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Mixed Boundary Value Problems for Steady-State Magnetohydrodynamic Equations of Viscous Incompressible Fluid

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Abstract—The inhomogeneous boundary value problem for the steady-state magnetohydrodynamic equations of viscous incompressible fluid under the Dirichlet conditions for the velocity and mixed boundary conditions for the electromagnetic field is considered. Sufficient conditions for the data that ensure the global solvability of this problem and the local uniqueness of its solution are found.

Keywords: magnetohydrodynamics, viscous fluid, inhomogeneous boundary value problem, mixed boundary conditions, solvability, uniqueness.

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1. INTRODUCTION AND STATEMENT OF THE MIXED BOUNDARY VALUE PROBLEM

In recent years, particular emphasis has been given to the study of fluid dynamic processes in domains with the boundaries consisting of a number of parts with different physical properties. The mathematical modeling of such processes requires the investigation of boundary value problems for the equation of fluid dynamics and magnetohydrodynamics (MHD) under mixed boundary conditions for the velocity and other components of the solution.

Beginning with the works [1, 2], the mathematical modeling of mixed boundary value problems for the Stokes and Navier–Stokes equations was studied in a number of papers, among which we note [3–5] and the references in [5], where the solvability of the corresponding mixed boundary value problems was investigated. The MHD equations were studied beginning with [6, 7] subject to the "standard" boundary conditions. These conditions correspond to the situation when the boundary is either a perfect conductor or a perfect dielectric, or when inhomogeneous analogs of these conditions are specified on the boundary. Of interest are the papers [8–12], where the solvability of the boundary value and control problems for the MHD equations were studied under the Dirichlet conditions for the velocity. In [13–15], boundary value problems under mixed boundary conditions for the velocity were studied. A number of studies were devoted to the solvability of boundary value problems for the MHD equations under the Dirichlet condition for the magnetic field (e.g., see [16]) or under the interface conditions on the fluid–nonconducting medium interface (see [7, 17]).

However, the more general case, which is often encountered in practice, when different parts of the boundary of the flow region have different electrophysical properties has long remained unstudied in the mathematical literature. The difficulty of investigating such problems is due to the fact that the solution can have singularities at the points where the parts with different electrophysical properties described by different boundary conditions meet. This considerably complicates the analysis of solvability of the corresponding problems. Due to these difficulties, the first results concerning the solvability of boundary value problems for the MHD equations subject to mixed boundary conditions for the electromagnetic field appeared only in 2014 in [18]. The mathematical apparatus developed in [18] considerably relied on the new results for the solvability of mixed boundary value problems for the div–curl systems established in [19–21].

The aim of this paper, which continues the study started in [18], is to analyze the global solvability of the inhomogeneous mixed boundary value problem for the MHD model of incompressible fluid considered in a bounded three-dimensional domain Ω with the boundary $\Sigma = \partial\Omega$ subject to inhomogeneous mixed boundary conditions for the electromagnetic field. Under the assumption that the boundary Σ consists of two parts Σ_τ and Σ_ν of which each has certain electrophysical properties, this problem is described (in the SI system of units) by the equations

$$\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \varkappa \operatorname{curl}\mathbf{H}\times\mathbf{H} = \mathbf{f}, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\nu_1\operatorname{curl}\mathbf{H} - \mathbf{E} + \varkappa\mathbf{H}\times\mathbf{u} = \nu_1\mathbf{j}, \quad \operatorname{div}\mathbf{H} = 0, \quad \operatorname{curl}\mathbf{E} = 0 \quad \text{in} \quad \Omega, \tag{1.2}$$

$$\mathbf{u}|_\Sigma = \mathbf{g}, \quad \mathbf{H}\cdot\mathbf{n}|_{\Sigma_\tau} = q, \quad \mathbf{H}\times\mathbf{n}|_{\Sigma_\nu} = \mathbf{q}, \quad \mathbf{E}\times\mathbf{n}|_{\Sigma_\tau} = \mathbf{k}. \tag{1.3}$$

Here $\mathbf{u}$ is the velocity vector; $\mathbf{H}$ is the magnetic field vector; $\mathbf{E} = \mathbf{E}'/\rho_0$, $p = P/\rho_0$, where $\mathbf{E}'$ is the electric field vector; P is the pressure; $\rho_0 = \text{const}$ is the density of the fluid; $\varkappa = \mu/\rho_0$, $\nu_1 = 1/\rho_0\sigma = \varkappa\nu_m$, where σ, μ, and ν or ν_m are the constant coefficients of electric conductivity, magnetic permeability, and kinematic or magnetic viscosity, respectively; $\mathbf{n}$ is the unit vector of the outward normal to $\partial\Omega$; $\mathbf{g}$, $\mathbf{q}$, q, and $\mathbf{k}$ are functions defined on the boundary Σ or on different parts Σ_ν and Σ_τ of Σ. Below we refer to problem (1.1)–(1.3) with the given functions $\mathbf{f}$, $\mathbf{j}$, $\mathbf{g}$, q, $\mathbf{q}$, and $\mathbf{k}$ as to problem 1.

