locally represented as

$$l(\nu) = 1 + \frac{\nu+1}{2} - \frac{1}{12}(\nu+1)^2 + \ldots. \tag{2.19}$$

It follows from (2.19) that

$$\lim_{\nu\to-1} l(\nu) = 1, \quad \lim_{\nu\to-2} l(\nu) = 0, \quad l(1) = \frac{j_{\nu,1}^2}{8} = 1.83525\ldots. \tag{2.20}$$

Recalling the well-known inequalities

$$j_{\nu,1}^2 < 2(\nu+1)(\nu+3), \quad \nu > -1,$$

$$j_{\nu,1}^2 > 2(\nu+1)(\nu+3), \quad -2 < \nu < -1,$$

(see Ismail and Muldoon [29, formulas (5.11), (5.12)]), we obtain

$$l(\nu) < 1 + \frac{1}{2}(\nu+1) = \frac{\nu+3}{2} \quad \text{for} \quad \nu > -2, \quad \nu \neq -1.$$

On the basis of computations, Ismail and Muldoon [29] constructed the graph of $j_{\nu,1}^2$ on the interval $(-2, 0)$, which is given in Fig. 1 (see also Kokologiannaki, Muldoon, Siafarikas [34]).

This graph shows that $j_{\nu,1}^2$ is a convex function of ν on $(-2, 0)$. For $3 < \nu < +\infty$, this property was proved earlier by Lewis and Muldoon [37].

In [9] Elbert and Laforgia proved that $j_{\nu,k}^2$ ($k = 1, 2, \ldots, \nu \geq 0$) is a convex function of ν on the interval $(-1, 0)$. Additionally, it was proved that $j_{\nu,1}^2$ cannot be convex on the entire interval $(-k, \infty)$ for $k = 2, 3, \ldots$ and it was conjectured that $j_{\nu,1}^2$ is convex for $-1 < \nu < 0$.

In [35] Kokologiannaki, Muldoon, and Siafarikas proved that $j_{\nu,1}^2$ decreases to a minimum and then increases as ν grows from -2 to -1. Additionally, the convexity of $j_{\nu,1}^2$ was proved on the interval $(-2, 0)$. Therefore, in view of Elbert and Laforgia's work [8], $j_{\nu,1}^2$ is convex on $(-2, \infty)$, since $dj_{\nu,1}/d\nu$ is a continuous function of ν (see Watson [54, Section 15.6]).

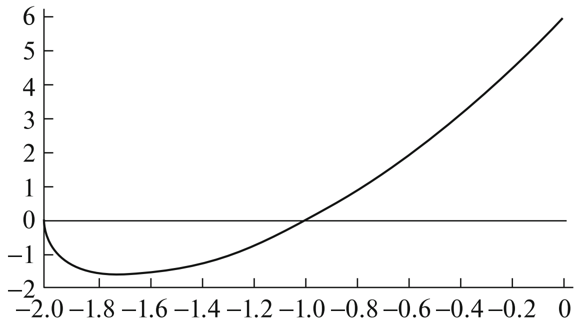

Fig. 1. Function $j_{\nu,1}^2$ for $-2 \le \nu \le 0$.

The following remarks were made about $l(\nu)$ in Ismail and Muldoon's work [29, p. 9]:

(i) The function $l(\nu)$ increases for $\nu > -2$ (for $\nu > -1$, this was previously known from [27, Theorem 2]).

(ii) The function $\dfrac{l(\nu)}{\sqrt{\nu+2}}$ decreases on the interval $(-2, -1)$ and increases for $\nu > -1$.

These remarks turn out to be valid and are rigorously proved in [35]. The proofs are based on the following implicit relation between $l = l(\nu)$ and ν:

$$H(l,\nu) = 1 - \frac{l}{1!} + \frac{l^2}{2!}\frac{(\nu+1)}{\nu+2} - \frac{l^3}{3!}\frac{(\nu+1)^2}{(\nu+2)(\nu+3)} + \ldots + \frac{(-1)^k l^k}{k!}\frac{(\nu+1)^{k-1}}{(\nu+2)\ldots(\nu+k+1)} + \ldots = 0, \tag{2.21}$$

which is derived from series (2.16) for the Bessel function $J_\nu(z)$. Assuming that

$$e_0(\nu) = 1, \quad e_k(\nu) = \frac{(\nu+1)^k}{(\nu+2)\ldots(\nu+k+1)}, \quad k = 1, 2, \ldots, \tag{2.22}$$

it follows from (2.21) that

$$H(l,\nu) = 1 - \frac{l}{1!} + \frac{l^2}{2!}e_1(\nu) - \frac{l^3}{3!}e_2(\nu) + \ldots \frac{(-1)^k l^k}{k!}e_{k-1}(\nu) + \ldots = 0. \tag{2.23}$$

Lemma 2.2. *The partial derivative* $H_l \equiv \partial H(l,\nu)/\partial t$ *is negative for* $-2 < \nu \le 1$ *and* $0 < l < 2$.

Lemma 2.3. *The partial derivative* $H_l \equiv \partial H(l,\nu)/\partial \nu$ *is positive for* $-2 < \nu < 1$ *and* $0 < l < 2$.

The proofs of these lemmas are omitted. They imply the following result.

Theorem 2.2. *The function* $l(\nu)$ *defined by formula* (2.1) *increases for* $-2 < \nu \le 1$.

The following result holds for the derivative $l'(\nu)$.

Lemma 2.4. *The function* $l'(\nu)$ *satisfies the inequalities*

(i) $l'(\nu) < \dfrac{l(\nu)}{2(\nu+2)}$ *for* $-2 < \nu < -1$;

(ii) $l'(\nu) > \dfrac{l(\nu)}{2(\nu+2)}$ *for* $-1 < \nu \le 1$.

Using this lemma and the inequalities

$$\frac{1}{j_{\nu,1}}\frac{dj_{\nu,1}}{d\nu} > \frac{1}{j_{\nu,1}^2}\left[1 + \left(1 + j_{\nu,1}^2\right)^{1/2}\right], \quad \nu > -1$$

(see Ifantis and Siafarikas [25]) and

$$j_{\nu,1}^2 < \frac{2(\nu+1)(\nu+5)(5\nu+11)}{7\nu+19}, \quad \nu > -1$$

(see Ismail and Muldoon [27]), the authors prove the following result.

Theorem 2.3. *The function* $\frac{j_{\nu,1}^2}{4(\nu+1)\sqrt{\nu+2}}$ *decreases from* $\sqrt{2}$ *to* 1 *for* $-2 < \nu < -1$ *and increases for* $\nu > -1$.

Applying Theorem 2.3, the authors derive the inequalities

$$4\sqrt{2}(\nu+1)\sqrt{\nu+2} < j_{\nu,1}^2 < 4(\nu+1)\sqrt{\nu+2} \quad ,-2 < \nu < -1.$$

The upper bound was previously known (see Ismail and Muldoon [29, formula (5.8)]). The lower bound is new, and it is sharp as ν tends to -2.

The main result of the work is as follows.

Theorem 2.4. *The function* $j_{\nu,1}^2$ *is convex for* $-2 < \nu \le 0$.

All these lemmas and theorems are proved in [35]; here, the proofs are omitted. This study was motivated by applications of the convexity of $j_{\nu,1}^2$ related to a hypothesis put forward in the quantum-mechanical work by Putterman, Kac, and Uhlenbeck [48], which states that the sequence of differences $j_{n,1}^2 - j_{n-1,1}^2$ increases with n, where $j_{n,1}$ is the first positive zero of $J_n(x)$, $n = 1, 2, \ldots$.

3. ON THE CONVEXITY OF ZEROS OF THE BESSEL FUNCTIONS $C_\nu(x)$

Now we analyze an important work by Elbert, Gatteschi, and Laforgia [7], where the concavity properties of the zeros $j_{\nu,\chi}$ of $C_\nu(x)$ are proved for various ν and χ, where $c_{\nu,k} \equiv j_{\nu,\chi}$ for $k-1 < \chi < k$ and $\alpha = (k-\chi)\pi$.

As we noted above, if $\chi > 0$, then $j_{\nu,\chi}$ is defined as the solution of the Watson integrodifferential equation

$$\frac{d}{d\nu} j = 2j\int_0^\infty K_o(2j\sinh t)e^{-2\nu t}dt, \quad j = j_{\nu,\chi}, \tag{2.3'}$$

with the boundary condition

$$\lim_{\nu\to-\chi+0} j_{\nu,\chi} = 0. \tag{2.4'}$$

The main result of [7] is the proof of the inequality

$$\frac{d^2}{d\nu^2} j_{\nu,\chi} < 0 \tag{3.1}$$

for $\nu \ge \nu_0$, provided that

$$j_{\nu_0,\chi} > \nu_0 + 1/2, \quad \text{where} \quad \nu_0 \ge -1/2.$$

This result means that $j_{\nu,\chi}$ is concave and supplements the result on the concavity of $j_{\nu,k}$ in ν proved by Elbert [6] for $-k < \nu < \infty$, $k = 1, 2, \ldots$. The proof makes use of χ-notation for zeros and the well-known formulas

$$\int_0^\infty K_o(u)e^{-au}du = \frac{\arccos a}{\sqrt{1-a^2}}, \quad |a| < 1, \tag{3.2}$$

$$\int_0^\infty K_o(u)e^{-u}du = 1 \tag{3.3}$$

(see Watson [54, p. 388]).

Substituting $2j\sinh t = u$ into Watson's formula (2.3') for $j = j_{\nu,\chi}$ yields

$$j' = \int_0^\infty K_o(u)\frac{e^{-2\nu\,\mathrm{arcsinh}(u/2j)}}{\sqrt{1+u^2/4j^2}}du, \tag{3.4}$$

where, for notational brevity, we set

$$j' = (d/d\nu)\, j, \quad j = j_{\nu,\chi}. \tag{3.5}$$

The following formula for the second ν-derivative of a zero j is well known (see Elbert [6, formula (16)]):

$$j'' = 2j\int_0^\infty K_o(2j\sinh t)\, e^{-2\nu t} I(t)\, dt, \tag{3.6}$$

where

$$I(t) = \frac{2\nu j'}{j}\operatorname{th} t + \frac{j'}{j}\operatorname{th}^2 t - 2t. \tag{3.7}$$

Recall the lemma from Elbert and Laforgia's work [9] stating that, if $0 \le \nu < \infty$ and $j_{\nu,\chi} \ge \nu + 1/4$, then

$$j' = \frac{d}{d\nu} j_{\nu,\chi} > 1. \tag{3.8}$$

The authors prove that, for inequality (3.8) to hold, the condition $j_{\nu,\chi} \ge \nu + 1/4$ can be replaced with the less restrictive one $j'_{0,x} > 1.$

Next, the functions $f(t, \nu)$ and $\phi = \phi(\nu)$ for $t > 0$ and $-\infty < \nu < \infty$ are defined as

$$f(t,\nu) = \frac{\nu \tanh t}{t} + \frac{1}{2}\frac{\tanh^2 t}{t}, \quad t > 0, \tag{3.9}$$

and

$$\phi(\nu) = \sup\{f(t,\nu),\, t > 0\}. \tag{3.10}$$

The function $f(t, \nu)$ is related to $I(t)$ from (3.7) by the formula $(2j'/j)tf = I + 2t$. By the definition of $f(t, \nu)$,

$$\lim_{t\to+0} f(t,\nu) = \nu, \quad \lim_{t\to\infty} f(t,\nu) = 0. \tag{3.11}$$

Differentiating f with respect to t gives

$$f'(t,\nu) = \frac{d}{dt} f(t,\nu) = \frac{t - \sinh t\cosh t}{t^2\cosh^2 t} + \frac{2t\, \tanh t - \sinh^2 t}{t^2\cosh^2 t},$$

whence $\lim_{t\to+0} f' = 1/2$.

To study the properties of f, it is convenient to introduce a function $g(t)$ by the formula

$$g(t) = \frac{2t\,\tanh t - \sinh^2 t}{2(\sinh t\,\cosh t - t)}, \quad t > 0. \tag{3.12}$$

This function decreases with increasing t, since

$$g'(t) = -t\frac{g_1(t)}{\cosh^2 t(\sinh t\cosh t - t)^2},$$

where $g_1(t) = t + \sinh^3 t\cosh t - \sinh t\cosh t$ is a positive function. This follows from the fact that $g_1(t) = g_1'(0)$ and $g_t''(t) = 2\sinh t\cosh t(\cosh^2 t + 7\sinh^2 t) > 0$ for $t > 0$. Moreover,

$$\lim_{t\to+0} g(t) = +\infty, \quad \lim_{t\to+\infty} g(t) = -1/2.$$

Lemma 3.1. *Let* $f(t, \nu)$ *be defined by formula* (3.9). *Then the following assertions are valid*:

(i) $f < 0$ *for* $\nu \le -1/2$;

(ii) f *has only one maximum for fixed* ν $(\nu > -1/2)$ *at* $t = t(\nu)$, *where* $t(\nu)$ *is the inverse of the function* $\nu = g(t)$ *defined by* (3.12);

(iii) $\max f(t, \nu) = f(t(\nu), \nu) > 0$ *for* $\nu > -1/2$ *and* $t > 0$.

Proof. For $\nu \le -1/2$, it follows from (3.9) that

$$f(t,\nu) \le -\frac{1}{2}\frac{\tanh t}{t} + \frac{\tanh^2 t}{2t} < 0, \quad t > 0,$$

which proves (i).

The function f' can be written in the form

$$f' = \frac{\sinh t \cosh t - t}{t^2 \cosh^2 t}(g(t) - \nu),$$

where $g(t)$ is defined by (3.12). Since $g(t)$ is a decreasing function, $f' > 0$ on the interval $(0, t(\nu))$ and $f' < 0$ for $t > t(\nu)$. Therefore, f has a minimum at $t = t(\nu)$. This proves (ii).

Assertion (iii) follows from the second relation in (3.11).

Now we analyze the behavior of $\phi(\nu)$.

Theorem 3.1. *Let* $\phi(\nu)$ *be defined by formula* (3.10). *Then the following assertions hold*:

(i) $\phi(\nu) = 0$ *for* $\nu \le -1/2$;

(ii) $\phi(\nu) = f(t(\nu), \nu)$ *for* $\nu > -1/2$;

(iii) $\phi(\nu)$ *strictly increases for* $\nu \ge -1/2$;

(iv) $\phi(\nu)$ *is strictly convex for* $\nu \ge -1/2$;

(v) $\phi(\nu) = \nu + \dfrac{3}{16\nu} + O(\nu^{-3}),\ \nu \to \infty$;

(vi) $\theta(\nu) = \phi(\nu) - \nu$ *is a strictly decreasing positive function for* $\nu \ge -1/2$.

Proof. For $\nu \le -1/2$, Lemma 3.1(i) implies that $f(t, \nu) < 0$ and, in view of (3.11), we obtain

$$\phi(\nu) = \sup\{f(t,\nu),\, t > 0\} = 0,$$

whence assertion (i) follows.

