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Numerical Continuation of Solution at a Singular Point of High Codimension for Systems of Nonlinear Algebraic or Transcendental Equations

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Abstract—Numerical continuation of solution through certain singular points of the curve of the set of solutions to a system of nonlinear algebraic or transcendental equations with a parameter is considered. Bifurcation points of codimension two and three are investigated. Algorithms and computer programs are developed that implement the procedure of discrete parametric continuation of the solution and find all branches at simple bifurcation points of codimension two and three. Corresponding theorems are proved, and each algorithm is rigorously justified. A novel algorithm for the estimation of errors of tangential vectors at simple bifurcation points of a finite codimension m is proposed. The operation of the computer programs is demonstrated by test examples, which allows one to estimate their efficiency and confirm the theoretical results.

Keywords: singular point, simple bifurcation point, codimension, Lyapunov–Schmidt reduction, bifurcation equation, Levin's method, continuation method, nonlinear algebraic and transcendental equations.

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1. INTRODUCTION

Many mathematical problems that model various physical phenomena are described by systems of nonlinear algebraic or transcendental equations (or can be reduced to such systems)

$$F(x) = 0, \tag{1}$$

where $x \in \mathbb{R}^{n+1}$, $F: \mathbb{R}^{n+1} \to \mathbb{R}^n$, and F is a sufficiently smooth function.

The system of equations (1) determines a locus of points in the space $\mathbb{R}^{n+1}$ of variables $\{x_1, x_2, \ldots, x_{n+1}\}$ that is called the solution curve of Eq. (1).

The numerical construction of this curve can be a challenging task because the curve can contain stable and unstable segments, as well as limit and bifurcation points (e.g., see [1–11]).

In [10, 11] when passing through a limit singular point x_0, i.e., a point at which the rank of the Jacobian matrix $J(x)$ of the function $F(x)$

$$J(x_0) = F'(x_0) = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial x_2} & \cdots & \dfrac{\partial F_1}{\partial x_{n+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial F_n}{\partial x_1} & \dfrac{\partial F_n}{\partial x_2} & \cdots & \dfrac{\partial F_n}{\partial x_{n+1}} \end{pmatrix}_{x=x_0} \tag{2}$$

is n, the technique of changing the parameter was used, which was first proposed in [12].

In [6], the numerical construction of the solution curve of system (1) using the parametric continuation method was investigated. It was shown that the transformation of the system to the best parameter (see [13]), which is the length of the solution curve arc, solves the problem of the construction of the curve at the limit singular points. In this case, no change of parameter is needed.

In [14], a more intricate and more important for applications case is considered: this is the construction of the numerical solution to the system of equations (1) in a neighborhood of the singular point x_0 at which the rank of the Jacobian matrix of the function F is less than n:

$$\operatorname{rank} F'(x_0) = n - 1. \tag{3}$$

Such a point is called *the bifurcation point of codimension one*. This paper is a logical continuation of paper [14]. It is devoted to the solution of a more difficult problem of constructing a numerical solution of the system of equations (1) subject to conditions (2), (3) in a neighborhood of the singular point at which the rank of the Jacobian matrix of the function F is $n - 2$ and $n - 3$. Such points are called the bifurcation points of codimensions two and three, respectively.

The localization of the bifurcation point and the analysis of the solution behavior in a neighborhood of such a point is a difficult problem. The main tools for its solution are local parametric continuation methods (see [15]). This manifests itself in that the calculation of the Jacobian matrix and its inversion for finding the nearest points of the solution is an essential feature of these methods. Various versions of the continuation method that make all the variables equivalent use a unified algorithm for the continuation of solution at regular and limit points of the solution set of nonlinear systems of equations. The analysis of the solution behavior at bifurcation points requires additional methods. As the main method, we use expansion in the Taylor series in the neighborhood of the singular point. It makes it possible to make up the bifurcation equation (the Lyapunov–Schmidt reduction) that determines both the number of branches of the solution and their behavior in the neighborhood of the singular point under examination. The complexity of the analysis depends on the degree of singularity of the Jacobian matrix F'.

The approximate construction and investigation of bifurcation equations, which were started by Lyapunov in [16] and Schmidt in [17], are studied in numerous works (e.g., see [1, 3, 4, 15, 18–21]). The studies that are closely related to the topic of the present paper are discussed in [14].

Book [1] is devoted to the computational aspects of bifurcation theory and to various methods for solving nonlinear equations, while [2] and [6] are devoted to the analysis of the continuation of solution at singular points of codimension one and two. It is assumed that the solutions to the system of equations (1) are described by smooth curves $x = x(\lambda)$, where λ is the curve arc length. This makes it possible to use the expansion in the Taylor series for the analysis of the behavior at the singular point x_0. It is assumed that the original space $\mathbb{R}^{n+1}$ is decomposed in the direct sum of two subspaces P^r and A^d. The space A^d is spanned by the vectors that are orthogonal to the rows of the Jacobian matrix $J_0 = J(x_0)$ the rank of which is r, and $d = n + 1 - r$ is the degree of singularity of the problem. This makes it possible to obtain two results: the tangent vector of any branch of the curve belongs to the space A^d, and the analysis of the original problem of dimension $n + 1$ is equivalent to the analysis of another problem of dimension d. This last problem is to solve the bifurcation equation. In the case of double ($\operatorname{rank} J_0 = n - 1$) and triple ($\operatorname{rank} J_0 = n - 2$) singularity, the bifurcation equation is constructed in the first approximation.