In the special case when $q = 0$, $\mathbf{q} = 0$, and $\mathbf{k} = 0$, the boundary conditions for the electromagnetic field in (1.3) correspond to the situation when the part Σ_τ of $\partial\Omega$ is a perfect conductor and the other part $\Sigma_\nu \subset \partial\Omega$ is a perfect dielectric. The solvability of the corresponding homogeneous boundary value problem was proved in [18]. In another special case when $\Sigma_\nu = \varnothing$ and $\Sigma_\tau = \Sigma$, the boundary conditions (1.3) take the form

$$\mathbf{u}|_\Sigma = \mathbf{g}, \quad \mathbf{H}\cdot\mathbf{n}|_\Sigma = q, \quad \mathbf{E}\times\mathbf{n}|_\Sigma = \mathbf{k}. \tag{1.4}$$

The boundary value problem (1.1), (1.2), (1.4) was investigated in [10, 11] and in [22], where a scheme for analyzing its solvability was proposed; using this scheme, the global solvability and the local uniqueness of the solution were proved under the condition that the vector field $\mathbf{g}$ is tangential.

The main difficulty in the analysis of problem (1.1)–(1.3) is due to the inhomogeneity of the boundary conditions for $\mathbf{u}$, $\mathbf{H}$, and $\mathbf{E}$ in (1.3). For this reason, before investigating the solvability of this problem, we construct vectors $\mathbf{u}_0$, $\mathbf{H}_0$, and $\mathbf{E}_0$ from certain functional classes that satisfy the corresponding boundary conditions in (1.3). Following the conventional terminology, we refer to $\mathbf{u}_0$, $\mathbf{H}_0$, and $\mathbf{E}_0$ as to liftings of the inhomogeneous boundary conditions. The main difficulty is in the construction of the magnetic lifting $\mathbf{H}_0$ that must satisfy the inhomogeneous mixed boundary conditions for $\mathbf{H}$ in (1.3). The key idea used in this paper for this purpose is to select the lifting $\mathbf{H}_0$ in the subspace of the space $L^3(\Omega)^3$ consisting of harmonic vector fields. Based on this selection of $\mathbf{H}_0$ and the possibility to construct the hydrodynamic lifting $\mathbf{u}_0 \in H^1(\Omega)^3$ with an arbitrarily small norm $\|\mathbf{u}_0\|_{L^4(\Omega)^3}$, which was proved in [11], we prove the global solvability of problem 1 and the local uniqueness of its solution using the scheme described in [11, 22].

2. FUNCTIONAL SPACES: PRELIMINARY RESULTS

Below we will use the Sobolev spaces $H^s(D)$, $s \in \mathbb{R}$, $H^0(D) \equiv L^2(D)$, where D denotes Ω, the boundary $\Sigma = \partial\Omega$, or a part of it Σ^0. The corresponding spaces of vector functions are denoted by $H^s(D)^3$ and $L^2(D)^3$. The scalar products and norms in the spaces $H^s(D)$ and $H^s(D)^3$ are denoted by $(\cdot,\cdot)_{s,D}$ and $\|\cdot\|_{s,D}$. The scalar products and norms in $L^2(\Omega)$ and $L^2(\Omega)^3$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|_\Omega$. Along with the space $H^{1/2}(\Sigma^0)$, we will use the subspace $H^{1/2}_{00}(\Sigma^0)$ consisting of the functions $u \in H^{1/2}(\Sigma^0)$ that, when continued by zero on $\Sigma\backslash\Sigma^0$, belong to $H^{1/2}(\Sigma)$. By $H^{-1/2}(\Sigma)$, we denote the dual space of $H^{1/2}(\Sigma)$ with respect to $L^2(\Sigma)$; $H^{-1/2}_{00}(\Sigma^0)$ is the dual space of $H^{1/2}_{00}(\Sigma^0)$ with respect to the space $L^2(\Sigma^0)$. For an arbitrary Hilbert space H, H^* is the dual of H.

As in [18, 20], we assume that the domain Ω and the decomposition of its boundary $\Sigma = \partial\Omega$ into the parts Σ_τ and Σ_ν satisfy the following conditions:

(i) Ω is a bounded domain in R^3, and its boundary $\partial\Omega$ consists of $m+1$ nonoverlapping closed C^2-surfaces $\Sigma_0, \Sigma_1, \ldots, \Sigma_m$ of which each has a finite area, where Σ_0 is the outer boundary of Ω.

(ii) Σ_τ is a nonempty open subset of $\partial\Omega$ consisting of $M+1$ nonoverlapping open components $\{\sigma_0, \sigma_1, \ldots, \sigma_M\}$, and there exists a positive number d_0 such that the distance $dist(\sigma_i, \sigma_j) \ge d_0 > 0$ for $i \ne j$ and $M \ge 1$. The boundary of each component σ_i is either the empty set or a $C^{1,1}$-curve. Define $\Sigma_\nu = \partial\Omega \backslash \bar{\Sigma}_\tau$.

In the general case, Ω may be a multiply connected domain, and we denote by $n \ge 1$ the number of "handles" of Ω. Recall that these numbers n and m are, respectively, the first and the second Betti numbers of Ω (e.g., see [22]).