Part (ii) of the theorem follows directly from definition (3.10) of $\phi(\nu)$ and property (iii) in Lemma 3.1. Concerning part (iii), for $-1/2 < \nu_1 < \nu_2$, it follows from (3.9) that

$$f(t,\nu_1) < f(t,\nu_2), \quad t > 0.$$

Applying this inequality with $t = t(\nu_1)$ and Theorem 3.1(ii) yields $\phi(\nu_1) < f(t(\nu_1), \nu_2)$.

In view of property (ii) of $f(t, \nu_2)$ in Lemma 3.1, we obtain for $t(\nu_1) > t(\nu_2)$ the inequality

$$f(t(\nu_1),\nu_2) < f(t(\nu_2),\nu_2) = \phi(\nu_2),$$

i.e., $\phi(\nu_1) < \phi(\nu_2)$, as required.

To prove property (iv), it suffices to show that

$$\phi(\alpha\nu_1 + \beta\nu_2) < \alpha\phi(\nu_1) + \beta\phi(\nu_2),$$

where $0 < \alpha, \beta < 1$, $\alpha + \beta = 1$, $\nu_1, \nu_2 > -1/2$, and $\nu_1 \ne \nu_2$.

The case $\nu_1 = \nu_2$ is dropped, since it is of no interest to us.

Using definitions (3.9) and (3.10), applying property (ii), and setting $t^* = t(\alpha\nu_1 + \beta\nu_2)$, we obtain

$$\begin{aligned}\phi(\alpha\nu_1 + \beta\nu_2) &= f(t^*,\alpha\nu_1 + \beta\nu_2) = \alpha f(t^*,\nu_1) + \beta f(t^*,\nu_2)\\ &\le \alpha f(t(\nu_1),\nu_1) + \beta f(t(\nu_2),\nu_2) = \alpha\phi(\nu_1) + \beta\phi(\nu_1).\end{aligned} \tag{3.13}$$

This inequality shows that $\phi(\nu)$ is a convex function for any ν. For $\nu > -1/2$, the convexity is strict since, by Lemma 3.1(ii), (3.13) holds as an equality only at $t^* = t(\nu_1)$ and $t^* = t(\nu_2)$ or, equivalently, at $g(t^*) = \nu_1$ and $g(t^*) = \nu_2$. However, this case $\nu_1 = \nu_2$ was excluded above. The asymptotic formula from (v) is derived using a power series expansion of $\tanh t$ about $t = 0$. Then conditions (3.13) and the definition of $t(\nu)$ imply that $t(\nu)$ tends to zero as $\nu \to \infty$ and we have the approximations

$$\tanh t = t - \frac{t^3}{3} + O(t^5), \quad t \to 0,$$

$$f(t,\nu) = \nu + \frac{1}{2}t - \frac{\nu}{3}t^2 + O(t^3) + \nu O(t^4), \quad t \to 0,$$

and

$$f' = \frac{1}{2} - \frac{2}{3}\nu t + O(t^2) + \nu O(t^3), \quad t \to 0.$$

For

$$[f'(t,\nu)]_{t=t(\nu)} = 0,$$

we obtain

$$t(\nu) = \frac{3}{4\nu} + O\left(\nu^{-3}\right) \to \infty$$

and $\phi(\nu) = f(t(\nu), \nu)$ has the required asymptotic expansion.

Finally, $\theta(\nu) = \phi(\nu) - \nu$ is a convex function, because so is $\phi(\nu)$. On the other hand, in view of the asymptotic formula obtained,

$$\lim_{\nu\to\infty}\theta(\nu) = 0.$$

Thus, $\theta'(\nu)$ increases and $\lim_{\nu\to\infty}\Theta'(\nu) = 0$. Therefore, $\theta'(\nu) < 0$ for $-1/2 < \nu < \infty$ and $\theta(\nu)$ falls off to zero as ν increases. The theorem is completely proved.

Now, we prove that $j_{\nu,\chi}$ is a concave function of ν, where χ is a parameter. The proof is based on the following lemmas.

Lemma 3.2. *The inequality* $j'_{\nu,\chi} > 1$ *holds in the following cases*:

(i) $j = j_{\nu,\chi} > 0$ *for* $\nu \le -1/2$;

(ii) $j \ge 1/2(\nu + 1/2)$ *for* $-1/2 < \nu < 0$;

(iii) $j \ge \nu + 1/4$ *for* $\nu \ge 0$.

The proof is omitted.

Lemma 3.3. *On the interval* $-1/2 < \nu < 0$, *it is true that*

$$j'_{\nu,\chi} < Q(\nu),$$

where

$$Q(\nu) = \frac{\pi}{2}\sqrt{(1+2\nu)^{1+2\nu}(1-2\nu)^{1-2\nu}}.$$

Proof. First, we show that

$$\max_{0\le x<\infty}\frac{e^{-2\nu\,\mathrm{ar}\sinh x}}{\sqrt{1+x^2}} = \frac{2}{\pi}Q(\nu)$$

for $-1/2 < \nu < 0$. Indeed, the function on the left-hand side has a maxima at $|2\nu|/\sqrt{\left(1-4\nu^2\right)}$, which is equal to $2Q(x)/\pi$. Moreover, at $\nu = -1/2$, this function strictly increases and approaches a value of 2. Therefore, in view of (3.4),

$$j' = \int_0^\infty K_o(u)\frac{e^{-2\nu\,\mathrm{arcsinh}(u/2j)}}{\sqrt{1+u^2/4j^2}}du < \frac{2}{\pi}Q(\nu)\int_0^\infty K_o(u)\,du.$$

Applying formula (3.2) and setting $a = 0$ yields the required result.

Below is the main result of [7].

Theorem 3.2. *Let, for some* ν *and* χ,

$$\left(j_{\nu,\chi}/j'_{\nu,\chi}\right) \ge \phi(\nu),$$

where $\phi(\nu)$ *is defined by* (3.10). *Then* $j''_{\nu,\chi} < 0$ *for these* ν *and* χ.

Proof. It follows from (3.6) and (3.7) that

$$j' = 2j\int_0^\infty K_o(2j\sinh t)e^{-2\nu t}\left[2\nu\frac{j'}{j}\tanh t + \frac{j'}{j}\tanh t - 2t\right]dt$$
$$= 4j'\int_0^\infty K_o(2j\sinh t)e^{-2\nu t}t\left[\nu\frac{\tanh t}{t} + \frac{\tanh^2 t}{2t} - \frac{j}{j'}\right]dt.$$

In view of (3.9), the function in square brackets is equal to $f(t, \nu) - j/j'$. Therefore, from properties (i) and (ii) of $f(t, \nu)$ and the definition of $\phi(\nu)$, it follows that

$$j'' < 4j'\left[\phi(\nu) - \frac{j}{j'}\right]\int_0^\infty K_o(2j\sinh t)e^{-2\nu t} \le 0,$$

which concludes the proof of the theorem.

The following consequences hold true.

Corollary 3.1. Under the condition $\phi(\nu) = 0$ for $\nu \le -1/2$, the function $j_{\nu,\chi}$ is concave for $\nu \le -1/2$ and any χ.

For this reason, in what follows we consider only the case $\nu > -1/2$.

Corollary 3.2. Let $-1/2 \le \nu \le 0$ and $j_{\nu,\chi} \ge Q(\nu)\phi(\nu)$, where $Q(\nu)$ and $\phi(\nu)$ are defined above. Then, for these ν and χ,

$$j''_{\nu,\chi} < 0.$$

The proof follows from Theorem 3.2 and Lemma 3.2.

Theorem 3.3. *Let* $\nu_0 > -1/2$ *and* $j_{\nu_0,\chi} \ge \nu_0 + 1/2$ *for some* ν_0 *and* χ. *Then*

(i) $j_{\nu,\chi} > j_{\nu_0,\chi} + \nu_0$ *for* $\nu > \nu_0$;

(ii) $j''_{\nu,\chi} < 0$ *for* $\nu \ge \nu_0$.

The proof is rather long, so it is omitted. Note, however, that the proof makes use of tabulated values of some of the functions involved.

Corollary 3.3. For $\nu \ge 1/2$, the function $j_{\nu,\chi}$ is concave on the interval $(-\chi, \infty)$.

Remark 3.1. As a special case, this corollary yields the concavity of the zeros, which was proved only for $\nu \ge 0$ in [37].

Proof of Corollary 3.3. Consider an inequality following from McMahon's expansion of $c_{\nu,\chi}$ (see Watson [54, p. 490], Kerimov [33]), namely,

$$c_{\nu,\chi} > \left(k + \frac{\nu}{2} - \frac{1}{4}\right)\pi - \alpha, \quad 0 \le \alpha < \pi, \quad -\frac{1}{2} < \nu < \frac{1}{2}.$$

In the case $\alpha = \pi/2$ and $k = 1$, which corresponds to the first zero $y_{\nu,1}$ of $Y_\nu(x)$, we have

$$y_{\nu,1} > \left(\frac{\nu}{2} + \frac{1}{4}\right)\pi, \quad -\frac{1}{2} < \nu < \frac{1}{2}.$$

Note that, in χ-notation, we have $y_{\nu,1} = j_{\nu,1/2}$, so

$$j_{\nu,1/2} > \left(\frac{\nu}{2} + \frac{1}{4}\right)\pi, \quad -\frac{1}{2} < \nu < \frac{1}{2},$$

and, for any $\nu_0 \in (-1/2, 1/2)$,

$$j_{\nu_0,1/2} > \left(\nu_0 + \frac{1}{2}\right)\frac{\pi}{2} > \nu_0 + \frac{1}{2} > 0.$$

Then, by Theorem 3.3, the function $j_{\nu,1/2}$ is convex for $\nu \ge \nu_0$. Therefore, $j_{\nu,1/2}$ is concave for $\nu > -1/2$.

The function $j_{\nu,\chi}$ satisfies one of the following inequalities for $\chi > 1/2$:

(i) $j_{\nu,\chi} > 0$ if $\nu \le -1/2$;

(ii) $j_{\nu,\chi} > j_{\nu,1/2}$ if $\nu > -1/2$.

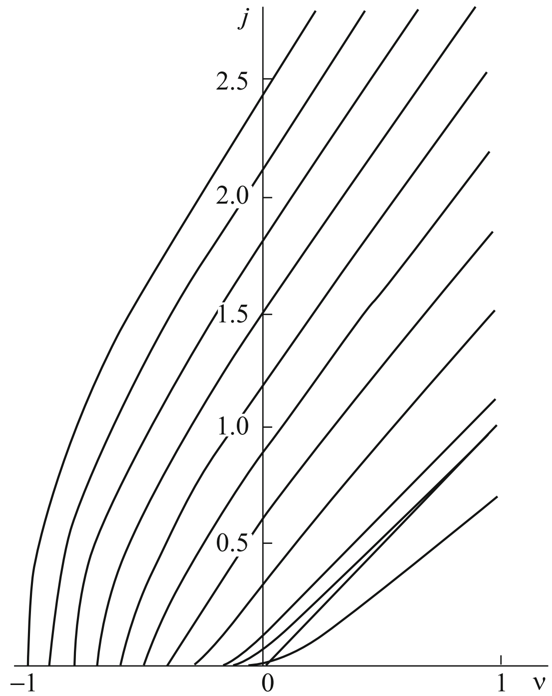

Fig. 2. Plots of $j_{\nu,\chi}$ for $0 < \chi < 1$, $\nu > -\chi$, and $-3 < \nu < 1$ (from Elbert, Gatteschi, and Laforgia, Appl. Anal. **16**, p. 277 (1983)).

In both cases, applying Theorem 3.3 yields the required result, i.e., Corollary 3.3 is proved. Other useful generalizations of the results proved in [36] can be obtained in the case of nonnegative ν.

Corollary 3.4. Let $\chi > \chi_o = 1 - (\alpha_0/\pi) = 0.344...$, where

$$\tan\alpha_0 = \frac{J_0(1/2)}{Y_0(1/2)}, \quad 0 < \alpha_0 < \pi.$$

Then the function $j_{\nu,\chi}$ is concave for $\nu \geq 0$.

Proof. Consider the cylinder function

$$C_\nu(\alpha_0, x) = J_\nu(x)\cos\alpha_0 - Y_\nu(x)\sin\alpha_0.$$

By the definition of the number α_0, the first positive zero $c_{0,1}$ of $C_0(\alpha_0, x)$ is equal to 1/2. Then, in χ-notation, $j_{0,\chi_0} = 1/2$. Applying Theorem 3.3 with $\nu_0 = 0$ and $j_{\nu_0,\chi_0} = 1/2$ yields the concavity of j_{ν,χ_0}. The concavity of j_{ν,χ_0} for $\chi > \chi_o$ and $\nu > 0$ follows from the inequality $j_{0,\chi} > j_{0,\chi_0}$. Corollary 3.4 is proved.

Now we determine the smallest χ_0 for which j_{ν,χ_0} is concave for every $\nu \geq 0$.

Remark 3.2. Corollary 3.4 shows that $c_{\nu,1}$ is concave for $0 \leq \alpha \leq \alpha_0 = (\pi/2) + 0.493$.

Laforgia and Muldoon [37] obtained this result only for $0 \leq \alpha \leq \pi/2$.

To analyze the behavior of the function $j_{\nu,\chi}$ for $0 < \chi < 1$ and $\nu > -\chi$, the authors computed its values for a number of χ such that $0 < \chi < 1$ and $\nu > -\chi$. The results were presented in the form of plots for $-1 < \nu < 1$, $\chi = 0.1$ 1/6, and $j = \nu$, 0.2, 0.5, −1 (see Fig. 2). Inspection of the plots suggests the following conclusions.

(1) The function $j_{\nu,\chi}$ is concave for $\chi \geq 1/2$ and $\nu \geq -\chi$ on the basis of Corollary 3.3.

(2) In the case $1/6 < \chi < 1/2$, the function $j_{\nu,\chi}$ is convex on the interval $[-\chi, \nu(\chi)]$ with some function $\nu = \nu(\chi)$ and is concave on the interval $(\nu(\chi), \infty)$. Concerning $\nu(\chi)$, we can assume that it decreases and satisfies the limiting relations

$$\lim_{\chi \to \frac{1}{2}-0} \nu(\chi) = -\frac{1}{2}, \quad \lim_{\chi \to \frac{1}{6}+0} \nu(\chi) = \infty. \tag{3.14}$$

(3) In the case $0 < \chi \leq 1/6$, the function $j_{\nu,\chi}$ is convex for all $\nu > -\chi$.

Cases (1) and (3) have not been proved; they are hypothesized and considered true. Their validity is confirmed by numerical experiments. It is stated in [7] that $j_{\nu,1/6} > \nu$, and this inequality is implicitly contained in Spigler's work [51, p. 79].

Concerning case (3), Elbert and Laforgia [16] proved that the function $c_{\nu,1}$ is convex only in the domain

$$\frac{1}{2} \le \nu < \infty \quad \text{if} \quad \pi - \frac{1}{2} + \varepsilon_0 < \alpha < \pi,$$

where $\varepsilon_0 = 0.163302$.

The convexity properties of the zeros $c_{\nu,\chi}$ of $C_\nu(x)$ were also studied by Elbert and Laforgia [16]. In [7] it was proved that $j_{\nu,\chi}$ is a concave function of ν for $\chi \ge 0.344...$ and $j_{\nu,\chi}^2$ is convex for $\chi \ge 0.7070...$. In [16], some new convexity properties of $j_{\nu,\chi}$ were investigated. First, the following result was proved.