An important approach when dealing with singular points is the parametric solution continuation method, where the best parameter, which is the arc length of the curve containing the singular point, is used. For example, in [3] various aspects of the application of the parametric continuation method with the length of the solution curve used as the parameter are discussed. The presentation briefly touches some issues concerning the detection and analysis of bifurcation points. Note that this approach is highly efficient, and it has already been used for the numerical parametric approximation of curves (see [22–25]), in iterative procedures of overcoming limit singular points in the numerical solution of nonlinear boundary value problems by the shooting method (see [26–28]), and for the numerical construction of curves (see [29–33]).

2. NUMERICAL CONTINUATION OF SOLUTION AT SINGULAR POINTS OF CODIMENSION TWO

In this section, we consider the numerical continuation of solution through certain singular points of codimension two on the solution curve of the system of nonlinear algebraic or transcendental equations (1) containing a parameter. An algorithm for constructing all the branches of the curve at a simple bifurcation point of codimension two is proposed, and the operation of a computer program implementing this algorithm is demonstrated using an example. This example confirms the theoretical results. A novel approach for finding the tangential vectors at bifurcation points of codimension two is proposed.

2.1. Behavior of the Solution Curve at a Simple Bifurcation Point of Codimension Two

Let the curve be given be system (1) and $F \in C^3(\mathbb{R}^{(n+1)})$.

Definition 2.1. The point x_0 is called a simple bifurcation point of codimension two of Eq. (1) if the following conditions are fulfilled:

(1) $F(x_0) = 0$,

(2) rank $F' = n - 2$,

(3) the Hessian matrices of the functions $V_1(\alpha_1, \alpha_2, \alpha_3) = w_1^{\mathrm{T}} F(x_0 + \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3)$ and $V_2(\alpha_1, \alpha_2, \alpha_3) = w_2^{\mathrm{T}} F(x_0 + \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3)$ have nonzero eigenvalues of different signs at the origin of coordinates, i.e., at $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Here, a_1, a_2, a_3 are linearly independent vectors in the space $N(F'(x_0))$, $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, and $w_1, w_2 \in N([F'(x_0)]^{\mathrm{T}})$, where $N(\cdot)$ is the null-space of the corresponding operator.

The point x_0 satisfies Condition (2); therefore, the dimension of the null-space of the operator $F'(x_0)$ is equal to three: dim $N(F'(x_0)) = 3$.

Let the vectors a_1, a_2, a_3 form a basis in this space. Let us orthonormalize and transpose the linearly independent rows of the matrix $F'(x_0)$. Denote the resulting vectors by $p_1, p_2, \ldots, p_{n-2}$. It is clear that any vector x can be represented as

$$x = \rho_1 p_1 + \ldots \rho_{n-2} p_{n-2} + \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3.$$

Consider the functions F_{i_k} ($i_k \in 1, 2, \ldots, n$; $k = 1, 2, \ldots, n-2$) whose gradient components form the linearly independent rows of the matrix $F'(x_0)$ at the point x_0. For the functions F_{i_k}, we introduce the notation $F_{i_k} = U_k$ ($k = 1, 2, \ldots, n-2$). Let w_1, w_2 be linearly independent nonzero vectors such that

$$w_i^{\mathrm{T}} F'(x_0) = 0^{\mathrm{T}}, \quad i = 1, 2. \tag{4}$$

The existence of nonzero vectors w_1 and w_2 is equivalent to the linear independence of the rows of the matrix $F'(x_0)$. Define the functions V_i by

$$V_i = w_i^{\mathrm{T}} F, \quad i = 1, 2.$$

This definition and equality (4) imply

$$\frac{\partial V_i}{\partial x_j}(x_0) = 0, \quad i = 1, 2, \quad j = 1, 2, \ldots, n+1.$$

Consider the system of equations

$$\begin{gathered} U_k(x) = 0, \quad k = 1, 2, \ldots, n-2, \\ V_i(x) = 0, \quad i = 1, 2. \end{gathered} \tag{5}$$

Lemma 2.1. *Finding a solution to system* (1) *is equivalent to finding a solution to system* (5).

This lemma is proved similarly to the proof of Lemma 1 in [14].

Let us parameterize this curve using the best parameter λ:

$$\begin{gathered} \rho = \rho(\lambda), \\ \alpha = \alpha(\lambda) \end{gathered} \tag{6}$$

that is measured from the bifurcation point x_0. In (6), we have $\rho = (\rho_1, \rho_2, \ldots, \rho_{n-2})^{\mathrm{T}}$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)^{\mathrm{T}}$.

Theorem 2.1. *At the simple bifurcation point* x_0, *it holds that*

$$\begin{gathered} \dot{\rho}_j(0) = 0, \quad j = 1, 2, \ldots, n-2, \\ a_1^{\mathrm{T}} V_i''(x_0) a_1 \dot{\alpha}_1^2(0) + a_2^{\mathrm{T}} V_i''(x_0) a_2 \dot{\alpha}_2^2(0) + a_3^{\mathrm{T}} V_i''(x_0) a_3 \dot{\alpha}_3^2(0) + 2 a_1^{\mathrm{T}} V_i''(x_0) a_2 \dot{\alpha}_1(0) \dot{\alpha}_2(0) \\ + 2 a_1^{\mathrm{T}} V_i''(x_0) a_3 \dot{\alpha}_1(0) \dot{\alpha}_3(0) + 2 a_2^{\mathrm{T}} V_i''(x_0) a_3 \dot{\alpha}_2(0) \dot{\alpha}_3(0) = 0, \quad i = 1, 2. \end{gathered} \tag{7}$$

This theorem is proved similarly to the proof of Theorem 1 and the corollary to it in [14].