Let $\mathcal{D}(\Omega)$ be the space of infinitely differentiable functions with compact support on Ω, $H_0^1(\Omega)$ be the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$, $V = \{\mathbf{v} \in H_0^1(\Omega)^3 : \operatorname{div}\mathbf{v} = 0\}$, $H^{-1}(\Omega)^3 = (H_0^1(\Omega)^3)^*$, $L_0^2(\Omega) = \{p \in L^2(\Omega) : (p,1) = 0\}$, $H(\operatorname{curl},\Omega) = \{\mathbf{v} \in L^2(\Omega)^3 : \operatorname{curl}\mathbf{v} \in L^2(\Omega)^3\}$, $H^1(\Omega, \Sigma_\tau) = \{\varphi \in H^1(\Omega) : \varphi|_{\Sigma_\tau} = 0\}$, and $C_{\Sigma_\tau 0}(\bar{\Omega})^3 := \{\mathbf{v} \in C^0(\bar{\Omega})^3 : \mathbf{v}\cdot\mathbf{n}|_{\Sigma_\tau} = 0, \mathbf{v}\times\mathbf{n}|_{\Sigma_\nu} = 0\}$. In addition to the spaces introduced above, we will use the space $H_{DC}(\Omega) = \{\mathbf{h} \in H(\operatorname{curl},\Omega) : \operatorname{div}\mathbf{h} \in L^2(\Omega)\}$ equipped with the Hilbert norm

$$\|\mathbf{h}\|_{DC}^2 := \|\mathbf{h}\|^2 + \left\|\operatorname{div}\mathbf{h}^2\right\| + \|\operatorname{curl}\mathbf{h}\|^2. \tag{2.1}$$

Any vector $\mathbf{v}$ defined on the boundary $\Sigma = \partial\Omega$ (or on its part $\Sigma^0 \subset \partial\Omega$) can be represented by the sum of its normal and tangential components $\mathbf{v}_n$ and $\mathbf{v}_T$ as $\mathbf{v} = \mathbf{v}_n + \mathbf{v}_T$. These components are determined by $\mathbf{v}_n = (\mathbf{v}\cdot\mathbf{n})\mathbf{n} \equiv v_n\mathbf{n}$ and $\mathbf{v}_T = \mathbf{v} - \mathbf{v}_n \equiv (\mathbf{n}\times\mathbf{v})\times\mathbf{n}$. Here, the scalar $v_n = \mathbf{v}\cdot\mathbf{n}$ is the normal component of the vector field $\mathbf{v}$ and $\mathbf{v}\times\mathbf{n}$ is the tangential vector that is orthogonal both to the normal $\mathbf{n}$ and to the vector $\mathbf{v}_T$. It is clear that $\mathbf{v}_T = 0$ on Σ^0 if and only if $\mathbf{v}\times\mathbf{n}|_{\Sigma^0} = 0$. Conventionally, the subscript T in the notation of the spaces $H_T^1(\Omega)$, $H_T^{1/2}(\Sigma)$, or $L_T^2(\Sigma^0)$ indicates that the corresponding space consists of the vector functions of $H^1(\Omega)^3$, $H^{1/2}(\Sigma)^3$, or $L^2(\Sigma^0)^3$ that are tangential to the boundary Σ or to its part $\Sigma^0 \subset \Sigma$.

Below we will use the following Green formulas (see [22, 23]):

$$\int_\Omega \Delta u v dx = -\int_\Omega \nabla u\cdot\nabla v d\sigma + \int_\Sigma \frac{\partial u}{\partial n} v d\sigma \quad \forall u \in H^2(\Omega), \quad v \in H^1(\Omega), \tag{2.2}$$

$$\int_\Omega \mathbf{v}\cdot\operatorname{grad}\varphi dx + \int_\Omega \operatorname{div}\mathbf{v}\varphi dx = \int_\Sigma \mathbf{v}\cdot\mathbf{n}\varphi d\sigma \quad \forall \mathbf{v} \in H^1(\Omega)^3, \quad \varphi \in H^1(\Omega), \tag{2.3}$$

$$\int_\Omega (\mathbf{v}\cdot\operatorname{curl}\mathbf{w} - \mathbf{w}\cdot\operatorname{curl}\mathbf{v}) dx = \int_\Sigma (\mathbf{v}\times\mathbf{n})\cdot\mathbf{w}_T d\sigma \quad \forall \mathbf{v}, \quad \mathbf{w} \in H^1(\Omega)^3. \tag{2.4}$$

Formulas (2.3) and (2.4) also hold in the case when $\mathbf{v} \in H_{DC}(\Omega)$ under the condition that the integrals on the right-hand sides of (2.3) and (2.4) denote the duality relation $\langle\cdot,\cdot\rangle_\Sigma$ between $H^{-1/2}(\Sigma)$ and $H^{1/2}(\Sigma)$ or between $H^{-1/2}(\Sigma)^3$ and $H^{1/2}(\Sigma)^3$. If $\varphi \in H^1(\Omega, \Sigma_\tau)$ or $\mathbf{w} \in C_{\Sigma_\tau 0}(\bar{\Omega})^3 \cap H^1(\Omega)^3$, the right-hand sides of (2.3) or (2.4) take the form $\int_{\Sigma_\nu} \mathbf{v}\cdot\mathbf{n}\varphi d\sigma$ or $\int_{\Sigma_\tau} (\mathbf{v}\times\mathbf{n})\cdot\mathbf{w}_T d\sigma$. Using this fact and (2.3), we will say as in [20] that the function $\mathbf{v} \in H_{DC}(\Omega)$ satisfies the condition $\mathbf{v}\cdot\mathbf{n} = 0$ weakly on Σ_ν if the left-hand side of (2.3) vanishes for every function $\varphi \in H^1(\Omega, \Sigma_\tau)$. Similarly, based on (2.4) we will say that the function $\mathbf{v} \in H(\operatorname{curl},\Omega)$ satisfies the condition $\mathbf{v}\times\mathbf{n} = 0$ weakly on Σ_τ if the left-hand side of (2.4) vanishes for every function $\mathbf{w} \in C_{\Sigma_\tau 0}(\bar{\Omega})^3 \cap H^1(\Omega)^3$.