Lemma 3.4. *For* $\chi \ge 0.7070...$, *it is true that*

$$1 < j' < \frac{j}{\nu+1}, \quad \nu \ge 0,$$

where $j = j_{\nu,\chi}$ *and* $j' = \dfrac{d}{d\nu} j_{\nu,\chi}$.

Proof. The lower bound from this lemma was obtained by Elbert, Gatteschi, and Laforgia [7]. To derive the upper bound, we note that it is equivalent to the inequality

$$h(\nu) = (\nu+1) - j < 0.$$

Furthermore,

$$h'(\nu) = (\nu+1)\, j''.$$

It is well known (see [7, p. 276] that $j_{\nu,\chi}$ is concave for $\chi \ge 0.344...$. Therefore, $h(\nu)$ decreases with decreasing $\nu \ge 0$ when $\chi \ge 0.344...$. To complete the proof of the upper bound, it remains to be shown only that $h(0) < 0$ or, equivalently,

$$j'_{0,\chi}/j_{0,\chi} < 1.$$

In view of Watson's formula (see [54, p. 508]) or (1.5), we have

$$j'_{0,\chi}/j_{0,\chi} = 2\int_0^\infty K_0\left(2 j_{0,\chi} \sinh t\right) dt.$$

Since $K_0(x)$ strictly decreases, while $j_{\nu,\chi}$ increases for $x > 0$, we conclude that $j'_{0,\chi}/j_{0,\chi}$ decreases with decreasing χ. However, by the definition of $\tilde{\chi} = 0.7070...$ (see Elbert and Laforgia [11]), $j'_{0,\chi}/j_{0,\chi} = 1$. Therefore, $j'_{0,\chi}/j_{0,\chi} < 1$ for $\chi > 0.344...$, which proves Lemma 3.4.

Now we recall the following lower bound for j'' (see [9, p. 74]):

$$j'' > \frac{1}{\nu + j}\left[\frac{\nu}{j}(j')^2 - j'\right], \quad \nu \ge 0, \quad \chi \ge 0.7070\ldots \tag{3.15}$$

Below is the main result of [16].

Theorem 3.4. *Let a function $f(\nu)$ be defined by the formula*

$$f(\nu) = \begin{cases} \log\dfrac{2}{\nu+1} - \dfrac{1}{4}, & 0 \le \nu \le 1, \\ \dfrac{1}{2}(\nu \log \nu - \nu), & \nu \ge 1. \end{cases}$$

Then $j_{\nu,\chi} + f(x)$ is a convex function of $\nu \ge 0$ for $\chi \ge \chi_0 = 0.7070...$. The function $f(\nu)$ can be replaced by $f(\nu) + a\nu + b$, where a and b are arbitrary constants.

Proof. We need to prove that

$$j'' + f''(\nu) > 0.$$

Using inequality (3.15) in the case $0 \le \nu \le 1$, we have to prove that

$$\frac{1}{\nu + j}\left[\frac{\nu}{j}(j')^2 - j'\right] + \frac{1}{(\nu+1)^2} \ge 0, \quad 0 \le \nu \le 1. \tag{3.16}$$

Consider a quadratic polynomial in j':

$$g(j') = \frac{\nu}{j}(j')^2 - j',$$

where $1 < j' < j/(\nu + 1)$, according to the lemma conditions. On this interval, the function $g(j')$ has a maximum at $j/(\nu + 1)$. Therefore,

$$g(j') \ge \frac{-j}{(\nu+1)^2}, \quad 0 \le \nu \le 1.$$

In view of (3.16), we need to prove only

$$-\frac{1}{(\nu+1)^2}\frac{1}{\nu + j} + \frac{1}{(\nu+1)^2} \ge 0.$$

This inequality holds, since it is equivalent to $\nu/((\nu + j)(\nu + 1)^2) \ge 0$, which is true. On the second interval $\nu \ge 1$, we again use (3.15). Thus, we need to prove the inequality

$$\frac{1}{\nu + j}\left[\frac{\nu}{j}(j')^2 - j'\right] + \frac{1}{4\nu} \ge 0, \quad \nu \ge 1, \tag{3.17}$$

where the quadratic polynomial $g(j')$ now has its minimum at $j/2\nu$. Substituting this value into (3.17) yields

$$\frac{1}{\nu + j}\left[\frac{\nu}{j}\frac{j^2}{4\nu^2} - \frac{j}{2\nu}\right] + \frac{1}{4\nu} = \frac{1}{4(\nu + j)} > 0,$$

whence the proof of the theorem follows.

Remark 3.3. This result includes, as an important special case, the zeros of $J_\nu(x)$, which correspond to $\chi = k = 1, 2, \ldots$. Moreover, the inequality

$$j'' + f''(\nu) \ge 0$$

is stronger than

$$j'' + 2 \ge 0,$$

which guarantees the convexity of $j + \nu^2$ as shown by Giordano and Laforgia [19]. Therefore, this result is a consequence of the theorem proved above.

As an application of the theorem, we consider the cases $\nu = 0$, $\nu = 1/2$, and $\nu = 1$. For $\nu = 1/2$, it is well known (see Elbert and Laforgia [9]) that $j_{1/2,\chi} = \chi\pi$. Therefore, Theorem 3.4 implies that

$$2\chi\pi + \log\frac{8}{9} < j_{0,\chi} + j_{1,\chi}.$$

On the other hand, since $j_{\nu,\chi}$ is a concave function of ν, we obtain

$$j_{0,\chi} + j_{1,\chi} < 2\chi\pi.$$

For example, for $\chi = 1$, we have

$$6.1654 < 6.2365 < 6.2832,$$

where the numerical values of $j_{0,1}$ and $j_{1,1}$ are taken from the tables given in Watson's book (see [54]).

The monotonicity of the zeros $j_{\nu,\chi}$ of $C_\nu(x)$ for $\nu \ge 0$ was investigated by Elbert and Laforgia [13].

Elbert and Laforgia [8] proved that $j_{\nu,\chi}$ is a strictly increasing function of χ. This property can briefly be written as

$$j_{\nu,\chi'} > j_{\nu,\chi}, \quad \chi' > \chi > 0, \quad \nu > -\chi. \tag{3.18}$$

Moreover, it is well known (see Sturm [53]) that the sequence $\{c_{\nu,k+1} - c_{\nu,k}\}_{k=1}^{\infty}$ is strictly decreasing for $|\nu| > 1/2$ and increasing for $|\nu| < 1/2$. In χ-notation, we have the sequence $\{j_{\nu,\chi+k+1} - j_{\nu,\chi+k}\}_{k=1}^{\infty}$, which is monotone. This property suggests that $j_{\nu,\chi}$ is a concave function of χ for $\nu \geq 1/2$ and convex for $0 \leq \nu \leq 1/2$. Thus, the following result holds.

Theorem 3.5. *The function* $j_{\nu,\chi}$ *is concave in* χ *if* $\nu \geq 1/2$ *and convex in* χ *if* $0 \leq \nu \leq 1/2$.

To prove this theorem, for $\nu \geq 0$ and $\chi > 0$, we first consider the derivative

$$\frac{\partial}{\partial\chi} j_{\nu,\chi} = I_{\nu,\chi} = I. \tag{3.19}$$

In the special case $\nu = 1/2$, we have $j_{1/2,\chi} = \chi\pi$, so

$$I_{1/2,\chi} = \pi.$$

Differentiating Watson's formula for $\frac{d}{d\nu} j_{\nu,\chi}$ with respect to χ yields

$$\frac{d}{d\nu} I = 2I\int_0^\infty K_0(2j\sinh t)e^{-2\nu t}dt + 2j\int_0^\infty K_0'(2j\sinh t)2I\sinh te^{-2\nu t}dt,$$

where $j = j_{\nu,\chi}$. Integrating by parts the second integral on the right-hand side gives

$$2I\int_0^\infty K_0'(2j\sinh t)2j\cosh t\tanh te^{-2\nu t}dt = 2I\left[K_0(2j\sinh t)\tanh te^{-2\nu t}dt\right]_0^\infty$$

$$= -2I\int_0^\infty K_0(2j\sinh t)\left[\frac{1}{\mathrm{c}\sinh^2 t} - 2\nu\tanh t\right]e^{-2\nu t}dt.$$

Recalling a well-known asymptotic formula for $K_0(u)$ (see formula (*) in Section 2), we see that the first term on the right-hand side vanishes. Therefore, in view of (3.19),

$$\frac{d}{d\nu} I = 2I\int_0^\infty K_0(2j\sinh t)\left(\tanh^2 t + 2\nu\tanh t\right)e^{-2\nu t}dt.$$

Combining this relation with (3.19) yields

$$I_{\nu,\chi} = \pi\exp\left\{2\int_{1/2}^{\nu}\left[\int_0^\infty K_0(2j_{\nu,\chi}\sinh t)\left(\tanh^2 t + 2\mu\tanh t\right)e^{-2\mu t}dt\right]d\mu\right\}. \tag{3.20}$$

A consequence of (3.20) is the following result.

Theorem 3.6. *For* $\nu \geq 0$ *and* $\chi' > \chi > 0$,

$$j_{\nu,\chi'} - j_{\nu,\chi}\begin{cases} > \pi(\chi' - k), & \nu > \frac{1}{2}, \\ = \pi(\chi' - \chi), & \nu = \frac{1}{2}, \\ < \pi(\chi' - \chi), & 0 \leq \nu < \frac{1}{2}. \end{cases} \tag{3.21}$$

Proof. It follows from (3.20) that $I_{\nu,\chi} \gtreqless \pi$ if $\nu \gtreqless 1/2$, respectively, and $j_{\nu,\chi'} - j_{\nu,\chi} = \int_\chi^{\chi'} I_{\nu,\lambda}d\lambda$, which implies assertion (3.21) in Theorem 3.6. To prove that $j_{\nu,\chi}$ is a concave function of χ for $\nu \geq 1/2$, we need to show that $I_{\nu,\chi}$ decreases for growing χ. Let $\chi' > \chi > 0$. Then, in view of (3.18), $j_{\nu,\chi'} > j_{\nu,\chi}$ for $\nu \geq 0$. On the other hand, $K_0(u)$ has the integral representation

$$K_0(u) = \int_0^\infty e^{-u}\cosh\xi d\xi$$

(see Watson [54, p. 46]), so $K_0(u)$ is a strictly decreasing function of u and, in view of (3.20), $I_{\nu,\chi} > \tilde{I}_{\nu,\chi'}$. This proves the concavity of $j_{\nu,\chi}$ in χ for $\nu \geq 1/2$.

The convexity of $j_{\nu,\chi}$ in χ in the case $0 \le \nu \le 1/2$ is proved in a similar manner. The proof of Theorem 3.6 is complete.

Theorem 3.7. *For* $\nu \ge 0$ *and* $\nu > 0$, *the function* $\log j_{\nu,\chi}$ *is concave in* ν *for fixed* χ *and concave in* χ *for fixed* ν.

Proof. The first part of the theorem follows from the inequality

$$\frac{j_{\nu+\varepsilon+\delta,\chi}}{j_{\nu+\varepsilon,\chi}} < \frac{j_{\nu+\delta,\chi}}{j_{\nu,\chi}}, \quad \varepsilon,\delta > 0,$$

so we need to prove that

$$H(\nu,\chi) = j_{\nu+\delta,\chi}/j_{\nu,\chi}$$

is a decreasing function of ν. The authors prove that $H(\nu + \varepsilon, \chi) < H(\nu, \chi)$, so the first part of the theorem is proved. To prove the second part of the theorem, it is necessary to show that

$$\frac{j_{\nu,\chi+r+h}}{j_{\nu,\chi+\nu}} < \frac{j_{\nu,\chi+h}}{j_{\nu,\chi}}, \quad r,h > 0,$$

which is equivalent to the fact that

$$L(\nu,\chi) = j_{\nu,\chi+h}/j_{\nu,\chi}$$

is a decreasing function of χ. The authors prove that $L(\nu, \chi)$ is a decreasing function of χ if $\nu > 1/2$.

In the case $0 \le \nu \le 1/2$, the proof is longer and implies that $L(\nu, \chi)$ decreases with increasing χ. The theorem is completely proved.

As a consequence, it is shown that, if ε, δ, h, $r > 0$ are such that $\varepsilon + r > 0$ and $h + \delta > 0$, then the determinant

$$T = \begin{vmatrix} j_{\nu,\chi} & j_{\nu+\delta,\chi+h} \\ j_{\nu+\varepsilon,\chi+r} & j_{\nu+\delta+\varepsilon,\chi+h+r} \end{vmatrix}$$

is negative, i.e., $T < 0$. This result generalizes Lorch's one (see [37, p. 223]), which states that, for ε, $\delta \ge 0$ and h, $r = 0, 1, 2, \ldots$ such that $\varepsilon + r > 0$ and $h + \delta > 0$, the determinant

$$T = \begin{vmatrix} c_{\nu,k} & c_{\nu+\delta,k+h} \\ c_{\nu+\varepsilon,k+\varepsilon} & c_{\nu+\delta+\varepsilon,k+h+r} \end{vmatrix}$$

is negative, i.e., $T < 0$.

The proof of the corollary is omitted.

Elbert, Laforgia, and Lorch [17] study some new monotonicity properties of zeros of the Bessel function $C_\nu(x)$. Writing the zeros $c_{\nu,k}$ in χ-notation, the authors prove the following assertions.

1. If $0 \le \beta \le 0.2202728$, $1/2 \le \gamma \le 1$, and $\chi = \nu - \alpha/\pi \ge 0.3648159$, then

$$[\nu + (2\gamma - 1)\beta/j_{\nu,\chi}]^\gamma \frac{d}{d\nu}[j_{\nu,\chi}]$$

is a increasing function of γ for $\nu > 0$. Moreover, $j_{\nu,\chi}$ is $c_{\nu,k}$ if $\chi = \nu - \alpha/\pi$, $k = 1, 2, \ldots$, so, if $\chi = k = 1, 2, \ldots$, then $j_{\nu,\chi} = j_{\nu,k}$ and $j_{\nu,k-1/2} = y_{\nu,k}$. (Unfortunately, I failed to find this work, so the results were taken from MR1143384(93a:33006), as reviewed by E. Ifantis.)

Elbert and Laforgia [14] proved that, if $j_{\nu,\chi}$ is a positive zero of $C_\nu(x)$, then $\log j_{\nu,\chi}/\nu$ is a convex function of ν for $\nu > 0$ and $\chi \ge \chi_0 = 0.7070\ldots$. This result includes an important special case of zeros $j_{\nu,k}$ of $J_\nu(x)$ corresponding to $\chi \equiv k = 1, 2, \ldots$.

Elbert, Gatteschi, and Laforgia [7] proved that, under certain conditions on χ, the function $j_{\nu,\chi}$ is concave in ν, i.e.,

$$j'' = \frac{d^2}{d\nu^2} j_{\nu,\chi} < 0 \tag{3.22}$$

for $\chi \ge \chi_0 = 0.344\ldots$ and $\nu \ge 0$.