Theorem 2.1 allows us to understand how a solution to the system can be constructed in the first approximation. In a small neighborhood of the simple bifurcation point, an approximate solution is given by the intersection of two cones K_1 and K_2

$$K_i = \sum_{k,j=1}^{3} a_k^{\mathrm{T}} V_i''(x_0) a_j \dot{\alpha}_k(0)\dot{\alpha}_j(0), \quad i = 1,2.$$

Let us analyze the matrices of the quadratic forms of the cones K_1 and K_2 using the same symbols K_1 and K_2 for the cones themselves. Without loss of generality, we assume that the apexes of the cones coincide with the origin of coordinates.

It is clear that if at least one of the quadratic forms is sign definite, then the singular point is isolated; indeed, in this case at least one equation in (7) has no other real roots except for the trivial one. However, the isolated point cannot be attained in the process of the well defined parametric continuation of the solution.

Consider the case when both quadratic forms are not sign definite. Then, each equation in (7) has a real set of solutions, and finding the solution of the system is reduced to finding the intersection of these sets.

An algorithm for solving this problem consists of the following steps.

1. Reduction of the cone K_1 to the canonical form $\tilde{K}_1 = T_1^{\mathrm{T}} K_1 T_1$. This is equivalent to finding the eigenvectors and eigenvalues of the quadratic form K_1.

2. Reduction of the cone K_1 to the normal form with the axis $0z$: $\bar{K}_1 = T_2^{\mathrm{T}} \tilde{K}_1 T_2$.

3. Making up the equation of the cone K_2 in the new coordinates: $\bar{K}_2 = (T_1T_2)^{\mathrm{T}} K_1 T_1 T_2$.

4. Finding the intersection of the cones $\bar{K}_1$ and $\bar{K}_2$ in the plane $z = 1$. Substitute the points $p = (\cos\phi, \sin\phi, 1)$ into the second cone $\bar{K}_2$ and check the residual. If the residual is less than a given ε, then we assume that the desired point is the intersection point p^* of the two cones.

5. Find the intersection points in the original coordinates and return to the original coordinates $P^* = T_1T_2p^*$.

6. The vectors OP^* can be considered to be approximations of the tangential vectors of the bifurcation branches.

Let us discuss this algorithm in more detail. As a result of the transformations made above, K_1 became a circular cone, and its axis coincides with the axis $0z$. The second cone K_2 generally remains elliptic.

Let us intersect the cones by the plane $z = 1$. The intersection line of this plane with K_1 is a circle, and its intersection with K_2 is an ellipse or a hyperbole, depending on the relative position of the cones. Thus, the problem of finding the real roots of the last two equations in (7) is reduced to finding the intersection points of a circle with an ellipse or a hyperbole on the plane $z = 1$. Various relative positions of a circle and an ellipse are shown in Fig. 1. It is seen that the number of real roots of the last two equations in (7) can be in the range from zero through four. Note that the sign indefiniteness of the quadratic forms does not guarantee the existence of real solutions to the system. This is happens in the cases shown in Figs. 1a–1c.

An unambiguous conclusion about the branching can be drawn only in the cases shown in Figs. 1j and 1k. In the first of them, two branches of the solution intersect at the bifurcation point, and both branches touch two common generatrices of the cones. Therefore, the solution from the singular point can be continued in four directions as shown in Fig. 2. In the plane passing through the common generatrices of the cones, the possible bifurcation pattern is shown in Fig. 3. In the second case, four branches of the solution touching four common generatrices of the cones intersect at the singular point. Here, the branches do not lie in the same plane, and the solution from the singular point can be continued in eight directions.

In the cases shown in Figs. 1d–1i, we have the touching cones. The corresponding common generatrix can be tangent to two or more branches of the solution that touch each other at the singular point. To find these solutions, the bifurcation equations with account for the higher order terms of the Taylor series must be considered, and this must be done in the plane that touches both cones on their common generatrix. This simplifies the analysis because the number of variables is reduced and becomes equal to two; i.e., for each common generatrix of the cones, a two-dimensional subspace (plane) is singled out, and the bifurcation equation should be examined in this subspace.