By $\mathbf{u}\times\mathbf{n}|_{\Sigma_\tau} \in H_{00}^{-1/2}(\Sigma_\tau)^3 \equiv [H_{00}^{1/2}(\Sigma_\tau)^3]^*$, we denote the natural restriction of the trace $\mathbf{u}\times\mathbf{n}$ of the function $\mathbf{u} \in H(\operatorname{curl},\Omega)$ to the part Σ_τ acting by the formula (see [19])

$$\langle \mathbf{u}\times\mathbf{n}|_{\Sigma_\tau}, \mathbf{v}\rangle_{\Sigma_\tau} = \langle \mathbf{u}\times\mathbf{n}, \tilde{\mathbf{v}}\rangle_\Sigma \quad \forall \mathbf{v} \in H_{00}^{1/2}(\Sigma_\tau)^3.$$

Here $\tilde{\mathbf{v}}$ is the continuation of the function $\mathbf{v} \in H_{00}^{1/2}(\Sigma_\tau)^3$ by zero to $\partial\Omega\backslash\Sigma_\tau$. This formula defines the operator of the partial tangential trace $\gamma_\tau|_{\Sigma_\tau} : H(\mathrm{rot},\Omega) \to H_{00}^{-1/2}(\Sigma_\tau)^3$ onto the boundary part Σ_τ, which assigns to each vector $\mathbf{u} \in H(\mathrm{rot},\Omega)$ the restriction $\gamma_\tau|_{\Sigma_\tau}\mathbf{u} = \mathbf{u} \times \mathbf{n}|_{\Sigma_\tau} \in H_{00}^{-1/2}(\Sigma_\tau)^3$ of its tangential trace to Σ_τ.

It has already been mentioned that the solvability of problem 1 will be proved following the scheme proposed in [11] and [22, Chapter 6]. This scheme includes the choice of the functional spaces corresponding to the boundary value problem under consideration, the elimination of the pair $(\mathbf{E}, p)$, the derivation of a weak formulation with respect to the remaining pair $(\mathbf{u}, \mathbf{H})$, which has the sense of the weak solution of the problem, the construction of the required liftings of the inhomogeneous boundary conditions, the proof of the existence of a weak solution and its local uniqueness, and the proof of the feasibility of this weak solution. The feasibility of the weak solution implies that the eliminated pair $(p, \mathbf{E})$ can be uniquely reconstructed given the weak solution $(\mathbf{u}, \mathbf{H})$ using the weak formulation of the boundary value problem. To describe the properties of the magnetic component $\mathbf{H}$ of the solution to problem (1.1), (1.2), (1.4), the following spaces are used in this scheme:

$$\mathcal{H}(m) = \{\mathbf{h} \in H_{DC}(\Omega) : \operatorname{div}\mathbf{h} = 0,\ \operatorname{curl}\mathbf{h} = 0 \text{ in } \Omega,\ \mathbf{h}\cdot\mathbf{n}|_\Sigma = 0\},$$

$$\mathcal{H}(e) = \{\mathbf{h} \in H_{DC}(\Omega) : \operatorname{div}\mathbf{h} = 0,\ \operatorname{curl}\mathbf{h} = 0 \text{ in } \Omega,\ \mathbf{h}\times\mathbf{n}|_\Sigma = 0\},$$

$$X_T = \{\mathbf{h} \in H_{DC}(\Omega) : \mathbf{h}\cdot\mathbf{n}|_\Sigma = 0\}, \quad V_T = \{\mathbf{h} \in X_T \cap \mathcal{H}(m)^\perp : \operatorname{div}\mathbf{h} = 0\}. \tag{2.5}$$

Here and in what follows, $S^\perp$ denotes the orthogonal complement of an arbitrary subspace $S \subset L^2(\Omega)^3$ in $L^2(\Omega)^3$. In particular, the space V_T plays the role of test functions for the magnetic field.

It is well known (e.g., see [22, Section 6.1]) that, under condition (i), in which the condition $\Sigma \in C^2$ may be replace with the condition $\Sigma \in C^{1,1}$, the space X_T can be continuously embedded into the space $H^1(\Omega)^3$ such that the coercivity inequality $\|\operatorname{curl}\mathbf{h}\|_\Omega^2 \geq \delta_1\|\mathbf{h}\|_{1,\Omega}^2$ holds for all $\mathbf{h} \in V_T$with a constant δ_1 that depends on Ω. In addition, the spaces $\mathcal{H}(m)$ and $\mathcal{H}(e)$ are finite dimensional and $\dim\mathcal{H}(m) = n$, $\dim\mathcal{H}(e) = m$, where n and m are the first and the second Betti numbers; furthermore, we have the following orthogonal decomposition of the space $L^2(\Omega)^3 : L^2(\Omega)^3 = \nabla H_0^1(\Omega) \oplus \operatorname{curl} V_T \oplus \mathcal{H}(e)$. These properties are used in [11, 22] in the proof of the solvability of the boundary value problem (1.1), (1.2), (1.4).