Elbert and Laforgia [11] showed that

$$j''' = \frac{d^3}{d\nu^3} j_{\nu,\chi} > 0, \quad \chi \ge 0.7070\ldots, \quad \nu \ge 0, \tag{3.23}$$

as a consequence of the inequality proved in the following lemma.

Lemma 3.5. *For* $\nu \geq 0$ *and* $\chi \geq \chi_0 = 0.7070...$, *the function* $j = j_{\nu,\chi}$ *satisfies the inequality*

$$(\nu + j)\, j' > \frac{\nu j'^2}{j} - j', \quad j' = \frac{d}{d\nu} j_{\nu,\chi}. \tag{3.24}$$

This lemma is used to prove that $j_{\nu,\chi}/(\nu + 1)$ and $j^2_{\nu,\chi}$ are convex functions of $\nu \geq 0$ for $\chi \geq 1/2$ and $\chi \geq \chi_0 = 0.7070...$, respectively.

Theorem 3.8. *For* $\nu > 0$ *and* $\chi \geq \chi_0 = 0.7070...$, *the function* $\log j_{\nu,\chi}/\nu$ *is convex in* ν.

Proof. We need to prove that

$$\frac{d^2}{d\nu^2}\log\frac{j_{\nu,\chi}}{\nu} = \frac{j''j - j'^2}{j^2} + \frac{1}{\nu^2} > 0.$$

By Lemma 3.5, it suffices to show that

$$\frac{\nu j'^2 - jj'}{j(\nu + j)} > \frac{\nu^2 j'^2 - j^2}{\nu^2 j}$$

or, equivalently,

$$\nu^3 j'^2 - \nu^2 jj' - (\nu + j)\nu^2 j'^2 + (\nu + j)\, j^2 = j\,(j - \nu j')(j + \nu + \nu j') > 0.$$

This inequality holds, since $j' > 0$, $j > 0$, and $(j - \nu j')' = -\nu j'' > 0$ in view of (3.22). Theorem 3.8 is completely proved.

Theorem 3.9. *For* $\nu > 0$ *and* $\chi \geq \chi_0 = 0.7070...$, *the function* $j^2_{\nu,\chi}$ *is convex in* ν.

Proof. We need to show that $(j^2)'' > 0$ or

$$j'' > -j'^2/j.$$

In the case of inequality (3.24), it suffices to show that

$$\frac{\nu j'^2 - jj'}{j(\nu + j)} > \frac{-j'^2}{j}$$

or

$$2\nu j' + (j' - 1)\, j > 0.$$

Since j > 0, it is sufficient to prove that $j' > 1$.

This inequality was proved by Elbert and Laforgia [9] for $\chi \geq \tilde{\chi}$, where $\tilde{\chi} \in (0, 1)$. However, it was shown in [7, p. 276] in implicit form that $j'_{\nu,\chi} > 1$ if $\chi \geq \tilde{\chi} = 0.344...$. Therefore, in the case under consideration ($\chi \geq 0.7070...$), we obtain the required result $(j^2)'' > 0$ for $\chi \geq \max\{\tilde{\chi}, \chi_0\} = \chi_0$. Theorem 3.8 is proved.

Remark 3.4. Theorem 3.8 supplements Elbert and Laforgia's result [9] stating that $j^2_{\nu,\chi}$ is convex if $\chi \geq \tilde{\chi}$, but only for $\tilde{\chi} \in (0, 1)$ without indicating its numerical value. The convexity of $j^2_{\nu,\chi}$ is proved in [9] relying only on the inequality $j'_{\nu,\chi} > 1$. However, it holds only for $\chi \geq \tilde{\chi} = 0.344...$.

Theorem 3.10. *For* $\chi \geq 1$, *the function* $j_{\nu,\chi}/(\nu + 1)$ *is convex in* $\nu \geq 0$.

Proof. We need to prove that

$$f(\nu) = (\nu + 1)^2 f'' - 2(\nu + 1) f' + 2j > 0.$$

Differentiating this function yields

$$f'(\nu) = (\nu + 1)^2 f''',$$

which is positive in view of (3.22). Therefore, it suffices to determine the sign of $f(0)$.

It is stated that $f(0) > 0$.

It follows from (3.24) that $j''_{0,\chi} > -j'_{0,\chi}/j_{0,\chi}$, where $j'_{0,\chi} = (d/d\nu)j_{\nu,\chi}|_{\nu=0}$ and $j''_{0,\chi} = (d^2/d\nu^2)j_{\nu,\chi}|_{\nu=0}$. Therefore, it suffices to show that

$$g(\chi) = -\frac{j'_{0,\chi}}{j_{0,\chi}} + 2j_{0,\chi}\left(1 - \frac{j'_{0,\chi}}{j_{0,\chi}}\right) > 0. \tag{3.25}$$

Relying on Watson's formulas (see Watson [55, p. 508]) at $\nu = 0$, we obtain

$$\frac{j'_{0,\chi}}{j_{0,\chi}} = 2\int_0^\infty K_0(2j_{0,\chi}\sinh t)\,dt.$$

Since $K_0(u)$ is a strictly decreasing function of u, we conclude that $j'_{0,\chi}/j_{0,\chi}$ decreases in χ. By the definition of χ_0 (see Elbert, Gatteschi, and Laforgia [7]), $j'_{0,\chi_0} = j_{0,\chi_0}$. Therefore, $g(\chi)$ in (3.25) is an increasing function of χ. Moreover, $j'_{0,1} = 1.54...$ and $j_{0,1} = 2.40...$ (see Laforgia and Muldoon [35, p. 383]), so $g(1) = 1.07... > 0$. This completes the proof of the theorem.

Remark 3.5. The proofs of the theorems in this paper are based on inequality (3.24) from Lemma 3.5.

The authors believe that this inequality is not quite sharp and has to be replaced by a sharper one. Accordingly, they hypothesize that

$$(\nu + \chi + j_{\nu,\chi})\, j''_{\nu,\chi} > (\nu + \chi)\frac{j'^2_{\nu,\chi}}{j_{\nu,\chi}} - j'_{\nu,\chi} \quad \text{for} \quad \nu > -\chi.$$

Remark 3.6. Numerical experiments show that the claim of Theorem 3.8 can be obtained so that the function $\log[j_{\nu,\chi}/(\nu + \chi)]$ is also convex. If the relation from the remark were proved, this would imply the convexity of $\log[j/(\nu + \chi)]$.

Giordano and Rodono [20] consider the zeros $j_{\nu,\chi} = c_{\nu,k}$ of $C_\nu(x)$, where $\chi = k - \alpha/\pi$. Some new monotonicity and convexity properties of $j_{\nu,\chi}$ are proved. Specifically, it is proved that

$$\nu\left(d^2c_{\nu,k}/d\nu^2 + \delta\right)/c_{\nu,k}$$

is an increasing function of $\nu \ge 0$ for corresponding values of δ and $k - \alpha/\pi \ge 0.7070...$.

Relying on this result, under certain conditions, they prove that $c_{\nu,k} + 1/2\delta\nu^2$ is a convex function of $\nu \ge 0$. Moreover, the monotonicity of $c^2_{\nu,k}/\nu$ is proved. The results strengthen some earlier obtained ones of this type.

Giordano and Laforgia [19] prove that $j_{\nu,\chi} + \alpha\nu^2$ with $\alpha \ge 1$ is a convex function of ν for $\nu \ge 0$ and $\chi \ge \chi_0 = 0.7070...$. For notational simplicity, we define

$$j'_{\nu,\chi} = \frac{dj_{\nu,\chi}}{d\nu}, \quad j''_{\nu,\chi} = \frac{d^2j_{\nu,\chi}}{d\nu^2}, \quad j'''_{\nu,\chi} = \frac{d^3j_{\nu,\chi}}{d\nu^3}.$$

Theorem 3.11. *For* $\nu \ge 0$ *and* $\chi \ge 0.7070...$, *the function*

$$f(\nu) = \frac{\nu}{j_{\nu,\chi}}(j''_{\nu,\chi} + \delta)$$

increases in ν *for any*

$$\delta \ge j'_{0,\chi}/(2j_{0,\chi}).$$

Proof. We need to prove that

$$(j''_{\nu,\chi} + \delta)\frac{d}{d\nu}\frac{\nu}{j_{\nu,\chi}} + \frac{\nu}{j_{\nu,\chi}}j'''_{\nu,\chi} > 0$$

for $\nu \ge 0$, $\chi \ge 0.7070...$, and $\delta \ge j''^2_{0,\chi}/(2j_{0,\chi})$.

For the considered values of ν and χ, we have $j'''_{\nu,\chi} > 0$ (see Elbert and Laforgia [11]). Therefore, the previous inequality holds if

$$(j''_{\nu,\chi} + \delta)\frac{d}{d\nu}\frac{\nu}{j_{\nu,\chi}} > 0.$$

Since $d(\nu/j_{\nu,\chi})/d\nu$ is positive for $\nu \geq 0$ and $\chi \geq 0.7070...$ (see Elbert and Laforgia [11, p. 76]), it suffices to show that $j''_{\nu,\chi} + \delta > 0$. We can use the lower bound

$$j''_{\nu,\chi} > -\frac{1}{2}\frac{j'^2_{\nu,\chi}}{j_{\nu,\chi}}, \quad \nu \geq 0,$$

which follows from the convexity of $j^{3/2}_{\nu,\chi}$ for $\chi = 0.7070...$ and $\nu \geq 0$ (see Elbert and Laforgia [11, p. 75], Elbert and Laforgia [16]).

For $\nu = 0$, we have

$$j''_{0,\chi} > -\frac{1}{2}\frac{j'^2_{0,\chi}}{j_{0,\chi}}$$

and, recalling that $j''_{\nu,\chi}$ increases for $\nu \geq 0$ (see Elbert and Laforgia [13]), we obtain

$$j''_{\nu,\chi} > -\frac{1}{2}\frac{j'^2_{0,\chi}}{j_{0,\chi}}, \quad \nu \geq 0.$$

Therefore, $j''_{\nu,\chi} + \delta$ is positive for at least all $\delta \geq j'^2_{0,\chi}/\left(2j_{0,\chi}\right)$, which proves the theorem.

Remark 3.7. In the important case of zeros $j_{\nu,k}$ of $J_\nu(x)$ corresponding to $\chi = k = 1, 2, ...,$ the values of $j'^2_{0,\chi}/j_{0,\chi}$ can be estimated using the differential inequality (see Ifantis and Siafarikas [24])

$$\frac{dj_{\nu,k}}{d\nu} < \frac{j_{\nu,k}}{\nu + l + 1} + \frac{2}{j_{\nu,k}}\sum_{n=0}^{l-1}\left\{1 - \frac{\nu + n + 1}{\nu + l + 1}\right\}h^2_{n,\nu+1}\left[\frac{1}{j_{\nu,k}}\right], \tag{3.26}$$
$$\nu > -1, \quad l = 0, 1, 2, \ldots,$$

where $h_{n,\nu}(x)$ $(n \geq 0)$ are Lommel polynomials (see Watson [54], Kerimov [31]). If $\chi = k = 1$, we have $j''_{0,1} = -0.350987...$ (see Laforgia and Muldoon [35, p. 383]). Therefore, Theorem 3.11 holds for every $\delta >$ 0.350987... .

Corollary 3.5. For $\nu \geq 0$ and $\chi \geq 0.7070...$, the function

$$f(\nu) = \frac{\nu}{j_{\nu,\chi}}(j''_{\nu,\chi} + \delta)$$

increases with ν for all $\delta \geq 1$.

By Theorem 3.1, it suffices to show that

$$\frac{1}{2}\frac{j'^2_{0,\chi}}{j_{0,\chi}} < 1.$$

Since $j'_{0,\chi} < 1/2\pi$ (see Elbert and Laforgia [10]), we obtain

$$\frac{1}{2}\frac{j'^2_{0,\chi}}{j_{0,\chi}} < \frac{j'_{0,\chi}}{j_{0,\chi}}.$$

Recalling that $j'_{0,\chi} \leq j_{0,\chi}$ for $\chi \geq 0.7070...$ (see Elbert and Laforgia [11, p. 75]), we complete the proof of Corollary 3.5.

Corollary 3.6. For $\nu \geq 0$ and $\chi \geq 0.7070...$, the function

$$j_{\nu,\chi} + \frac{1}{2}\delta\nu^2$$

is convex in ν for all

$$\delta \geq j'^2_{0,\chi}/(2j_{0,\chi}).$$

Note that, in the case $\chi = k = 1$, the function $j_{\nu,\chi} + \beta\nu^2$ is convex for $\beta \geq 0.175493...$ and $\nu \geq 0$, where $j_{\nu,1}$ is the first positive zero of $J_\nu(x)$ (see Remark 3.7).

Remark 3.8. The function $j_{\nu,k} + \beta\nu^2$ is convex for $\beta \geq (2 + j^2_{\nu,k})/[2(\nu + k)j^2_{\nu,k}]$, $\nu > 0$, where $j_{\nu,k}$ is the kth positive zero of $J_\nu(x)$.

This result can be derived by combining the relation

$$\frac{d^2}{d\nu^2}\left(j_{\nu,k} + \beta\nu^2\right) = j''_{\nu,k} + 2\beta$$

with the inequalities

$$j''_{\nu,k} > \frac{\nu j'^2_{\nu,k} - j_{\nu,k}j'_{\nu,k}}{(\nu + j_{\nu,k})\, j_{\nu,k}}$$

(see Elbert and Laforgia [14, p. 2]),

$$j'_{\nu,k} < \frac{1}{\nu+2}\left[j_{\nu,k} + \frac{2}{j_{\nu,k}}\right]$$

(see Ifantis and Siafarikas [24, p. 140]), and

$$j'_{\nu,k} > \frac{2}{j_{\nu,k}} + \frac{8(\nu+1)^2}{j^3_{\nu,k}}$$

(see Ismail and Muldoon [28, p. 196]).

We see that the estimate for β depends on the zeros $j_{\nu,k}$, for which numerous lower and upper bounds are available. Therefore, this estimate can be sharpened for large ν. For example, if $k = 1$, we see that $j_{\nu,1} + \beta\nu^2$ is a convex function for $\beta \geq (2 + j^2_{\nu,1})[2(\nu + 2)j^2_{\nu,1}]^{-2}$. This estimate is less than 0.175492 for any $\nu \geq 1.5$. For $\nu = 1.5$, we have $\beta \geq 0.1570079$, and, for $\nu = 2$, $\beta \geq 0.13444789$.

Before proving the monotonicity properties of $j^2_{\nu,\chi}/\nu$, we prove the following result.