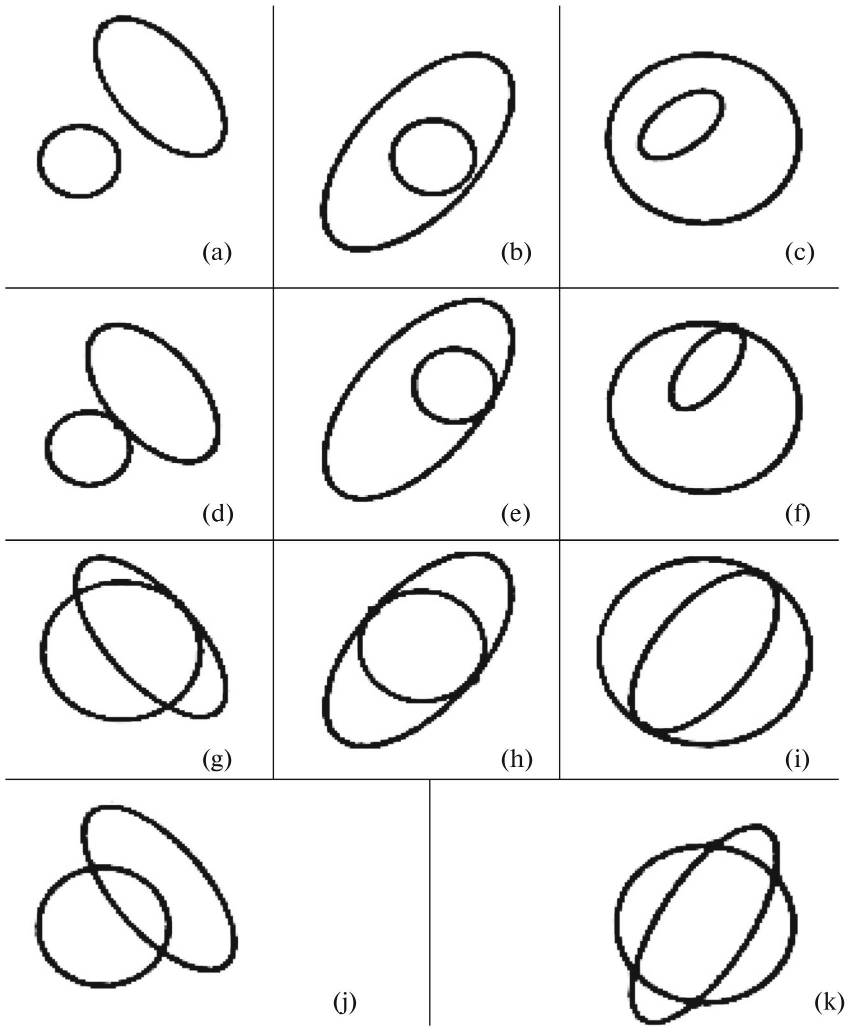

Fig. 1.

Example.

Consider the function $F:\mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$F(x,y,z) = \begin{pmatrix} x^2 - 2yz \\ \sqrt{2}xy - \sqrt{2}xz - yz + \dfrac{y^2}{2} + \dfrac{z^2}{2} \end{pmatrix}.$$

The set of solutions to Eq. (1) with this function is the intersection of two cones.

The matrices of these cones have the form

$$K_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$K_2 = \begin{pmatrix} 0 & \dfrac{\sqrt{2}}{2} & -\dfrac{\sqrt{2}}{2} \\ \dfrac{\sqrt{2}}{2} & \dfrac{1}{2} & -\dfrac{1}{2} \\ -\dfrac{\sqrt{2}}{2} & -\dfrac{1}{2} & \dfrac{1}{2} \end{pmatrix}.$$

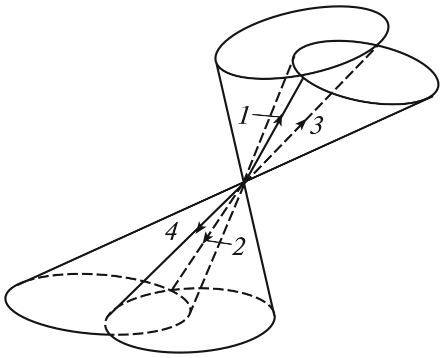

Fig. 2.

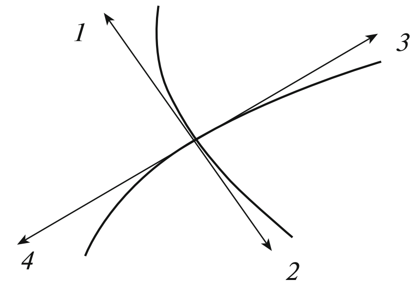

Fig. 3.

Let us illustrate the operation of the algorithm described above. It was implemented in MATLAB.

Reduce K_1 to the canonical form

$$\tilde{K}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.7071 & -0.7071 \\ 0 & 0.7071 & 0.7071 \end{pmatrix}.$$

The cone K_1 reduced to the normal form with the axis OZ is

$$\bar{K}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and

$$T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The cone K_2 in the new coordinates is

$$\bar{K}_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us find the intersection points of $\bar{K}_1$ and $\bar{K}_2$ in the plane $z = 1$. The tolerable residual is set to $\varepsilon = 0.01$. Then,

$$p_1^* = (0.4484, 0.8938, 1.0000),$$

$$p_2^* = (-0.4484, -0.8938, 1.0000).$$

In the original coordinates, the cone intersection points are given by $P_i^* = T_1 T_2 p_i^*$. For example,

$$P_1^* = (0.4484, 0.0751, 1.3391).$$

The vectors OP_i^* $(i = 1, 2)$ can be considered to be approximations of the tangent vectors of the bifurcation branches.

2.2. Finding All Vectors at a Bifurcation Point Using the Determination of the Tangent Vectors

Consider another method for constructing all branches at a bifurcation point of codimension two.

The sequence of steps below produces the intersection curve of two cones in explicit form.

1. Find the change of coordinates T that reduces the first cone to the canonical form. Let its axis coincide with the axis $0z$; then, the equation of the cone is

$$x^2 + y^2 - z^2 = 0.$$

2. Parameterize this cone as

$$x = tu, \quad y = \pm t\sqrt{1-u^2}, \quad z = t, \tag{8}$$

where $t \in \mathbb{R}$, $-1 \le u \le 1$.

3. Use the matrix T to make up the equation of the second cone in the new coordinates:

$$b_{11}x^2 + b_{12}xy + b_{22}y^2 + b_{23}xz + \ldots = 0. \tag{9}$$

Substitute (8) into Eq. (9). Since both cones pass through the origin of coordinates and, therefore, the equations are homogeneous, the parameter t remains undefined.

4. To find the parameter u, the irrational equation $F(u) = 0$ should be solved:

$$b_{11}u^2 \pm b_{12}u\sqrt{1-u^2} + b_{22}(1-u^2) + b_{23}u + \ldots = 0.$$

Let u_i be the roots of this equation.

5. The parametric equation of the intersection curve is then written as

$$x = tu_i, \quad y = \pm t\sqrt{1-u_i^2}, \quad z = t.$$

6. The tangent vectors to the branches of the curve are found by the formulas

$$v_i = (u_i, \sqrt{1-u_i^2}, 1).$$

Now, the tangent vectors can be used to obtain approximations of the next point after the bifurcation point on the branch of interest.