To apply the scheme described in [22] for the analysis of solvability of problem (1.1)–(1.3), denote by $H_{DC\Sigma_\tau}(\Omega)$ the closure of the space $C_{\Sigma_\tau 0}(\overline{\Omega})^3 \cap H^1(\Omega)^3$ with respect to the norm $\|\cdot\|_{DC}$ defined in (2.1) and define the following spaces:

$$\mathcal{H}_{\Sigma_\tau}(\Omega) = \{\mathbf{h} \in H_{DC}(\Omega) : \operatorname{div}\mathbf{h} = 0, \operatorname{curl}\mathbf{h} = 0 \text{ in } \Omega,\ \mathbf{h}\cdot\mathbf{n}|_{\Sigma_\tau} = 0,\ \mathbf{h}\times\mathbf{n}|_{\Sigma_\nu} = 0\},$$

$$\mathcal{H}_{\Sigma_\nu}(\Omega) = \{\mathbf{h} \in H_{DC}(\Omega) : \operatorname{div}\mathbf{h} = 0,\ \operatorname{curl}\mathbf{h} = 0 \text{ in } \Omega,\ \mathbf{h}\cdot\mathbf{n}|_{\Sigma_\nu} = 0,\ \mathbf{h}\times\mathbf{n}|_{\Sigma_\tau} = 0\},$$

$$V_{\Sigma_\tau}(\Omega) = \{\mathbf{v} \in H_{DC\Sigma_\tau}(\Omega) : \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\} \cap H_{\Sigma_\tau}(\Omega)^\perp. \tag{2.6}$$

In particular, the space $\mathcal{H}_{\Sigma_\tau}(\Omega)$ consists of the vector fields $\mathbf{h} \in H_{DC}(\Omega)$ that are harmonic in Ω and for which the normal component $\mathbf{h}\cdot\mathbf{n}$ and the tangential component $\mathbf{h}\times\mathbf{n}$ vanish (in the weak sense defined above) on the boundary parts Σ_τ and Σ_ν, respectively. A similar sense with Σ_τ replaced with Σ_ν and vice a versa has the space $\mathcal{H}_{\Sigma_\nu}(\Omega)$.

These spaces were defined and studied in [20, 21], and they were substantially used in [18] for the investigation of the boundary value problem for system (1.1), (1.2) subject to homogeneous boundary conditions in (1.3). In the case $\Sigma_\nu = \varnothing$, the spaces $\mathcal{H}_{\Sigma_\tau}(\Omega)$, $\mathcal{H}_{\Sigma_\nu}(\Omega)$, and $V_{\Sigma_\tau}(\Omega)$ turn, respectively, into the spaces $\mathcal{H}(m)$, $\mathcal{H}(e)$, and V_T defined in (2.5). It is important for the further consideration that in the general case, when $\Sigma_\tau \neq \varnothing$ and $\Sigma_\nu \neq \varnothing$, the spaces $H_{DC\Sigma_\tau}(\Omega)$, $V_{\Sigma_\tau}(\Omega)$, $\mathcal{H}_{\Sigma_\tau}(\Omega)$, and $\mathcal{H}_{\Sigma_\nu}(\Omega)$ have the properties similar to the properties of the spaces X_T, V_T, $\mathcal{H}(m)$, and $\mathcal{H}(e)$, respectively. This enables us to use these spaces in the proof of the solvability of the mixed boundary value problem (1.1)–(1.3). For the further reasoning, it is convenient to formulate the main properties of spaces (2.6) in the following lemma. The proofs of all its assertions can be found in [20].

Lemma 2.1. *Let conditions* (i) *and* (ii) *be fulfilled. Then*

(1) *the spaces* $\mathcal{H}_{\Sigma_\tau}(\Omega)$ *and* $\mathcal{H}_{\Sigma_\nu}(\Omega)$ *have finite dimensions*;

(2) *the continuous embedding* $H_{DC\Sigma_\tau}(\Omega) \subset H^1(\Omega)^3$ *holds, and the norm* $\|\cdot\|_{DC}$ *is equivalent to the norm* $\|\cdot\|_{1,\Omega}$ *in the space* $H_{DC\Sigma_\tau}(\Omega)$;

(3) *there exist constants* δ_1, C_0, *and* C_1 *depending on* Ω *and* Σ_τ *with which the following inequalities hold true*:

$$\|\mathrm{rot}\,\mathbf{h}\|_\Omega^2 \geq \delta_1 \|\mathbf{h}\|_{1,\Omega}^2 \quad \forall \mathbf{h} \in V_{\Sigma_\tau}(\Omega), \tag{2.7}$$

$$\|\mathbf{h}\|_{\Sigma_\tau} \leq C_0 \|\mathbf{h}\|_{1,\Omega}, \quad \|\mathrm{curl}\,\mathbf{h}\|_\Omega \leq C_1 \|\mathbf{h}\|_{1,\Omega} \quad \forall \mathbf{h} \in V_{\Sigma_\tau}(\Omega); \tag{2.8}$$

(4) *the following orthogonal decomposition of the space* $L^2(\Omega)^3$ *holds*:

$$L^2(\Omega)^3 = \nabla H^1(\Omega, \Sigma_\tau) \oplus \mathrm{curl}\, H_{DC\Sigma_\tau}(\Omega) \oplus \mathcal{H}_{\Sigma_\nu}(\Omega); \tag{2.9}$$

(5) $\nabla\varphi \times \mathbf{n} = 0$ *weakly on* Σ_τ *for every* $\varphi \in H^1(\Omega, \Sigma_\tau)$, *and it holds that*

$$\mathrm{curl}\, H_{DC\Sigma_\tau}(\Omega) \equiv \mathrm{curl}\, V_{\Sigma_\tau}(\Omega). \tag{2.10}$$

Relation (2.9) implies that any vector $\mathbf{h} \in L^2(\Omega)^3$ can be represented in the form $\mathbf{h} = \nabla\varphi + \mathrm{curl}\,\mathbf{v} + \mathbf{e}$, where $\varphi \in H^1(\Omega, \Sigma_\tau)$, $\mathbf{v} \in H_{DC\Sigma_\tau}(\Omega)$, and $\mathbf{e} \in \mathcal{H}_{\Sigma_\nu}(\Omega)$ are certain functions that are uniquely determined by $\mathbf{h}$.