Lemma 3.6. *For* $\nu > 0$ *and* $\chi \geq 0.7070...$, *there exists a unique value* ν_χ *such that*

$$j'_{\nu_\chi} = j_{\nu,\chi}/2\nu_\chi.$$

Proof. In view of Elbert and Laforgia's work [11, p. 76], the function $j_{\nu,\chi}/\nu$ decreases for $\nu > 0$ and $\chi \geq 0.7070...$. Since $j_{\nu,\chi}$ is concave, $j'_{\nu,\chi}$ also decreases for $\nu \geq 0$ and $\chi \geq 0.7070...$ (see Elbert, Gatteschi, and Laforgia [7])

Now consider the function

$$g(\nu) = \frac{j_{\nu,\chi}}{2\nu} - j'_{\nu,\chi}, \quad \nu > 0.$$

By applying the asymptotic formula for $c_{\nu,k}$,

$$c_{\nu,k} = c_{0,k} + \alpha_{1,k}\nu + \alpha_{2,k}\nu^2 + 0\left(\nu^3\right), \quad \nu \to 0,$$

with

$$\alpha_{1,k} = 2c_{0,k}M(c_{0,k}),$$

$$\alpha_{2,k} = 2c_{0,k}M(c_{0,k})[M(c_{0,k}) + c_{0,k}M'(c_{0,k})] - \frac{\pi}{4}c_{0,k}\left[J_\nu(x)\frac{\partial^2 Y_\nu(x)}{\partial\nu^2} - Y_\nu(x)\frac{\partial^2 J_\nu(x)}{\partial\nu^2}\right]_{\nu=0,\ x=c_{0,k}},$$

where

$$M(x) = \frac{\pi^2}{8}\{J_0^2(x) + Y_0^2(x)\}, \quad k = 1, 2, \ldots$$

(see Laforgia and Muldoon [35, p. 385]) and using the estimate $1 < j'_{\nu,\chi} < 1/2\pi$ as $\nu \to 0+$, we see that $g(\nu)$ tends to $+\infty$. As $\nu \to +\infty$, the function $g(\nu)$ tends to $-1/2$, since

$$\lim_{\nu\to\infty} j'_{\nu,\chi} = \lim_{\nu\to\infty} j_{\nu,\chi}/\nu = 1$$

for zeros $j_{\nu,\chi}$ with $\chi \geq 0.7070\ldots$ (see Elbert and Laforgia [9, pp. 208, 76] and Elbert and Laforgia [14, p. 3]). Therefore, there is at least one positive ν_χ such that

$$j'_{\nu,\chi} = j_{\nu,\chi}/2\nu_\chi.$$

Let us show that $g(\nu)$ decreases as ν grows from 0 to $+\infty$. Differentiating $g(\nu)$ yields

$$g'(\nu) = \frac{j'_{\nu,\chi}\nu - j_{\nu,\chi}}{2\nu^2} - j''_{\nu,\chi} < 0.$$

For our purposes, we need the following estimate for $j''_{\nu,\chi}$ (see Elbert and Laforgia [11, p. 74]):

$$j''_{\nu,\chi} > \frac{\nu j'^2_{\nu,\chi} - j_{\nu,\chi}j'_{\nu,\chi}}{(\nu + j_{\nu,\chi})j_{\nu,\chi}}, \quad \nu \geq 0 \quad \text{and} \quad \chi \geq 0.7070\ldots.$$

Thus, it suffices to show that

$$\frac{\nu j'^2_{\nu,\chi} - j_{\nu,\chi}j'_{\nu,\chi}}{(\nu + j_{\nu,\chi})j_{\nu,\chi}} > \frac{j'_{\nu,\chi}\nu - j_{\nu,\chi}}{2\nu^2}$$

or, equivalently,

$$\frac{1}{\nu + j_{\nu,\chi}} - < \frac{1}{2\nu},$$

in view of $j'_{\nu,\chi}\nu - j_{\nu,\chi} < 0$ (see Elbert and Laforgia [11, p. 76]). This holds, since $j_{\nu,\chi} > \nu$ (see [11, p. 80]). Therefore, there is only one ν_χ such that $g(\nu_\chi) = 0$. This proves Lemma 3.5.

Now we obtain the inequalities

$$j'_{\nu,\chi} < j_{\nu,\chi}/2\nu, \quad 0 < \nu < \nu_\chi, \quad j'_{\nu,\chi} \geq j_{\nu,\chi}/2\nu, \quad \nu \geq \nu_\chi. \tag{3.27}$$

Remark 3.9. In the important special case of zeros $j_{\nu,k}$ of $J_\nu(x)$, we can determine an interval I_k such that $\nu_\chi \in I_k$. Inequality (3.26) can be used to deriver an upper bound for $j'_{\nu,k}$, while the inequalities

$$\frac{dj_{\nu,k}}{d\nu} > \frac{2}{j_{\nu,k}}\sum_{n=0}^{m}\left\{h_{n,\nu+1}\left(\frac{1}{j_{\nu,k}}\right)\right\}^{-2}, \quad m = 0, 1, 2, \ldots, \tag{3.28}$$

can be used to deriver a lower bound for $j'_{\nu,k}$ (see Ismail and Muldoon [28, p. 195]).

For example, in the case $k = 1$, using inequality (3.26) with $l = 3$, we derive $j'_{2,1} < 1.2714$, while mathematical tables yield $1/4j_{2,1} = 1.2839$. Therefore, $j'_{\nu,1} < j_{\nu,1}/(2\nu)$ for $0 < \nu \leq 2$. From (3.28) with $m = 3$, we have $j'_{2.5,1} > 1.2234\ldots$ and $j_{\nu,1}/(2\nu)$ for $\nu = 2.5$ is equal to $1.15269\ldots$. For $\nu \geq 2.5$, we can obtain $j'_{\nu,1} > j_{\nu,1}/(2\nu)$.

Clearly, the value of ν_1 satisfying $j'_{\nu_1,1} > j_{\nu_1,1}/(2\nu_1)$ lies on interval (2.25).

Theorem 3.12. *For* $\nu > 0$ *and* $\chi \geq 0.7070...$, *the function* $j^2_{\nu,\chi}/\nu$ *decreases on the interval* $(0, \nu_\chi)$ *and increases on the interval* $(\nu_\chi, +\infty)$, *where* ν_χ *is defined in Lemma* 3.5.

Proof. Differentiating $j^2_{\nu,\chi}/\nu$ with respect to ν yields

$$\frac{d}{d\nu}\frac{j^2_{\nu,\chi}}{\nu} = \frac{2j_{\nu,\chi}\nu - j^2_{\nu,\chi}}{\nu^2}.$$

Applying Lemma 3.5 and using inequalities (3.27), we see that the function $j^2_{\nu,\chi}/\nu$ decreases on $(0, \nu_\chi)$ and increases on $(\nu_\chi, +\infty)$.

Remark 3.10. Since

$$\frac{d^2}{d\nu^2}\frac{j^2_{\nu,\chi}}{\nu} = 2\frac{j_{\nu,\chi}j''_{\nu,\chi}\nu^2 + [j'_{\nu,\chi}\nu - j_{\nu,\chi}]^2}{\nu^3},$$

the function $j^2_{\nu,\chi}/\nu$ is convex if

$$j''_{\nu,\chi} > -\frac{1}{j_{\nu,\chi}}\left[j'_{\nu,\chi} - \frac{j_{\nu,\chi}}{\nu}\right]^2$$

or

$$j'_{\nu,\chi} > -\frac{j'^2_{\nu,\chi}}{j_{\nu,\chi}} + \frac{1}{\nu}\left[2j'_{\nu,\chi} - \frac{j_{\nu,\chi}}{\nu}\right]. \tag{3.29}$$

Since $j^2_{\nu,\chi}$ is a convex function of ν for $\nu \geq 0$ and $\chi \geq 0.344...$, we see that (3.29) holds if $j'_{\nu,\chi} < j_{\nu,\chi}/2\nu$, i.e., for $0 < \nu < \nu_\chi$.

For example, in the case $\chi = k = 1$, we have $\nu_1 \in (2, 2.5)$ (see Remark 3.9). Therefore, $j^2_{\nu,\chi}/\nu$ is a convex function for at least $0 < 2 \leq \nu_1$.

4. ON THE VARIATION WITH RESPECT TO A PARAMETER OF ZEROS OF BESSEL FUNCTIONS OF THE FIRST KIND

Under this title, Ismail and Muldoon published an important work [28], where the Hellmann–Feynman theorem in a discrete setting is used to obtain representations for the derivatives (with respect to a parameter) of positive zeros of a family of entire functions. With the help of these representations, the zeros are proved to be monotone functions of the parameters involved.

This family includes the Bessel functions $J_\nu(x)$ and their basic analogue, namely, the q-Bessel functions $j^{(2)}_\nu(x; q)$. A consequence of this theory is that $j^2_{\nu,1}/(\nu + 1)$ increases with ν when $\nu \in (-1, \infty)$. The Lommel polynomials and generalized Lommel polynomials are used to derive improved lower bounds for the smallest zero $j_{\nu,1}$ of $J_\nu(x)$.

Now, this work will be described in more detail. In previous sections, we used classical formulas for derivatives of the positive zeros $j_{\nu,k}$ of the Bessel function $J_\nu(z)$ and the positive zeros $c_{\nu,k}$ of the cylinder function $C_\nu(x)$, namely, Schläfli's formula

$$\frac{dj}{d\nu} = 2\nu\left[jJ^2_{\nu+1}(j)\right]^{-1}\int_0^j t^{-1}J^2_\nu(j)\,dt, \quad j = j_{\nu,k} \tag{4.1}$$

and Watson's formula

$$\frac{dc}{d\nu} = 2c\int_0^\infty K_0(2c\sinh t)e^{-2\nu t}dt, \quad c = c_{\nu,k}. \tag{4.2}$$

In [23] Ifantis and Siafarikas derived a new formula of this kind:

$$\frac{dj}{d\nu} = j\left[\sum_{n=1}^{\infty} J_{\nu+n}^2(j)\right] \Big/ \left[\sum_{n=1}^{\infty}(\nu+n) J_{\nu+n}^2(j)\right]. \tag{4.3}$$

In [37] (see above) Lewis and Muldoon showed how formula (4.1) can be brought to a form to which the Hellmann–Feynman theorem for the approximate solution of a boundary value problem for differential equations can be applied. Watson's formula (4.2) is more useful than (4.1), since it applies to the general cylinder functions $C_\nu(x)$ and has a convenient form (in view of the fact that $K_0(u)$ is a positive decreasing function of u). That is why formula (4.2) has been the starting point for many works devoted to Bessel function zeros. However, formula (4.2) has not been put into a general setting; i.e., we do not know a formula that would apply to a general class of differential equations and would reduce to (4.2) in a special case. Formula (4.3) yields more general results. For example, it can be proved that, for a fixed k, the ratio

$$j_{\nu,k}/(\nu+1) \tag{4.4}$$

decreases on the interval $(-1, \infty)$. This result is not easy to obtain from formula (4.2).

A new proof of formula (4.3) that is more convenient than that given by Ifantis and Siafarikas [23] was presented in [28]. It was shown that formula (4.3) can be interpreted as a version of the Hellmann–Feynman theorem as applied to a discrete setting arising from the three-term recurrence relation for cylinder functions.

Below, the abstract Hellmann–Feynman theorem is briefly considered as applied to the study of zeros of the Bessel functions $J_\nu(x)$.

Let $\{H_\nu\}$ be a sequence of symmetric operators on a space equipped with a positive definite inner product $\langle\cdot\rangle$ depending on a parameter ν. Let λ_ν and ψ_ν be the eigenvalues and eigenvectors of the operator H_ν, i.e.,

$$H_\nu\psi_\nu = \lambda_\nu\psi_\nu, \tag{4.5}$$

$$\langle H_\nu\psi_\nu, \psi_\nu\rangle_\nu = \lambda_\nu < \psi_\nu, \psi_\nu \tag{4.6}$$

or

$$\lambda_\nu = \langle H_\nu\psi_\nu, \psi_\nu\rangle_\nu / \langle(\psi_\nu, \psi_\nu)_\nu\rangle. \tag{4.7}$$

Formally differentiating (4.6) with respect to ν, we obtain

$$\begin{aligned}&\left\langle\left(\frac{\partial}{\partial\nu}H_\nu\right)\psi_\nu, \psi_\nu\right\rangle_\nu + \left\langle H_\nu\frac{\partial}{\partial\nu}\psi_\nu, \psi_\nu\right\rangle_\nu + \left\langle H_\nu\psi_\nu, \frac{\partial}{\partial\nu}\psi_\nu\right\rangle_\nu + \frac{\partial}{\partial\mu}\langle H_\nu\psi_\nu, \psi_\nu\rangle_\mu\Big|_{\mu=\nu}\\ &= \left(\frac{\partial}{\partial\nu}\lambda_\nu\right)\langle\psi_\nu, \psi_\nu\rangle_\nu + \lambda_\nu\left\langle\frac{\partial}{\partial\nu}\psi_\nu, \psi_\nu\right\rangle_\nu + \lambda_\nu\left\langle\psi_\nu, \frac{\partial}{\partial\nu}\psi_\nu\right\rangle_\nu + \lambda_\nu\frac{\partial}{\partial\nu}\langle\psi_\nu, \psi_\nu\rangle_\mu\Big|_{\mu=\nu}.\end{aligned}$$

The last terms on the left- and right-hand sides are equal. Using the symmetry of H_ν and relation (4.5), we see that two middle terms on the left-hand side cancel with those on the right-hand side, so we obtain

$$\frac{\partial}{\partial\nu}\lambda_\nu = \left\langle\left(\frac{\partial}{\partial\nu}H_\nu\right)\psi_\nu, \psi_\nu\right\rangle_\nu / \langle\psi_\nu, \psi_\nu\rangle_\nu. \tag{4.8}$$

Relation (4.8) expresses the Hellmann–Feynman theorem.

To derive a formula for the second derivative of an eigenvalue, we assume that the inner product is independent of ν and that the eigenfunctions ψ_ν are normalized by $\|\psi_\nu\| = 1$. Then differentiating the equation

$$\frac{\partial\lambda_\nu}{\partial\nu} = \left\langle\frac{\partial H_\nu}{\partial x}\psi_\nu, \psi_\nu\right\rangle,$$

using the symmetry of $\partial H_\nu/\partial\nu$, which follows from the symmetry of H_ν, and the result

$$(\partial H_\nu/\partial\nu)\psi_\nu = -H_\nu(\partial\psi_\nu/\partial\nu) + \psi_\nu(\partial\lambda_\nu/\partial\nu) + \lambda_\nu(\partial\psi_\nu/\partial\nu)$$

of differentiating (4.5), we obtain

$$\frac{d}{d\nu^2}\lambda_\nu = \left\langle\left(\frac{\partial^2}{\partial\nu^2}H_\nu\right)\psi_\nu, \psi_\nu\right\rangle + 2\left\langle(\lambda_\nu - H_\nu)\frac{\partial}{\partial\nu}\psi_\nu, \frac{\partial}{\partial\nu}\psi_\nu\right\rangle. \tag{4.9}$$

This formula provides the second derivative version of the Hellmann–Feynman theorem.

An important consequence of formula (4.8) is that, under certain conditions, $d\lambda_\nu/d\nu > 0$. Accordingly, formula (4.9), combined with Rayleigh's principle, yields the inequality $d^2\lambda_\nu/d\nu^2 > 0$ for the largest eigenvalue.