3. NUMERICAL CONTINUATION OF SOLUTION AT SINGULAR POINTS OF CODIMENSION THREE

Using the Lyapunov–Schmidt reduction, the problem is reduced to constructing the intersection of three quadric surfaces in $\mathbb{R}^4$. The homogeneity of equations makes it possible to reduce the number of variables. Next, Levin's method described in [34] is used to write the intersection of two surfaces in a parametric form. This parameterization is used for substituting into the third equation. Thus, the problem is reduced to solving a nonlinear equation in one unknown in a bounded domain. A computational algorithm for passing through simple bifurcation points of codimension three is proposed.

3.1. The Behavior of the Solution Curve at a Simple Bifurcation Point

Let the curve be specified by Eq. (1) and the function $F \in C^3(\mathbb{R}^{(n+1)})$.

Let a_i be linearly independent vectors in $N(F'(x_0))$, and $w_i \in N([F'(x_0)^{\mathrm{T}})$, where $N(\cdot)$ is the null-space of the corresponding operator.

Let rank $F'(x_{(0)}) = n - m$. Then, the dimension of the null-space of the operator $F'(x_0)$ at the point x_0 is $m + 1$: dim $N(F'(x_0)) = m + 1$.

Let the vectors a_i form a basis in this space. Let us orthonormalize and transpose the linearly independent rows of the matrix $F'(x_0)$. The resulting vectors will be denoted by $p_1, p_2, \ldots, p_{n-m}$. It is clear that any vector can be represented in the form

$$x = \rho_1 p_1 + \ldots + \rho_{n-m}p_{n-m} + \alpha_1 a_1 + \ldots + \alpha_{m+1}a_{m+1}.$$

Consider the functions F_{i_k} ($i_k \in$ 1, 2, ..., n; k = 1, 2, ..., $n - m$) the gradient components of which form at the point x_0 linearly independent rows of the matrix $F'(x_0)$. For the functions F_{i_k}, we introduce the notation $F_{i_k} = U_k$, k = 1, 2, ..., $n - m$. Let w_i be linearly independent nonzero vectors such that

$$w_i^{\mathrm{T}} F'(x_0) = 0^{\mathrm{T}}, \quad i = 1, \dots, m. \tag{10}$$

The existence of nonzero vectors w_i is equivalent to the linear independence of the rows of $F'(x_0)$. Define the functions V_i by

$$V_i = w_i^{\mathrm{T}} F, \quad i = 1, \dots, m.$$

It follows from the definition of these functions and from equality (10) that

$$\frac{\partial V_i}{\partial x_j}(x_0) = 0, \quad i = 1, \dots, m, \quad j = 1, 2, \dots, n+1.$$

Consider the system of equations

$$\begin{gathered} U_k(x) = 0, \quad k = 1, 2, \dots, n - m, \\ V_i(x) = 0, \quad i = 1, 2, \dots, m. \end{gathered} \tag{11}$$

Lemma 3.1. *Solving system* (1) *is equivalent to solving system* (11).

The proof is similar to the proof of Lemma 1 in [14].

Definition 3.1. The point x_0 is called a simple bifurcation point of codimension m of Eq. (1) if the following conditions are fulfilled:

(1) $F(x_0) = 0$,

(2) rank $F' = n - m$,

(3) the Hessian matrices of the functions $V_i(\alpha_1, \alpha_2, \dots, \alpha_{m+1})$ (i = 1, 2, ..., m) have nonzero eigenvalues of different signs at the origin of coordinates, i.e., at $\alpha_1 = \alpha_2 = \dots = \alpha_{m+1} = 0$.

Let us parameterize the curve using the best parameter λ by formula (6), where the parameter is measured from the bifurcation point x_0.

Take into account the fact that in (6)

$$\rho = (\rho_1, \rho_2, \dots, \rho_{n-m})^{\mathrm{T}}, \quad \alpha = (\alpha_1, \alpha_2, \dots, \alpha_{n-m})^{\mathrm{T}}.$$

Theorem 3.1. *At the simple bifurcation point* x_0, *it holds that*

$$\begin{gathered} \dot{\rho}_k(0) = 0, \quad k = 1, 2, \dots, n - m, \\ \sum_{k,j=1}^{m+1} \alpha_k^{\mathrm{T}} V_i''(x_0) a_j \dot{\alpha}_k(0) \dot{\alpha}_j(0) = 0, \quad i = 1, 2, \dots, m. \end{gathered} \tag{12}$$

This theorem is proved similarly to the proof of Theorem 1 and the corollary to it in [14].

3.2. Levin's Method

Levin's method makes it possible to explicitly represent the curve of the intersection of two quadric surfaces.

Assume that $P(x_1, x_2, x_3) = 0$ and $Q(x_1, x_2, x_3) = 0$ are two different quadric surfaces. Construct the function

$$R(\lambda, x_1, x_2, x_3) = P(x_1, x_2, x_3) - \lambda Q(x_1, x_2, x_3), \quad \lambda \in \mathbb{R}.$$

For brevity, we will write $R(\lambda)$ meaning $R(\lambda, x_1, x_2, x_3)$. In addition, the matrix related to the quadric surface

$$P(x_1, x_2, x_3) = \sum_{i,j=1}^{3} a_{ij} x_i x_j + 2\beta_1 x_1 + 2\beta_2 x_2 + 2\beta_3 x_3 + \beta_4 = 0, \tag{13}$$

will be denoted by the same symbol P as the surface itself.