In addition to the spaces $H_T^1(\Omega)$, $H(\mathrm{curl}, \Omega)$, and $H_{DC}(\Omega)$, we define their subspaces

$$H_T^1(\Omega, \mathrm{div}) = \{\mathbf{v} \in H_T^1(\Omega) : \mathrm{div}\,\mathbf{v} = 0\}, \quad H^0(\mathrm{curl}, \Omega) = \{\mathbf{E} \in H(\mathrm{curl}, \Omega) : \mathrm{curl}\,\mathbf{E} = 0\},$$

$$L_{DC}^3(\Omega) = L^3(\Omega)^3 \cap \{\mathbf{H} \in H_{DC}(\Omega) : \mathrm{div}\,\mathbf{H} = 0\} \cap H_{\Sigma_\tau}^\perp(\Omega),$$

equipped, respectively, with the norms

$$\|\mathbf{v}\|_{H_T^1(\Omega,\mathrm{div})} = \|\mathbf{v}\|_{1,\Omega}, \quad \|\mathbf{E}\|_{H^0(\mathrm{rot},\Omega)} = \|\mathbf{E}\|_\Omega, \quad \|\mathbf{H}\|_{L_{DC}^3(\Omega)} = \|\mathbf{H}\|_{L^3(\Omega)^3} + \|\mathrm{curl}\,\mathbf{H}\|_\Omega. \tag{2.11}$$

These spaces will play the key role in the investigation of solvability of problem 1 in the sense that it is in these spaces $H_T^1(\Omega, \mathrm{div})$, $L_{DC}^3(\Omega)$, and $H^0(\mathrm{curl}, \Omega)$ where the velocity $\mathbf{u}$ and the vectors $\mathbf{H}$ and $\mathbf{E}$ of the magnetic and electric fields that form, together with the pressure $p \in L_0^2(\Omega)$, the solution of problem 1 will be sought. For the investigation of the properties of the electric lifting, we will also need the subspace

$$H_{\Sigma_\tau}^0(\mathrm{curl}, \Omega) := \{\mathbf{e} \in H^0(\mathrm{curl}, \Omega) : \mathbf{e} \times \mathbf{n}|_{\Sigma_\tau} \in L_T^2(\Sigma_\tau)\},$$

equipped with the norm $\|\mathbf{e}\|_{H_{\Sigma_\tau}^0(\mathrm{curl},\Omega)} = \|\mathbf{e}\|_\Omega + \|\mathbf{e} \times \mathbf{n}\|_{\Sigma_\tau}$.

We pay special attention to the space $L_{DC}^3(\Omega)$, where the magnetic component $\mathbf{H}$ of the solution to problem 1 will be sought. By definition, it consists of the solenoidal vector functions of the space $H_{DC}(\Omega)$ introduced above that belong to the space $L^3(\Omega)^3$ and are orthogonal in the sense of $L^2(\Omega)$ to all the elements of the finite dimensional space $\mathcal{H}_{\Sigma_\tau}(\Omega)$. The comparison with the space $H_T^1(\Omega, \mathrm{div})$ in which the velocity $\mathbf{u}$ will be sought shows that the magnetic component $\mathbf{H}$ of the solution has weaker differential properties than the velocity. However, this weakening of the differential properties of $\mathbf{H}$ is feasible because it will be proved in Section 3 that problem 1 has a weak solution $(\mathbf{u}, \mathbf{H}) \in H_T^1(\Omega, \mathrm{div}) \times L_{DC}^3(\Omega)$ and that this solution is locally unique. At the same time, the idea of finding the magnetic component $\mathbf{H}$ in the class $L_{DC}^3(\Omega)$ makes it possible to considerably weaken the differential properties of the functions q and $\mathbf{q}$ that appear in the mixed boundary conditions for $\mathbf{H}$ in (1.3) by choosing q and $\mathbf{q}$ in the corresponding subspaces of L^2 spaces (see conditions (iii) below.

Recall that, due to the embedding theorems, the space $H^1(\Omega)$ is continuously embedded into the space $L^r(\Omega)$ for $r \le 6$ and, with a certain constant C_r, which depends on Ω and $r > 1$, it holds that

$$\|\mathbf{v}\|_{L^r(\Omega)^3} \le C_r \|\mathbf{v}\|_{1,\Omega} \quad \forall \mathbf{v} \in H^1(\Omega)^3. \tag{2.12}$$

Along with Lemma 2.1, we will use properties of the bilinear and trilinear forms related to the linear and nonlinear terms in Eqs. (1.1)–(1.3). We formulate them in the form of the following lemma. A proof of all the assertions of this lemma follows from the results obtained in [22, 23].