Now consider a space of sequences in which the inner product of two sequences $u = \{u_i\}$ and $v = \{v_i\}$ is defined by the formula

$$\langle u, v\rangle = \sum_{n=0}^{\infty} a_{n+1}(\nu)\, u_n v_n, \tag{4.10}$$

where $\{a_n(\nu)\}$, $n = 1, 2, \ldots$, is a sequence of entire functions of ν and, for $n > 0$, $a_n(\nu)$ is a positive function for $\nu \in I = (a, b)$. The interval I is independent of n. Assume also that, uniformly in n, $n > 0$, the sequence $\{a_n(\nu)\}$ is bounded below by a positive number on each closed subinterval $[c, d]$ of I. Let L be the normed linear space of sequences u with a finite norm $[\langle u, v\rangle]^{1/2}$. The operator H_ν on L is defined by the infinite matrix

$$(H_\nu)_{i,j} = [\delta_{i,j+1} + \delta_{i+1,j}]/a_{i+1}(\nu), \quad i, j = 0, 1, \ldots. \tag{4.11}$$

The operator H_ν is symmetric. Consider a family of real-valued functions $f_\nu(x)$ satisfying the recurrence relation

$$[a_n(\nu)/x]\, f_{\nu+n}(x) = f_{\nu+n+1}(x) + f_{\nu+n-1}(x). \tag{**}$$

Let ξ_ν be a zero of $f_\nu(x)$. If the vector

$$u_\nu = \{f_{\nu+1}(\xi_\nu), f_{\nu+2}(\xi_\nu), \ldots\}$$

belongs to the space L, then u_ν is an eigenvector corresponding to the eigenvalue $1/\xi_\nu$, i.e.,

$$H_\nu u_\nu = (1/\xi_\nu)\, u_\nu.$$

Then the Hellmann–Feynman theorem in a discrete setting yields the formula

$$\frac{d}{d\nu}(1/\xi_\nu) = \left\langle \left(\frac{\partial}{\partial\nu} H_\nu\right) u_\nu, v_\nu \right\rangle / (u_\nu, u_\nu),$$

and, with the use of (4.10), we have

$$\frac{\partial \xi_\nu}{\partial \nu} = \xi_\nu \left[\sum_{n=1}^{\infty} a_n'(\nu)\, f_{\nu+n}^2(\xi_\nu)\right] \Big/ \left[\sum_{n=1}^{\infty} a_n(\nu)\, f_{\nu+n}^2(\xi_\nu)\right]. \tag{4.12}$$

These formulas are used to simplify the derivation of (4.3). For this purpose, we consider the well-known recurrence relation for $J_\nu(x)$:

$$(2\nu/x)\, J_\nu(x) = J_{\nu+1}(x) + J_{\nu-1}(x).$$

The function $x^{-\nu}J_\nu(x)$ is entire in x and analytic in ν for $\nu \in (-1, \infty)$, so its zeros are differentiable functions of ν. In view of $a_\nu(\nu) = 2(n + \nu)$ and $I = (-1, \infty)$, Eq. (4.12) implies (4.3), since the differentiated series converge uniformly in ν on compact ν-intervals. Considering the recurrence relation

$$\frac{2(\nu+n)}{(\nu+1)x} J_{\nu+n}((\nu+1)x) = J_{\nu+n+1}((\nu+1)x) + J_{\nu+n-1}((\nu+1)x),$$

which is satisfied by $J_\nu\{x(\nu + 1)\}$, we can apply what was said above to (4.4) with $a_n(\nu) = 2(\nu + n)/(\nu + 1)$. Here, $a_1'(\nu) = 0$ and $a_n'(\nu) < 0$ for $n = 2, 3, \ldots$. Then

$$j_{\nu,1}/(\nu+1) \tag{4.13}$$

is a decreasing function of ν on the interval $(-1, \infty)$.

By applying a more complicated method, this result was proved by Elbert [5] and Ifantis and Siafarikas [23].

Next, the authors simplify the sum in the denominator of formula (4.3). Recall the formula

$$4x^{-2}\sum_{n=0}^{\infty}(\nu+2n)J_{\nu+2n}^{2}(x)=J_{\nu-1}^{2}(x)-J_{\nu-2}(x)J_{\nu}(x) \tag{4.14}$$

(see Watson [54, Section 5.51]), which implies that

$$4x^{-2}\sum_{n=0}^{\infty}(\nu+n+1)J_{\nu+n+1}^{2}(x)=J_{\nu}^{2}(x)-J_{\nu+1}(x)J_{\nu-1}(x)+J_{\nu+1}^{2}(x)-J_{\nu}(x)J_{\nu+2}(x). \tag{4.15}$$

It follows from (4.13) that $J_{\nu+1}(j)=-J_{\nu-1}(j)$ for $j=j_{\nu,\,k}$. Combining this relation with identity (4.15) gives

$$\sum_{n=0}^{\infty}(\nu+n)J_{\nu+n}^{2}(j)=\frac{1}{2}j^{2}J_{\nu+1}^{2}(j). \tag{4.16}$$

In Ismail's work [27], relation (4.16) and the formula

$$2\sum_{n=0}^{\infty}J_{\nu+n}^{2}(x)=J_{\nu}^{2}(x)+2\nu\int_{0}^{x}t^{-1}J_{\nu}^{2}(t)\,dt,\qquad \nu>0,$$

from Watson's book [54, Section 5.51]) are used to prove the equivalence of (4.1) and (4.3). With the help of (4.16), formula (4.3) can be written in the simplified form

$$\frac{dj}{d\nu}=(2/j)\left[\sum_{n=0}^{\infty}J_{\nu+n}^{2}(j)\right]/J_{\nu+1}^{2}(j),\qquad j=j_{\nu,k}. \tag{4.17}$$

Now we describe a method based on using Lommel polynomials for the study of the zeros of $J_\nu(x)$. Instead of the Lommel polynomials, the authors use the class of more general polynomials $\{\varphi_n^\nu(x)\}$ defined by the recurrence formula

$$\begin{gathered}\varphi_0^{\nu}(x)=1,\qquad \varphi_1^{\nu}(x)=c_{\nu+1}x,\\ xc_{\nu+n+1}\varphi^{\nu}(x)=\varphi_{n+1}^{\nu}(x)+\varphi_{n-1}^{\nu}(x),\qquad n>0.\end{gathered} \tag{4.18}$$

It is assumed that $c_{\nu+n}>0$ for $n\ge 1$. For $c_\nu=2(\nu-1)$, the polynomials $\{\varphi_n(x)\}$ coincide with the generalized Lommel polynomials $\{h_{n,\,\nu}(x)\}$ (see Watson [54], Kerimov [31, 32]).

The polynomials $\{\varphi_n(x)\}$ satisfy the relation

$$xc_{\nu}\varphi^{\nu}(x)=\varphi_{n+1}^{\nu-1}(x)+\varphi_{n-1}^{\nu+1}(x). \tag{4.19}$$

Let

$$\begin{gathered}E^{\nu}(x)=\lim_{n\to\infty}\left[x^{n}\varphi_{n}^{\nu}(1/x)/\prod_{j=1}^{n}c_{j+\nu}\right],\\ f_{\nu}(x)=x^{\nu}E_{\nu}(x)/\Gamma_{\nu}(\nu+1),\end{gathered} \tag{4.20}$$

where $\Gamma_c(\nu)$ is the gamma function associated with the sequence (c_ν), i.e., a function satisfying

$$\Gamma_c(\nu+1)=c_\nu\Gamma(\nu),\qquad \Gamma_c(1)=1.$$

Relations (4.16) and (4.20) imply the recurrence formula

$$[c_\nu/x]f_\nu(x)=f_{\nu+1}(x)+f_{\nu-1}(x),$$

which has the form of (**). If $c_\nu=2\nu$, then $f_\nu(x)=J_\nu(x)$. It is assumed that $c_{n+\nu}>0$ for $n\ge 1$ and the series

$$\sum_{n=1}^{\infty}[c_{\nu+n}c_{\nu+n+1}]^{-1} \tag{4.21}$$

converges. Thus, the system $\{\varphi_n^\nu(x)\}$ is orthogonal on a bounded set E that is symmetric about $x = 0$ and its only limit point is $x = 0$.

The functions $\varphi_n(x)$ are orthogonal with respect to a discrete probability measure $d\mu$ whose masses are located at the reciprocals of the zeros of $x^{-\nu}f_\nu(x)$ and, in certain cases, at the point $x = 0$. Then the orthogonality condition for the polynomials $\{\varphi_n^\nu(x)\}$ is

$$\int_E \varphi_m^\nu(x)\varphi_n^\nu(x)\,d\mu(x) = (c_{n+1}/c_{n+\nu+1})\,\delta_{m,n}. \tag{4.22}$$

Now we define a matrix operator H_ν by the relation

$$(H_\nu)_{i,j} = (\delta_{i,j+1} + \delta_{i+1,j})/c_{n+\nu+1}, \quad i, j = 0, 1, \dots . \tag{4.23}$$

The inner product of two sequences u and v is defined as

$$\langle u, v\rangle = \sum_{n=0}^{\infty} c_{n+\nu+1} u_n v_n.$$

The operator H_ν is symmetric on the normed space L with the norm $\|u\|^2 = \langle u, v\rangle$, and the Hellmann–Feynman theorem is applicable to H_ν. For $u = \{u_i$: $i = 0, 1, \dots\}$ to be an eigenvector corresponding to an eigenvalue λ, it is necessary that $\{u_i\}$ be a multiple of $\{\varphi_i^\nu(\lambda)\}$. However, such a vector u belongs to L if and only if λ supports a mass of $d\mu$. This is the case, which follows from moment theory (see Shohat and Tamarkin [50, Corollary 2.6, p. 45]). Therefore, the nonzero eigenvalues of H_ν are the reciprocals of the zeros of $x^{-\nu}f_\nu(x)$. The function $x^{-\nu}f_\nu(x)$ is even, since $\varphi_n^\nu(-x) = (-1)^n\varphi_n^\nu(x)$. Let ξ_ν be a positive zero of $f_\nu(x)$. Since $x^{-\nu}f_\nu(x)$ is an analytic function of x and ν, the zero ξ_ν is differentiable in ν.

Theorem 4.1. *Let $\{c_{n+\nu}$: $n = 1, 2, \dots\}$ be a sequence of positive differentiable functions of* ν, $\nu \in I$, *and assume that series* (4.21) *converges. If $\{f_\nu(x)\}$ are analytic functions of x and ν and $x = \xi_\nu \neq 0$ is a zero of $f_\nu(x)$, then*

$$\frac{d}{d\nu}\xi_\nu = \xi_\nu\left[\sum_{n=0}^{\infty}\left\{\varphi_n^\nu(1/\xi_\nu)\right\}^2\frac{d}{d\nu}c_{n+\nu+1}\right]\Big/\left[\sum_{n=0}^{\infty}c_{n+\nu+1}\left\{\varphi_n^\nu(1/\xi_\nu)\right\}^2\right]. \tag{4.24}$$

This theorem shows that, as functions of ν, the zeros of $f_\nu(x)$ increase (decrease) if $c_{n+\nu}$ increases (decreases) with ν for all $n = 1, 2, \dots$.

The mass $M(\xi_\nu)$ of $d\mu$ located at $1/\xi_\nu$ is given by the formula (see Shohat and Tamarkin [50, p. 45, Corollary 2.6])

$$M(\xi_\nu) = c_{\nu+1}\Big/\left[\sum_{n=0}^{\infty}c_{n+\nu+1}\left\{\varphi_n^\nu(1/\xi_\nu)\right\}^2\right]. \tag{4.25}$$

Since the Lommel polynomials $\{h_{n,\nu+1})(x)\}$ correspond to $c_\nu = 2\nu$, we set $\xi_\nu = j_{\nu,k}$ and find from (4.24) that

$$\frac{d}{d\nu}j_{\nu,k} = (2/j_{\nu,k})\sum_{n=0}^{\infty}\{h_{n,\nu+1}(1/j_{\nu,k})\}^2, \tag{4.26}$$

which is known as the Lommel formula. The Lommel polynomials are related to the Bessel functions $J_\nu(x)$ by the formula

$$J_{\nu+m}(z) = J_\nu(z)h_{m,\nu}(1/z) - J_{\nu-1}(z)h_{m-1,\nu+1}(1/z), \tag{4.27}$$

where m takes nonnegative integer values (see Watson [54, Section 9.6], Kerimov [31]). Note that (4.27) implies the equivalence of formulas (4.18) and (4.26).

What was described at the beginning of this section is related to the above exposition based on Lommel polynomials. An analogue of relation (4.27), which can be proved by induction, is

$$f_{\nu+m}(x) = f_\nu(x)\varphi_m^{\nu-1}(1/x) - f_{\nu-1}(f)\varphi_{m-1}^\nu(1/x). \tag{4.28}$$

This shows that (4.24) implies (4.12) if we set $a_n(\nu) = c_{n+\nu}$.

The Lommel formula yields the sequential lower bounds

$$j_{\nu,k} \frac{d}{d\nu} j_{\nu,k} > 2, \tag{4.29}$$

which also follow from (4.17),

$$j_{\nu,k} \frac{d}{d\nu} j_{\nu,k} > 2\left[1 + 4(\nu+1)^2 / j_{\nu,k}^2\right], \tag{4.30}$$

and

$$j_{\nu,k} \frac{d}{d\nu} j_{\nu,k} > 2\left[2 - 4(\nu+1)(\nu+3)/j_{\nu,k}^2 + 16(\nu+1)^2(\nu+2)^2 / j_{\nu,k}^4\right]. \tag{4.31}$$

Computations at $k = 1$ show that these bounds are quite sharp in the interval $-1 < \nu \le 2$. They may be contrasted with other bounds, such as

$$\frac{d}{d\nu} j_{\nu,k} > 1, \quad \nu \ge 0, \tag{4.32}$$

which was proved by Elbert and Laforgia [9], and

$$\frac{d}{d\nu} j_{\nu,k} < j_{\nu,k}/(\nu+1), \quad \nu > -1, \tag{4.33}$$

proved by Elbert [6] using (4.2) and by Ifantis and Siafarikas [23] using (4.3).

For comparison purposes, we also present some inequalities obtained for the first zero $j_{\nu,1}$ with the help of the Rayleigh function

$$\sigma_\nu^{(m)} = \sum_{n=1}^{\infty} j_{\nu,n}^{-2m} \tag{4.34}$$

(see Watson [54, p. 502] and Kerimov [31]). These inequalities have the form

$$\left[\sigma_\nu^m\right]^{-1/m} < j_{\nu,1}^2 < \sigma_\nu^m / \sigma_\nu^{(m+1)}, \quad m = 1, 2, \dots . \tag{4.35}$$

Here, σ_ν^m is a rational function, which can be calculated using by the recurrence formula

$$\begin{gathered}(\nu+n)\sigma_\nu^{(n)} = \sum_{k=1}^{n-1} \sigma_\nu^{(k)} \sigma_\nu^{(n-k)}, \quad n = 2, 3, \dots, \\ \sigma_\nu^{(1)} = 1/[4(\nu+1)]\end{gathered} \tag{4.36}$$

(see Kerimov [31]).