Parameterization of simple ruled quadrics

Surface	Canonical equation	Parameterization
Straight line	$a_1x_1 + a_2x_2 = 0$	$X(u) = [0, 0, u]$
Simple plane	$x_1 = 0$	$X(u, v) = [0, u, v]$
Double plane	$a_1x_1^2 = 0$	$X(u, v) = [0, u, v]$
Parallel planes	$a_1x_1^2 = 1$	$X(u, v) = \left[\frac{1}{\sqrt{a_1}}, u, v\right]$
		$X(u, v) = \left[-\frac{1}{\sqrt{a_1}}, u, v\right]$
Intersecting planes	$a_1x_1^2 - a_2x_2^2 = 0$	$X(u, v) = \left[\frac{u}{\sqrt{a_1}}, \frac{u}{\sqrt{a_2}}, v\right]$
		$X(u, v) = \left[\frac{u}{\sqrt{a_1}}, -\frac{u}{\sqrt{a_2}}, v\right]$
Hyperbolic paraboloid	$a_1x_1^2 - a_2x_2^2 - x_3 = 0$	$X(u, v) = \left[\frac{u+v}{2\sqrt{a_1}}, \frac{u+v}{2\sqrt{a_2}}, uv\right]$
Parabolic cylinder	$a_1x_1^2 - x_2 = 0$	$X(u, v) = [u, a_1u^2, v]$
Hyperbolic cylinder	$a_1x_1^2 - a_2x_2^2 = 1$	$X(u, v) = \left[\frac{1}{2\sqrt{a_1}}\left(u + \frac{1}{u}\right), \frac{1}{2\sqrt{a_2}}\left(u + \frac{1}{u}\right), v\right]$

Then, (13) can be rewritten in matrix form $X^{\mathrm{T}}PX = 0$, where $X = (x_1, x_2, x_3, 1)^{\mathrm{T}}$ and P is a symmetric fourth-order matrix

$$P = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \beta_1 \\ \alpha_{12} & \alpha_{22} & \alpha_{23} & \beta_2 \\ \alpha_{13} & \alpha_{23} & \alpha_{33} & \beta_3 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}.$$

The submatrix of P corresponding to the quadratic form will be denoted by P_u:

$$P_u = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{pmatrix}.$$

Theorem 3.2 (see [34]). *The solution to the equation $R(\lambda) = 0$ is either an empty set or contains one surface from the table.*

Levin's method consists of the following steps.

1. From the solution to the equation $R(\lambda) = 0$, the simple surface in the table is found, which is associated with the value of the multiplier λ that is a root of the equation $\det R_u(\lambda) = 0$.

2. Using the linear transformation T, this simple surface is reduced to the canonical form. For this surface, the corresponding parameterization from the table is substituted into the equation

$$X^{\mathrm{T}}T^{-1}PTX = a(u)v^2 + b(u)v + c(u) = 0.$$

3. In the domain of feasible values of the variable u, the parameter v is represented in terms of u, and $v = v(u)$ is substituted into the parameterization of the simple surface

$$X(u) = X(u, v(u)).$$

Thus, we obtain a parameterization of the intersection of the surfaces $P = 0$ and $Q = 0$.

4. Using the transformation T, return to the original space; i.e., the desired parameterization has the form $TX(u)$.

3.3. Implementation of the Continuation Algorithm at a Simple Bifurcation Point

The system of equations (12) implies that its solution is reduced to finding the intersection of three quadric surfaces

$$F_i(x_1, x_2, x_3, x_4) = 0, \quad i = 1, 2, 3.$$

The left-hand side of each equation is a homogeneous function.

The method consists of the following steps.

1. Make the change of variables $x_i' = \frac{x_i}{x_4}$, which reduces the system dimension.

2. Using Levin's method, find the intersection of two quadratic forms in parametric form.

3. Substitute this parameterization into the third equation. Thus, the problem is reduced to solving an equation in one unknown.

3.4. The Error of Calculating the Tangent Vectors: The Case rank $F'(x_0) = n - m$. The Error of Calculating the Quadratic Form

An algorithm for calculating the tangent vectors was described above. Below, we will use the same notation.

Find the subspace in which the bifurcation occurs.

This problem is equivalent to finding the vectors $a_1, a_2, \ldots, a_{m+1}$ that form a basis in this subspace. These vectors can be found from the condition

$$F'(x_0)a_i = 0, \quad i = 1, 2, \ldots, m+1. \tag{14}$$

Here $F'(x_0)$ is the Jacobian matrix and x_0 is the singular point. System (14) can be solved as follows (see [35]). Extend the matrix $F'(x_0)$ by zero rows and denote the resulting square matrix by A. Use the decomposition

$$A^{\mathrm{T}} = QR, \tag{15}$$

where Q is an orthogonal matrix and R is an upper triangular matrix

$$R = \begin{pmatrix} R_1 \\ O \end{pmatrix}.$$

Here, R_1 is a matrix of size $(n - m) \times (n + 1)$, and O is a zero matrix of size $(m + 1) \times (n + 1)$. Using (15) and the equality $Q^{\mathrm{T}} = Q^{-1}$, we obtain

$$AQ = (AQ_1|AQ_2) = R^{\mathrm{T}} = (R_1^{\mathrm{T}}|Q^{\mathrm{T}}).$$

Here we took into account that Q can be represented as a union of two block matrices Q_1 of size $(n + 1) \times (n - m)$ and Q_2 of size $(n + 1) \times (m + 1)$. The vertical line within the parenthesis separates the blocks of the matrix. The matrix Q_2 contains the last $m + 1$ columns of Q, which are the orthonormal vectors a_1, $a_2, \ldots, a_{m+1}$.