Lemma 2.2. *Under conditions* (i) *and* (ii)*, there exist positive constants* δ_0, γ_0', γ_0, γ_1', γ_1, γ_2, *and* β *depending on* Ω *such that*

$$(\nabla \mathbf{v}, \nabla \mathbf{v}) \ge \delta_0 \|\mathbf{v}\|_{1,\Omega}^2 \quad \forall \mathbf{v} \in H_0^1(\Omega)^3, \tag{2.13}$$

$$|((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w})| \le \gamma_0' \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \|\mathbf{w}\|_{L^4(\Omega)^3} \le \gamma_0 \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \|\mathbf{w}\|_{1,\Omega} \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in H_T^1(\Omega), \tag{2.14}$$

$$|(\operatorname{rot} \mathbf{u} \times \mathbf{v}, \mathbf{w})| \le \gamma_1' \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \|\mathbf{w}\|_{L^4(\Omega)^3} \le \gamma_1 \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \|\mathbf{w}\|_{1,\Omega} \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in H^1(\Omega)^3, \tag{2.15}$$

$$\begin{gathered} |(\operatorname{curl} \boldsymbol{\Psi} \times \mathbf{H}, \mathbf{u})| \le \|\operatorname{curl} \boldsymbol{\Psi}\|_{\Omega} \|\mathbf{H}\|_{L^3(\Omega)^3} \|\mathbf{u}\|_{L^6(\Omega)^3} \le \gamma_2 \|\mathbf{H}\|_{L^3(\Omega)^3} \|\mathbf{u}\|_{1,\Omega} \|\operatorname{curl} \boldsymbol{\Psi}\|_{\Omega} \\ \forall \mathbf{H} \in L^3(\Omega)^3, \quad \mathbf{u} \in H^1(\Omega)^3, \quad \boldsymbol{\Psi} \in H(\operatorname{curl}, \Omega), \end{gathered} \tag{2.16}$$

$$\sup_{\mathbf{v} \in H_0^1(\Omega)^3, \mathbf{v} \ne 0} \frac{-(\operatorname{div} \mathbf{v}, p)}{\|\mathbf{v}\|_{1,\Omega}} \ge \beta \|p\|_{\Omega} \quad \forall p \in L_0^2(\Omega). \tag{2.17}$$

Here, in particular, $\gamma_2 = C_6$, where C_6 *is the constant appearing in* (2.12) *for* $r = 6$. *Furthermore, the following identities hold true*:

$$((\mathbf{u}\cdot\nabla)\mathbf{v}, \mathbf{w}) = -((\mathbf{u}\cdot\nabla)\mathbf{w}, \mathbf{v}) \quad \forall \mathbf{u} \in H_T^1(\Omega, \operatorname{div}), \quad \mathbf{v} \in H_0^1(\Omega)^3, \quad \mathbf{w} \in H^1(\Omega)^3, \tag{2.18}$$

$$\begin{gathered} (\mathbf{H} \times \mathbf{u}, \operatorname{curl} \Psi) = (\operatorname{curl} \boldsymbol{\Psi} \times \mathbf{H}, \mathbf{u}) = -(\operatorname{curl} \boldsymbol{\Psi} \times \mathbf{u}, \mathbf{H}) \\ \forall \boldsymbol{\Psi} \in H(\operatorname{curl}, \Omega), \quad \mathbf{u} \in L^6(\Omega)^3, \quad \mathbf{H} \in L^3(\Omega)^3. \end{gathered} \tag{2.19}$$

3. PROOF OF THE SOLVABILITY OF PROBLEM 1

In this section, we prove the solvability of problem 1. In the investigation of solvability, an important role is played by the products of spaces $X = H_0^1(\Omega)^3 \times V_{\Sigma_\tau}(\Omega)$ and $Z = V \times V_{\Sigma_\tau}(\Omega) \subset X$ (they will be used as test spaces for the pair $(\mathbf{u}, \mathbf{H})$) and by the dual spaces $X^* = H^{-1}(\Omega)^3 \times V_{\Sigma_\tau}(\Omega)^*$ and $Z^* = V^* \times V_{\Sigma_\tau}(\Omega)^*$. X and Z are the Hilbert spaces equipped with the conventional graph norm $\|(\mathbf{u}, \mathbf{H})\|_X = (\|\mathbf{u}\|_{1,\Omega}^2 + \varkappa \|\mathbf{H}\|_{1,\Omega}^2)^{1/2}$. Recall that here $\varkappa = \mu\rho_0^{-1} \equiv \nu_1/\nu_m$ is a dimensional parameter appearing in the first equation in (1.2). The parameter $\varkappa$ is introduced into the norm $\|(\mathbf{u}, \mathbf{H})\|_X$ to equate the dimensions of both terms appearing in the norm. The elements of the space X^* (or Z^*) have the form $\mathbf{F} = (\mathbf{f}, \mathbf{q})$, where $\mathbf{f} \in H^{-1}(\Omega)^3$, $\mathbf{q} \in V_{\Sigma_\tau}(\Omega)^*$ (or $\mathbf{f} \in V^*$, $\mathbf{q} \in V_{\Sigma_\tau}(\Omega)^*$) and, by definition,

$$\langle (\mathbf{f}, \mathbf{q}), (\mathbf{v}, \boldsymbol{\Psi}) \rangle_{X^* \times X} = \langle \mathbf{f}, \mathbf{v} \rangle_{H^{-1}(\Omega)^3 \times H_0^1(\Omega)^3} + \langle \mathbf{q}, \boldsymbol{\Psi} \rangle_{V_{\Sigma_\tau}(\Omega)^* \times V_{\Sigma_\tau}(\Omega)}.$$

Reasoning as in [22], it is easy to verify that

$$\|\mathbf{F}\|_{X^*} := \|(\mathbf{f}, \mathbf{q})\|_{X^*} \le \|\mathbf{f}\|_{-1,\Omega} + \varkappa^{-1/2} \|\mathbf{q}\|_{V_{\Sigma_\tau}(\Omega)^*}. \tag{3.1}$$

In addition, for any $\mathbf{F} \in X^*$, it holds that $\|\mathbf{F}\|_{Z^*} \le \|\mathbf{F}\|_{X^*}$.