On applying several explicit formulas for $\sigma_\nu^{(n)}$ given by Watson [54, p. 502] and Kerimov [31], the first few of the formulas (4.35) yield the successively improving bounds

$$4(\nu+1) < j_{\nu,1}^2 < 4(\nu+1)(\nu+2), \tag{4.37}$$

$$4(\nu+1)(\nu+2)^{1/2} < j_{\nu,1}^2 < 2(\nu+1)(\nu+3), \tag{4.38}$$

$$2^{5/3}(\nu+1)[(\nu+2)(\nu+3)]^{1/3} < j_{\nu,1}^2 < 8(\nu+1)(\nu+2)(\nu+4)/(5\nu+11), \tag{4.39}$$

$$j_{\nu,1}^2 < 2(\nu+1)(\nu+5)(5\nu+11)/(7\nu+19), \tag{4.40}$$

$$j_{\nu,1}^2 < 8(7\nu+19)(\nu+1)(\nu+2)(\nu+3)(\nu+6)/\left(42\nu^3 + 36\nu^2 + 1026\nu + 946\right). \tag{4.41}$$

Although these inequalities, which are valid for $\nu > -1$, are implicitly contained in (4.35), they were not well known and were sometimes rediscovered by different methods. For example, inequality (4.37) was proved by Ronveaux and Moussiaux [49] by applying a differential equation technique based on homographic transformations of the Riccati phase equations. Inequality (4.38) was proved by Elbert [5].

Now we show that $j_{\nu,k}^2/(\nu+1)$ increases as ν grows in the interval $(-1, \infty)$. In view of the representation

$$j_{\nu,k}^2 = 2(\nu+1)[1+O(\nu+1)] \quad \text{as} \quad \nu \to -1 \tag{4.42}$$

(see Piessens [47]), this result is quite sharp for ν close to -1. We need to prove that

$$2(\nu+1)\frac{d}{d\nu}j_{\nu,1} > j_{\nu,1}. \tag{4.43}$$

In view of (4.30), it suffices to show that

$$4(\nu+1)\,j_{\nu,1}^{-2}\left[1+4(\nu+1)^2\,j_{\nu,1}^{-2}\right] > 1. \tag{4.44}$$

However, it follows from (4.28) that (4.44) (and, hence, (4.43)) holds for $-1 < \nu < 1$. For $\nu \ge 1$, inequality (4.42) can be proved taking into account (4.44), since $j_{\nu,1}/(\nu + 1)$ decreases, $\nu > -1$, and $j_{1,1}/2$ is equal to 1.92..., which is less than 2. Thus, the following result holds true.

Theorem 4.2. *The function* $j_{\nu,1}^2/(\nu+1)$ *is an increasing function of* ν *on the interval* $(-1, \infty)$.

With the use of inequalities (4.29), (4.30), and (4.42) and the orthogonality of Lommel polynomials, the authors derive upper and lower bounds for $j_{\nu,1}$.

It is well known that, if $\{p_n(x)\}$ is a family of discrete orthogonal polynomials, then the largest zero of $p_n(x)$ converges to the largest mass point of the measure. The largest zero of $p_n(x)$ increases with n, because the zeros of $p_n(x)$ and $p_{n+1}(x)$ interlace. Similarly, the second largest zeros of $p_n(x)$ increase with n and converge to the second largest mass point.

The Lommel polynomials $\{h_{n,\nu+1}(x)\}$ are orthogonal with respect to a discrete measure whose masses are supported at $\pm 1/j_{\nu,n}$, $n = 1, 2, \ldots$. An explicit formula for the Lommel polynomials (see Watson [54, Section 9.61], Kerimov [31]) is

$$h_{n,\nu}(x) = \sum_{k=0}^{[n/2]}(-1)^k\binom{n-k}{k}(\nu+k)_{n-2k}(2x)^{n-2k}, \tag{4.45}$$

where

$$(a)_n = a(a+1)\ldots(a+n-1), \quad n > 0, \quad (a)_0 = 1.$$

The largest zero of $h_{3,\nu+1}$ is $[2(\nu + 1)(\nu + 3)]^{-1/2}$. Therefore,

$$[2(\nu+1)(\nu+3)]^{-1/2} < 1/j_{\nu,1},$$

which is the right-hand inequality in (4.38).

A similar consideration of $h_{5,\nu+1}(x)$ leads to the improved bounds

$$j_{\nu,1} < [A/(1+B)]^{1/2}, \quad j_{\nu,2} < [A(1-B)]^{1/2}, \quad \nu > -1, \tag{4.46}$$

where

$$\begin{gathered} A = 2(\nu+1)(\nu+5), \\ B = \left[1-\frac{3}{4}(\nu+1)(\nu+5)/(\nu+2)(\nu+4)\right]^{1/2}. \end{gathered} \tag{4.47}$$

The first inequality in (4.46) is due to Schafheitlin (see Watson [54, p. 487]) and is also obtained by considering $h_{5,\nu+1}(x)$. This inequality is sharper than the right-hand inequality in (4.38).

In Watson's book [54, p. 487], this inequality was said to be sharper than

$$j_{\nu,1} < \left\{\frac{4}{3}(\nu+1)(\nu+5)\right\}^{1/2}. \tag{4.48}$$

Numerical experiments show that (4.46) is sharper than the upper bound in (4.40) for all $\nu > -1$ and sharper than the upper bound in (4.41) for $\nu > -0.54$. However, (4.46) is not better for large ν than the next bound, after (4.41), in the Rayleigh sequence; that bounds behaves, for large ν, like $14\nu^2/11$, while (4.46) behaves like $4\nu^2/3$. More accurate lower bounds for $j_{\nu,1}$ and $j_{\nu,2}$ can be obtained by applying the Cardan and Ferrari formulas to find the largest and second largest zeros of the Lommel polynomials of degrees 7 and 9. However, the corresponding computations require much effort.

The next issue to be discussed is how a lower bound, such as (4.29), (4.30), or (4.31), for the derivative of the zero $j_{\nu,1}$ can be combined with an lower bound for $j_{\nu,1}$ in order to obtain a sharper lower bound for $j_{\nu,1}$. We may choose any lower limit of integration (exceeding or equal to -1) but the most interesting results are obtained with the choices -1 and 0.

As a first example, we note that, from (4.42), $j_{\nu,1} \to 0$ as $\nu \to -1$, and then integrate (4.29) to derive the lower bound in (4.37):

$$j_{\nu,1}^2 > 4(\nu+1).$$

If we use 0 rather than -1 as a lower limit of integration, then

$$j_{\nu,1}^2 > j_{\nu,1}^2 + 4\nu, \quad \nu > 0, \tag{4.49}$$

which is weaker than the well-known inequality

$$j_{\nu,1} > j_{0,1} + \nu, \quad \nu > 0 \tag{4.50}$$

(see Laforgia and Muldoon [35]).

Combining (4.30) with the lower bound in (4.37), we obtain

$$j_{\nu,1}^3 \frac{d}{d\nu} j_{\nu,1} > 8(\nu+1) + 8(\nu+1)^2,$$

which, after integration with the lower limit -1, gives the lower bound

$$j_{\nu,1} > 2(\nu+1)^{1/2}\left\{1 + \frac{2}{3}(\nu+1)\right\}^{1/4}, \quad \nu > -1, \tag{4.51}$$

which is better than the lower bound in (4.37) and better for small ν than the lower bound

$$j_{\nu,1} > [\nu(\nu+2)]^{1/2}, \quad \nu \ge 0 \tag{4.52}$$

(see Watson [54, Section 15.3, (5)]).

If inequality (4.30) is used again, but with lower bound (4.51), then

$$j_{\nu,1}^4 > 19.2\nu[1 + 2(\nu+1)/3]^{3/2} + 19.2 + 32(\nu+1)^3/3, \quad \nu > -1, \tag{4.53}$$

which is better than (4.51), as can be seen if we write (4.51) in the form

$$j_{\nu,1}^4 > 16(\nu+1)^2 + 32(\nu+1)^3/3, \quad \nu > -1,$$

and use

$$1 + \frac{2}{3}(\nu+1) > (\nu+1)^{2/3}, \quad \nu > -1.$$

Inequality (4.53) is sharper than the lower bound in (4.37), but is not sharper than the lower bounds in (4.38) and (4.39).

Inequalities (4.30) and (4.31) can also be used to derive bounds that are sharper than (4.50). For example, multiplying (4.30) by $j_{\nu,1}^2$, applying (4.50), and then integrating, we obtain the lower bound

$$j_{\nu,1}^4 > j_{0,1}^4 + 13\frac{1}{3}\nu^3 + 8(j_{0,1}+4)\nu^2 + 8\left(j_{0,1}^2 + 4\right)\nu, \quad \nu \ge 0. \tag{4.54}$$

Inequality (4.38) yields a better lower bound. To see this, we write (4.38) in the form

$$j_{\nu,1}\frac{d}{d\nu} j_{\nu,1} > 2\left[1 + 4(\nu+1)^2/j_{\nu,1}^2 + \left\{4(\nu+1)(\nu+2)/j_{\nu,1}^2 - 1\right\}^2\right].$$

Using (4.38), we obtain

$$4(\nu+1)(\nu+2)/j_{\nu,1}^2 - 1 > 2^{1/3}(\nu+1)^{3/2}(\nu+3)^{-1/2}/j_{\nu,1}.$$

These inequalities yield

$$j_{\nu,1}^3 \frac{d}{d\nu} j_{\nu,1} > 2\left[j_{\nu,1}^2 + 4(\nu+1)^2 + 2(\nu+1)^3/(\nu+3)\right], \tag{4.55}$$

and, after using (4.51), we obtain

$$j_{\nu,1}^3 \frac{d}{d\nu} j_{\nu,1} > 8(\nu+1)\left[1 + \frac{2}{3}(\nu+1)\right]^{1/2} + 8(\nu+1)^2 + 4(\nu+1)^3/(\nu+3).$$

Integrating this inequality from -1 to ν gives

$$j_{\nu,1}^4 > \frac{96}{5}\left[1 + \nu\{1 + 2(\nu+1)/3\}^{3/2}\right] + 16\nu^3 + 32\nu^2 + 80\nu + 64 - 128\ln\{(\nu+3)/2\}, \quad \nu > -1. \tag{4.56}$$

This inequality is sharper than (4.53) for all $\nu > -1$. On the other hand, using (4.50) rather than (4.51) in (4.55) and integrating with the lower limit 0, we obtain

$$j_{\nu,1}^3 > j_{0,1}^4 + 18\frac{2}{3}\nu^3 + 8(j_{0,1} + 4)\nu^2 + 8\left(j_{0,1}^2 + 10\right)\nu - 128\ln(1 + \nu/3), \quad \nu \ge 0. \tag{4.57}$$

Numerical experiments show that (4.57) is sharper than (4.54).

It is of interest to compare (4.54) and (4.57), which are valid on $(0, \infty)$ and become sharp at $\nu = 0$, with (4.50), the only other inequality of this kind known to the authors. These inequalities are sharper than (4.50) for $0 < \nu < 6.5$ and $0 < \nu < 11$, respectively, but (4.50) is sharper for large values of ν. All the inequalities obtained are valid only for small values of ν. For large ν, asymptotic expansions are known.

5. ON ZEROS OF q-BESSEL FUNCTIONS

The basic Bessel (or q-Bessel) functions of the first and second kinds were first defined by Jackson (see [30]) by the series

$$J_\nu^{(1)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n (x/2)^{\nu+2n}}{(q;q)_n (q^{\nu+1};q)_n}, \quad 0 < q < 1, \tag{5.1}$$

$$J_\nu^{(2)}(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n (x/2)^{\nu+2n}}{(q;q)_n (q^{\nu+1};q)_n} q^{n(\nu+1)}, \quad 0 < q < 1, \tag{5.2}$$

where

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{j=0}^{n-1}\left(1 - aq^j\right), \quad (a;q)_\infty = \prod_{j=0}^{\infty}\left(1 - aq^j\right).$$

Before analyzing the zeros of these functions, we present some theoretical results concerning q-Bessel functions, q-Lommel polynomials, relations between them, etc. All of them can be found in [26, 28].

The author of [26] additionally investigates the zeros of the function $J_{\nu+\alpha x}(x)$, which is related to $J_\nu^{(1)}(x;q)$ and $J_\nu^{(2)}(x;q)$.

Consider the q-gamma function (see Askey [2], Jackson [30])

$$\Gamma_q(x) = (q;q)_\infty (1-q)^{1-x} / \left(q^{-x};q^{-1}\right)_\infty, \quad 0 < q < 1, \tag{5.3}$$

and

$$\Gamma_q(x) = \left(q^{-1};q^{-1}\right)_\infty (-1+q)^{1-x} q^{x(x-1)/2} / \left(q^{-x};q^{-1}\right)_\infty, \quad q > 1. \tag{5.4}$$

As $q \to 1$, we have $\Gamma_q(x) \to \Gamma(x)$. It follows that

$$J_\nu^{(1)}(x(1-q);q) \to J_\nu(x)$$

as $q \to 1$, since $\left(q^{\nu+1}; q\right)_\infty / (q; q)_\infty$ is equal to

$$(1-q)^{-\nu} / \Gamma_q(\nu+1).$$

Both $J_\nu^{(1)}(x; q)$ and $J_\nu^{(2)}(x; q)$ satisfy the recurrence relations

$$q^\nu J_{\nu+1}^{(k)}(x; q) = \frac{2\left(1-q^\nu\right)}{x} J_\nu^{(k)}(x;q) - J_{\nu-1}^{(k)}(x;q), \quad k = 1, 2. \tag{5.5}$$

Integrating (5.5) yields a relation similar to that between the Bessel functions $J_\nu(x)$ and the Lommel polynomials $\{R_{n,\nu}(x)\}$. Namely, the formula

$$q^{m\nu+m(m-1)/2} J_{\nu+m}^{(k)}(x; q) = R_{m,\nu}(x; q) J_\nu^{(k)}(x; q) - R_{m-1,\nu+1}(x; q) J_{\nu-1}^{(k)}(x; q), \quad k = 1, 2, \tag{5.6}$$

relates the q-Bessel functions to the q-Lommel polynomials $R_{m,\nu}(x; q)$.

In turn, the q-Lommel polynomials $\{R_{n,\nu}(x; q)\}$ satisfy the recurrence relation

$$\begin{gathered} R_{n+1,\nu}(x; q) = \frac{2\left(1-q^{\nu+n}\right)}{x} R_{n,\nu}(x; q) - q^{n+\nu-1} R_{n-1,\nu}(x; q), \quad n = 1, 2, \ldots, \\ R_{0,\nu}(x; q) = 1, \quad R_{1,\nu}(x; q) = 2\left(1-q^\nu\right)/x. \end{gathered} \tag{5.7}$$

Setting

$$h_{n,\nu}(x; q) = R_{n,\nu}(1/x; q), \tag{5.8}$$

we obtain the generalized q-Lommel polynomials, which satisfy the recurrence

$$\begin{gathered} h_{n+1}(x; q) = 2x\left(1-q^{n+\nu}\right) h_{n,\nu}(x; q) - q^{n+\nu-1} h_{n-1,\nu}(x; q), \quad n = 1, 2, \ldots, \\ h_{0,\nu}(x; q) = 1, \quad h_{1,\nu}(x; q) = 2x\left(1-q^\nu\right). \end{gathered} \tag{5.9}$$

The functions $J_\nu^{(1)}(x; q)$ and $J_\nu^{(2)}(x; q)$ are related by

$$\left(-x^2/q; q\right)_\infty J_\nu^{(1)}(x; q) = J_\nu^{(2)}(x; q), \quad |x| < 2 \tag{5.10}$$

(see Hahn [21]). Concerning the zeros of $J_\nu^{(1)}(x; q)$ and $J_\nu^{(2)}(x; q)$, it is only mentioned in [21] that $J_\nu^{(2)}(x; q)$ has an infinite number of real zeros.