The approximation vectors Q_i of the vectors a_i are found by formulas similar to (14):

$$AQ_i = (R^{\mathrm{T}})_i, \quad i = n - m, \ldots, n, \tag{16}$$

where $(R^{\mathrm{T}})_i$ $(i = n - m, \ldots, n)$ are the last columns of the matrix R^{T}.

We may consider the deviation of the vectors from the space formed by the vectors a_i, (i.e., from the null-space $N(J)$) as the error in determining these vectors. This is related to the fact that the vectors a_i are determined not uniquely. The deviation of the ith vector can be found by the formula

$$Q_i - \sum_{k=1}^{m+1}(Q_i, a_k)a_k = \varepsilon_i, \quad i = n - m, \ldots, n. \tag{17}$$

The number $\varepsilon = \max_i|\varepsilon_i|$ can be considered as a measure of deviation of the system of vectors Q_i from the space $N(J)$.

Theorem 3.3. *The error* ε *satisfies the equality*

$$\varepsilon = \max_i \max_j |\varepsilon_{ij}|, \quad i = n-m, \ldots n, \quad j = 1, 2, \ldots, n-m,$$

where $\varepsilon_{ij} = (Q_i, p_j)$ *and* p_j $(j = 1, 2, \ldots, n-m)$ *are orthonormal linearly independent row vectors of the matrix* J.

Proof. Since a_i form a basis in the space $N(J)$, the error vectors ε_i do not belong to $N(J)$. Therefore, ε_i are orthogonal to $N(J)$.

Let p_j be vectors such that $|p_j| = 1$ and $(p_j, a_i) = 0$ for $i = 1, 2, \ldots, m+1$ and $j = 1, 2, \ldots, n-m$.

Multiply equality (17) by p_j. Since p_j are orthogonal to a_i, we have

$$\varepsilon_{ij} = (Q_i, p_j) = (\varepsilon_i, p_j), \quad i = n-m, \ldots, n \quad j = 1, 2, \ldots, n-m.$$

In addition, the preceding equality implies the decomposition

$$\varepsilon_i = \sum_{j=1}^{n-m} \varepsilon_{ij} p_j, \quad i = n-m, \ldots, n.$$

Define the norm

$$\|\varepsilon_i\| = \max_j \varepsilon_{ij}, \quad i = n-m, \ldots, n, \quad j = 1, 2, \ldots, n-m.$$

Then, the error can be determined by the formula

$$\varepsilon = \max_i \|\varepsilon_i\| = \max_i \max_j |\varepsilon_{ij}|, \quad i = n-m, \ldots, n, \quad j = 1, 2, \ldots, n-m.$$

The theorem is thus proved.

The vector v that is tangent to the curve at the bifurcation point can be represented by

$$v = \sum \alpha_i a_i = \sum \tilde{\alpha}_i \tilde{a}_i + \sum \Delta\alpha_i \tilde{a}_i + \sum \tilde{\alpha}_i \Delta a_i + \sum \Delta\alpha_i \Delta a_i,$$

where $\alpha_i = \tilde{\alpha}_i + \Delta\alpha_i$, $a_i = \tilde{a}_i + \Delta a_i$, and $\tilde{a}_i$ and $\tilde{\alpha}_i$ are the exact values of the vectors to be found and of the expansion coefficients.

Therefore, the error of determining the tangent vector can be bounded above as follows:

$$\left\|v - \sum \alpha_i a_i\right\| \le \sum c_{1i} |\Delta\alpha_i| + \sum c_{2i} \|\Delta a_i\| + \sum |\Delta\alpha_i| \|\Delta a_i\|.$$

Here, $c_{1i} = \|\tilde{a}_i\|$, $c_{2i} = \|\tilde{\alpha}_i\|$.

The number ε_i is the error of the computation of the vector a_i; i.e., it is equal to $\|\Delta a_i\|$.

The error $|\Delta\alpha_i|$ of the computation of the coefficient α_i can be found based on the following considerations. To find α_i, we should solve the system of equations

$$V(\alpha) = 0,$$

$$\|\alpha\| = 1.$$

Therefore, $\alpha_i = \alpha_i(a_1, \ldots, a_{m+}1)$, and the approximate expression for the error can be obtained from the approximation

$$\Delta\alpha_i \approx \sum \frac{\partial \alpha_i}{\partial a_j} \Delta a_j.$$

In the computations, the derivatives $\dfrac{\partial \alpha_i}{\partial a_j}$ may be replaced by their finite difference approximations.

To clarify the situation, consider a simpler case obtained from the general case when the rank of the Jacobian matrix rank $F'(x_0) = n-1$, i.e., when $m = 1$. Then, a corollary to Theorem 3.3 is as follows.

Corollary. The error ε satisfies the equality

$$\varepsilon = \max_i \max_j |\varepsilon_{ij}|, \quad i = n-1, n, \quad j = 1, 2, \ldots, n-1,$$

where $\varepsilon_{ij} = (Q_i, p_j)$ and p_j $(j = 1, 2, \ldots, n-1)$ are the orthonormal linearly independent row vectors of the matrix J.

In this case, the vector v that is tangent to the curve at the bifurcation point can be written as

$$v = \alpha_1 a_1 + \alpha_2 a_2 = \tilde{\alpha}_1\tilde{a}_1 + \Delta\alpha_1\tilde{a}_1 + \tilde{\alpha}_1\Delta a_1 + \tilde{\alpha}_2\tilde{a}_2 + \Delta\alpha_2\tilde{a}_2 + \tilde{\alpha}_2\Delta a_2 + \Delta\alpha_1\Delta a_1 + \Delta\alpha_2\Delta a_2,$$

where $\alpha_i = \tilde{\alpha}_i + \Delta\alpha_i$, $a_i = \tilde{a}_i + \Delta a_i$, $\tilde{a}_i$ and $\tilde{\alpha}_i$ are the exact values of the vectors to be found and of the expansion coefficients.