Series (5.1) for $J_\nu^{(1)}(x; q)$ converges only for $|x| < 2$; additionally, there is formula (5.10). Therefore, it cannot be expected that the zeros of $J_\nu^{(1)}(x; q)$ have the same properties as those of $J_\nu^{(2)}(x; q)$.

It is proved in [26] that all nonzero zeros (which are infinitely many) of $J_\nu^{(2)}(x; q)$ are real and simple and that the zeros $j_{\nu,n}(q)$ of $J_\nu^{(2)}(x; q)$ and $J_{\nu+1}^{(2)}(x; q)$ interlace. Similar properties are also possessed by the zeros of $J_{\nu+ax}(x)$.

Due to relation (5.10), the function $J_\nu^{(1)}(x; q)$ has only a finite set of nonzero zeros for $|x| < 2$, and they coincide with zeros of the entire function $x^{-\nu} J_\nu^{(2)}(x; q)$.

For q-Lommel polynomials, there is an explicit formula:

$$R_{m,\nu}(x;q) = \sum_{j=0}^{[m/2]} \frac{(x/2)^{2j-m} (-1)^j \left(q^\nu; q\right)_{m-j} (q; q)_{m-j}}{(q; q)_j \left(q^\nu; q\right)_j (q; q)_{m-2j}} q^{j(j+\nu-1)}. \tag{5.11}$$

Similarly, the generalized q-Lommel polynomials are given by the explicit formula

$$h_{n,\nu}(x; q) = \sum_{j=0}^{[n/2]} \frac{(2x)^{n-2j} (-1)^j \left(q^\nu; q\right)_{n-j} (q; q)_{n-j}}{(q; q)_j \left(q^\nu; q\right)_j (q; q)_{n-2j}} q^{j(j+\nu-1)}. \tag{5.12}$$

This formula is an exact analogue of the corresponding formula from Watson's book [54, p. 296, formula (3)] for the generalized Lommel polynomials $h_{n,\nu}(x)$, i.e., of formula (4.4).

Let $\{j_{\nu,k}(q)\}$ denote the positive zeros of the even function $x^{-1}J_\nu^{(2)}(x;q)$ ordered in the form

$$0 < j_{\nu,1}(q) < j_{\nu,2}(q) < \ldots < j_{\nu,n}(q) < \ldots. \tag{5.13}$$

Relying on the above theoretical results regarding q-Bessel functions and q-Lommel polynomials, the following theorems are proved in [26], which are given below without detailed proofs.

Theorem 5.1. *The basic Bessel functions $J_\nu^{(2)}(x;q)$ and $J_{\nu+1}^{(2)}(x;q)$ have no common zeros, except for possibly the point $z = 0$, when ν takes real values.*

Theorem 5.2. *All zeros of $x^{-\nu}J_\nu^{(2)}(x;q)$ are real and simple for $\nu > -1$. There are infinitely many such zeros, and their only limit point is a point at infinity.*

Theorem 5.3. *Between any two consecutive positive zeros of $x^{-\nu}J_\nu^{(2)}(x;q)$, the function $x^{-\nu-1}J_{\nu+1}^{(2)}(x;q)$ has exactly one zero when $\nu \geq -1$.*

Theorem 5.4. *All zeros of the function $x^{-\nu-ax}J_{\nu+ax}(x)$ are real and simple for $\nu > -1$ and real a.*

There are infinitely many such zeros, and their only limit point is a point at infinity. Between any two consecutive zeros of $x^{-\nu-ax}J_{\nu+ax}(x)$, the function $x^{-\nu-ax-1}J_{\nu+ax-1}(x)$ has only one zero.

The variation in the positive zeros $j_{\nu,k}(q)$ of $J_\nu^{(2)}(x;q)$ with respect to ν and q is studied in the second part of [28, pp. 201–206].

The function $J_\nu^{(2)}(x;q)$ is analytic for $x > 0$, $\nu > -1$, and $0 < q < 1$. Therefore, its positive zeros are differentiable with respect to ν and q.

Returning to recurrence formula (**) for $f_\nu(x)$ and setting $f_\nu(x) = q^{\nu^2/4}J_\nu^{(2)}(x;q)$, we obtain the recurrence formula

$$\left[2\left(1-q^{\nu}\right)q^{(1-2\nu)/4}/x\right]f_\nu(x) = f_{\nu+1}(x) + f_{\nu-1}(x). \tag{5.14}$$

Now ξ_ν is a zero of $J_\nu^{(2)}(x;q)$, and $a_n(\nu) = 4q^{1/4}\sinh[(n+\nu)w]$, where $w = -(\ln q)/2 > 0$.

The function $a_n(\nu)$ increases in ν, so the positive zeros of $J_{\nu+1}^{(2)}(x;q)$ increase with ν, $\nu > -1$, provided that the vector $u = \{f_{\nu+1}(\xi_\nu), f_{\nu+2}(\xi_\nu)\}$ belongs to a suitable inner product space. However, we have the asymptotic formulas

$$q^{1/4}a_n(\nu) \sim 4q^{(n+\nu)/2}, \quad J_{\nu+n+1}^{(2)}(x;q) \sim (x/2)^{n+\nu+1}/(q;q)_\infty$$

as $n \to \infty$, so the series defining the norm $\|u\|$ converge. It follows that the zeros of a q-Bessel function are the eigenvalues of a symmetric operator and all of them are real for $\nu > -1$.

Concerning the variation of zeros with respect to q, the authors consider the formula

$$a_n(q) = 2q^{1/4}\left[q^{-(n+\nu)2} - q^{(n+\nu)/2}\right]. \tag{5.15}$$

Then

$$\frac{\partial}{\partial q}a_n(q) = -q^{-3/4}\cosh[(n+\nu)w][2(n+\nu) - \tanh\{(n+\nu)w\}], \tag{5.16}$$

where $w = -((\ln q))/2$. It can be seen that $2r > \tanh(rw)$ for $r > 0$ and $e^{-4} < q < 1$. From this and (5.15), it follows that $a_n(q)$, $n > 0$, decreases as q increases on the interval $(e^{-4}, 1)$. Another result can be obtained by considering $2q^{-1/4}j_{\nu,k}(q)$ rather than the zeros $j_{\nu,k}(q)$. With a suitable scale, we derive the recurrence relation

$$\left[\left(q^{-\nu/2} - q^{\nu/2}\right)/y\right]f_\nu(y) = f_{\nu+1}(y) + f_{\nu-1}(y). \tag{5.17}$$

In view of the notation $a_n(q) = q^{-\nu/2} - q^{\nu/2}$, we see that $a_n(q)$ $(n > 0)$ decreases for increasing $q : 0 < q < 1$ and $\nu > -1$.

In view of what was said above, the following result holds.

Theorem 5.5. *Let* $\nu > -1$. *The positive zeros* $j_{\nu,k}(q)$ *of* $J_\nu^{(2)}(x; q)$ *increase with* ν *for* $0 < q < 1$ *and decrease in* q *for* $e^{-4} < q < 1$, *but* $q^{-1/4}j_{\nu,k}(q)$ *increases with* ν *and decreases in* q *for* $0 < q < 1$.

The function $a_n(\nu)/\sinh[(\nu+1)w]$ decreases with increasing ν on $(-1, \infty)$. It follows that $j_{\nu,1}(q)/(q^{-\nu-1} - q^{\nu-1})$ is a decreasing function of ν if $\nu \in (-1, \infty)$ and $q \in (0, 1)$.

In [26] Ismail defined the q-Lommel polynomials $h_{n,\nu}(x; q)$ as generated by the recurrence relations

$$h_{0,\nu}(x; q) = 1, \quad h_{1,\nu}(x; q) = 2x\left(1 - q^{\nu}\right), \tag{5.18}$$

$$h_{n+1,\nu}(x; q) = 2x\left(1 - q^{n+\nu}\right)h_{n,\nu}(x; q) - q^{n+\nu-1}h_{n-1,\nu}(x; q), \quad n > 0. \tag{5.19}$$

They are orthogonal with respect to a purely discrete measure whose masses are located at the reciprocals of the zeros of $J_{\nu-1}^{(2)}(x; q)$. Inequalities of form (4.30) and (4.46) can be extended to q-Lommel polynomials if we consider the zeros of the q-Lommel polynomials of degrees 3 and 5, respectively. The polynomials $h_{n,\nu}(x; q)$ are given by the explicit formula

$$h_{n,\nu}(x; q) = \sum_{j=0}^{[n/2]} \frac{(2x)^{n-2j}(-1)^j (q; q)_{n-j}\left(q^{\nu}; q\right)_{n-j} q^{j(j+\nu-1)}}{(q; q)_j \left(q^{\nu}; q\right)_j (q; q)_{n-2j}}$$

(see Ismail [26, formula (3.6)]).

Thus,

$$h_{1,\nu+4}(x; q) = 4x^2\left(1 - q^{\nu+1}\right)\left(1 - q^{\nu+2}\right) - q^{\nu+1},$$

so we obtain the inequality

$$j_{\nu,1}^2 < 4\left(1 - q^{\nu+1}\right)\left(1 - q^{\nu+2}\right)q^{-\nu-1},$$

which, in the limit as $q \to 1$, yields the upper bound in (4.27). Considering the zeros of the polynomial $h_{3,\nu+4}(x; q)$ yields the inequality

$$j_{\nu,1}^2 < 4\left(1 - q^{\nu+1}\right)\left(1 - q^{\nu+3}\right)/\left[q^{\nu+1}(1+q)\right],$$

which, in the limit as $q \to 1$, leads to the upper bound in (4.28).

Next, the authors examine the convexity of the reciprocals of the zeros $j_{\nu,1}$ and $j_{\nu,1}(q)$ for $J_\nu(x)$ and $J_\nu^{(2)}(x; q)$. This issue is first investigated in the general case of an entire function $f_\nu(x)$ with the help of results obtained previously for zeros of $J_\nu(x)$. To use the general formula (4.8) in this case, inner product (4.10) has to be replaced with an inner product independent of ν.

The most suitable variant is the usual inner product

$$\langle u, v\rangle = \sum_{n=0}^{\infty} u_n v_n, \quad u = \{u_0, u_1 \ldots\}, \quad v = \{v_0, v_1 \ldots\}. \tag{5.20}$$

Let

$$g_{n+\nu}(x) = [a_n(v)]^{1/2} f_{n+\nu}(x). \tag{5.21}$$

Then the recurrence formula (**) becomes

$$x^{-1}g_{n+\nu}(x) = \gamma_n(\nu)\, g_{n+\nu+1}(x) + \gamma_{n-1}(\nu)\, g_{n+\nu-1}(x), \tag{5.22}$$

where

$$\gamma_n(\nu) = [a_n(\nu)a_{n+1}(\nu)]^{1/2}, \quad n = 1, 2, \ldots. \tag{5.23}$$

Now the operator T_ν is defined by the formulas

$$(T_\nu)_{m,n} = \gamma_n\delta_{m,n+1} + \gamma_m\delta_{m+1,n}, \quad m, n = 0, 1, \ldots. \tag{5.24}$$

The operator T_ν is symmetric, and its second derivative is given by

$$(T_\nu'')_{m,n} = \gamma_n''\delta_{m,n+1} + \gamma_m''\delta_{m+1,n}, \quad m,n = 0,1,\ldots, \tag{5.25}$$

where the primes denote derivatives with respect to ν. Note that the second term on the right-hand side of (4.9) is nonnegative if λ_ν is the largest eigenvalue of T_ν.

Theorem 5.6. *Let* $a_n(\nu) = c_{n+\nu+1}$ *and* Λ *be the largest eigenvalue of the operator* T_ν. *If the assumptions of Theorem* 4.1 *hold and* $\gamma_n'' > 0$ *for* $n = 1, 2, \ldots$, *then* Λ *is a convex function of* ν.

Corollary 5.1. Under the assumption of Theorem 4.2, the reciprocal of the smallest positive zero of $f_\nu(x)$ is a convex function of ν.

Taking the logarithmic derivative of $f_\nu(x)$, one can obtain the identity

$$\begin{aligned}4\gamma_n''/\gamma_n &= [\{a_n'(\nu)/a_n(\nu)\} + \{a_{n+1}'(\nu)/a_{n+1}(\nu)\}]^2 + 2[a_n'(\nu)/a_n(\nu)]^2 \\ &+ 2[a_{n+1}'(\nu)/a_{n+1}(\nu)]^2 - 2[a_n''(\nu)/a_n(\nu) + \{a_{n+1}''(\nu)/a_{n+1}(\nu)\}].\end{aligned} \tag{5.26}$$

It follows from (5.26) that $\gamma_n''(\nu)$ is positive if $a_n(\nu) = 2(n+\nu)$ or $a_n(\nu) = 4q^{1/4}\sinh((n+\nu)w)$, where $w = ((\ln q))/2$.

This proves the following result.

Theorem 5.7. *Both* $1/j_{\nu,1}$ *and* $1/j_{\nu,1}(q)$ *are convex functions of* ν *when* $\nu > -1$.

The convexity of $1/j_{\nu,1}$ can be derived from the formula

$$(1/j_{\nu,1})'' = j_{\nu,1}''/j_{\nu,1}^2 + 2(j')^2/j^3, \quad j = j_{\nu,1},$$

on using $j_{\nu,1} > 0$ and the result $j_{\nu,1}'' < 0$ for $\nu > -1$ from [6].

A finite-dimensional version of this theorem is also proposed.

In this case, T_ν is replaced by an $N \times N$ matrix obtained by deleting the rows $N, N+1, \ldots$ and the columns $N, N+1, \ldots$ from T_ν. The new matrix is defined on $\mathbb{R}^N$, and its eigenvalues are the zeros of $\psi_N^*(x)$. It follows that the largest zero of $\psi_N^*(x)$ is a convex function of ν if $j_n'' \geq 0$ for $1 \leq n \leq N$. In this case, $\psi_N^*(x)$ is the Lommel polynomial $h_{N,\nu+1}(x)$ or the q-Lommel polynomial $h_{N,\nu+1}(x;q)$ if $\nu > -1$.

As $N \to \infty$, the largest zeros of $h_{N,\nu+1}(x)$ and $h_{N,\nu+1}(x;q)$ converge to $1/j_{\nu,1}$ and $1/j_{\nu,1}(q)$, respectively, which proves Theorem 5.7.

CONCLUSIONS

This paper continued the study concerning the properties of positive real zeros of first and second kind Bessel functions begun in the first and second parts of this work. Specifically, we discussed the monotonicity, convexity, and concavity of zeros of the Bessel functions $J_\nu(x)$, $Y_\nu(x)$, and $C_\nu(x)$, squared zeros, zeros of q-Bessel functions, etc.

This study was conducted as carefully as in the previous parts of the work.

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