Therefore, the error of determining the tangent vector can be bounded above as follows:

$$\|v - (\alpha_1 a_1 + \alpha_2 a_2)\| \le c_{11}|\Delta\alpha_1| + c_{21}\|\Delta a_1\| + c_{12}|\Delta\alpha_2| + c_{22}\|\Delta a_2\| + |\Delta\alpha_1|\|\Delta a_1\| + |\Delta\alpha_2|\|\Delta a_2\|.$$

Here, $c_{1i} = \|\tilde{a}_i\|$, $c_{2i} = \|\tilde{\alpha}_i\|$.

The number ε_i is the error of the computation of the vector a_i; i.e., it is equal to $\|\Delta a_i\|$.

The error $|\Delta\alpha_i|$ of the computation of the coefficient α_i can be found based on the following considerations. To find the coefficients α_i, the following system of equations should be solved:

$$H_{11}\alpha_1^2 + 2H_{12}\alpha_1\alpha_2 + H_{22}\alpha_2^2 = 0,$$
$$\alpha_1^2 + \alpha_2^2 = 1.$$

Let, for definiteness, $H_{11} \ne 0$. Then, using the change of variables $t = \frac{\alpha_1}{\alpha_2}$, we solve the quadratic equation $H_{11}t^2 + 2H_{12}t + H_{22} = 0$. The roots t_i ($i = 1, 2$) depend on $H_{ij} = H_{ij}(a_1, a_2)$.

Finally, we obtain

$$\alpha_1 = t_i\alpha_2,$$
$$\alpha_2 = \frac{1}{\sqrt{1 + t_i^2}}, \quad i = 1, 2.$$

Therefore, $\alpha_i = \alpha_i(a_1, a_2)$, and the approximate expression for the error can be found from the approximation

$$\Delta\alpha_i \approx \frac{\partial\alpha_i}{\partial a_1}\Delta a_1 + \frac{\partial\alpha_i}{\partial a_2}\Delta a_2.$$

In the computations, the derivatives $\frac{\partial\alpha_i}{\partial a_j}$ may be replaced by their finite difference approximations.

3.5. Test Example of the Operation of the Program of Passing through Bifurcation Points of Codimension Three

Consider the function $F: \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$F(x, y, z, t) = \begin{pmatrix} x^2 + y^2 + z^2 - 2t^2 \\ x^2 + y^2 - t^2 \\ x^2 - t^2 \end{pmatrix}.$$

We are interested in the set of solutions to Eq. (1) with this function. It is clear that the origin of coordinates is a bifurcation point of codimension three for the Jacobian matrix of this system of equations. Since the function is homogeneous, the program reduces the number of variables by dividing each equation by t^2, and the system of equations (1) takes the form

$$F(x, y, z) = \begin{pmatrix} x^2 + y^2 + z^2 - 2 \\ x^2 + y^2 - 1 \\ x^2 - 1 \end{pmatrix} = 0. \tag{18}$$

Next, using Levin's method, the intersection curve of the surfaces specified by the first two equations in (18) is found. The auxiliary surface

$$R(\lambda) = (x^2 + y^2 + z^2 - 2) + \lambda(x^2 + y^2 - 1) = 0. \tag{19}$$

is determined. The matrix of this surface is constructed:

$$R = \begin{pmatrix} 1+\lambda & 0 & 0 & 0 \\ 0 & 1+\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2-\lambda \end{pmatrix}.$$

The determinant of the matrix of the surface quadratic form is calculated:

$$\det R_u(\lambda) = (1+\lambda)^2.$$

Then, the equation $\det R_u(\lambda) = 0$ is solved:

$$\lambda = -1.$$

The simple surface

$$z^2 - 1 = 0$$

is constructed, which is obtained upon the substitution of $\lambda = -1$ into Eq. (19). An appropriate parameterization

$$X = (u, v, \pm 1)$$

for this surface is selected from the table. By substituting this parameterization into the second equation $x^2 + y^2 - 1 = 0$ of system (18), we obtain

$$u^2 + v^2 - 1 = 0.$$

The parameter v is found as a function of the variable u:

$$v = \pm\sqrt{1-u^2}.$$

Thus, the program yields a parameterization of the intersection curve of two surfaces:

$$X = (u, \pm\sqrt{1-u^2}, \pm 1).$$

Upon the substitution of this parameterization into the third equation $x^2 - 1 = 0$, we obtain

$$X = (1, 0, \pm 1).$$

As the final result, the program produces the desired parameterization of the intersection curve of three surfaces

$$Y = (t, 0, \pm t, t).$$

CONCLUSIONS

Conditions under which the numerical solution can be continued beyond the bifurcation points of curves specified in the $(n+1)$-dimensional space by a system of equations whose Jacobian matrix at these points has the rank $n - 2$ or $n - 3$ are obtained. Algorithms and computer programs are developed that implement the procedure of discrete parametric continuation of the solution and find all branches at simple bifurcation points of codimension two and three. Corresponding theorems are proved, and each algorithm is rigorously justified. New results that complement and improve the results obtained by the authors earlier in [1–4, 7] are obtained.

A novel algorithm for the estimation of errors of tangential vectors at simple bifurcation points of codimension m is proposed.

Test examples demonstrate the efficient operation of computer programs implementing the proposed algorithms.

It follows from the above presentation that, in order to find the branches of solution at a singular point of codimension greater than three, a large number of situations must be examined. This makes the problem tedious and computationally complex. The study of this issue is not finished yet.

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