Define the following bilinear form:

$$a((\mathbf{u}, \mathbf{H}), (\mathbf{v}, \boldsymbol{\Psi})) = \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + \nu_1(\operatorname{curl} \mathbf{H}, \operatorname{curl} \boldsymbol{\Psi}) \equiv \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + \nu_m \varkappa(\operatorname{curl} \mathbf{H}, \operatorname{curl} \boldsymbol{\Psi}). \tag{3.2}$$

Due to (2.11), (2.7), and (2.13), the bilinear form a is continuous on the space $H^1_T(\Omega)\times H(\mathrm{curl},\Omega)$ and coercive on the subspace $X \equiv H^1_0(\Omega)^3 \times V_{\Sigma_\tau}(\Omega) \subset H^1_T(\Omega) \times H(\mathrm{curl},\Omega)$. In addition, due to (2.7) and (2.13), it holds that

$$a((\mathbf{v},\mathbf{\Psi}),(\mathbf{v},\mathbf{\Psi})) \ge \nu_*(\|\mathbf{v}\|^2_{1,\Omega} + \varkappa\|\mathbf{\Psi}\|^2_{1,\Omega}) \quad \forall(\mathbf{v},\mathbf{\Psi})\in X, \quad \nu_* = \min(\delta_0\nu,\delta_1\nu_m). \tag{3.3}$$

Let the continuous bilinear form $\hat{a} : X\times X \to R$ satisfy the following "δ smallness" condition on Z:

$$|\hat{a}((\mathbf{v},\mathbf{\Psi}),\mathbf{v},\mathbf{\Psi}))| \le \delta(\|\mathbf{v}\|^2_{1,\Omega} + \varkappa\|\mathbf{\Psi}\|^2_{1,\Omega}) \quad \forall(\mathbf{v},\mathbf{\Psi})\in Z, \quad 0\le\delta<\nu_*. \tag{3.4}$$

For the arbitrary element $\mathbf{F}\in X^*$, consider the variational problem of finding an element $(\mathbf{u},\mathbf{H})\in Z$ such that

$$a((\mathbf{u},\mathbf{H}),(\mathbf{v},\mathbf{\Psi})) + \hat{a}((\mathbf{u},\mathbf{H}) \quad \forall(\mathbf{v},\mathbf{\Psi})) = \langle\mathbf{F},(\mathbf{v},\mathbf{\Psi})\rangle \quad \forall(\mathbf{v},\mathbf{\Psi})\in Z. \tag{3.5}$$

The next lemma is a consequence of the Lax–Milgram theorem.

Lemma 3.1. *Let under conditions* (i) *and* (ii)*, the bilinear forms a and $\hat{a}$ which are continuous on X, satisfy conditions* (3.3) *and* (3.4). *Then*, (1) *the bilinear form $a + \hat{a}$ is continuous and coercive on Z with the constant* $\nu_* - \delta$; (2) *problem* (3.5) *has a unique solution* $(\mathbf{u},\mathbf{H})\in Z$ *for any element* $\mathbf{F}\in X^*$*, and the following bound holds*:

$$\|(\mathbf{u},\mathbf{H})\|_X \le (\nu_*-\delta)^{-1}\|\mathbf{F}\|_{X^*}.$$

Suppose that, in addition to (i) and (ii), it holds that

(iii) $\mathbf{f}\in H^{-1}(\Omega)^3$, $\mathbf{j}\in L^2(\Omega)^3$, $\mathbf{g}\in H^{1/2}_T(\Sigma)$, $q\in L^2(\Sigma_\tau)$, $\mathbf{q}\in L^2_T(\Sigma_\nu)$, $\mathbf{k}\in(\gamma|_{\Sigma_\tau})H^0_{\Sigma_\tau}(\mathrm{rot},\Omega)$.

The investigation of solvability of problem 1 is considerably complicated by the inhomogeneity of the boundary conditions in (1.3) both for the velocity and for electromagnetic field. As was mentioned in Section 1, the analysis of solvability of problem 1 is preceded by the construction of the liftings of the inhomogeneous boundary conditions in (1.3). The construction of the lifting for the velocity is based on the following lemma, which was proved in [11].

Lemma 3.2. *Under condition* (i)*, for any function* $\mathbf{g}\in H^{1/2}_T(\Sigma)$ *and any number* $\varepsilon>0$*, there exists a vector function* $\mathbf{u}_\varepsilon\in H^1_T(\Omega,\mathrm{div})$ *such that* $\mathbf{u}_\varepsilon=\mathbf{g}$ *on* Σ *and it holds that*

$$\|\mathbf{u}_\varepsilon\|_{1,\Omega} \le C_\varepsilon\|\mathbf{g}\|_{1/2,\Sigma}, \quad \|\mathbf{u}_\varepsilon\|_{L^4(\Omega)^3} \le \varepsilon\|\mathbf{g}\|_{1/2,\Sigma}. \tag{3.6}$$

Here, the constant C_ε depends on ε and Ω.

Remark 3.1. It should be emphasized that Lemma 3.1 on the existence of the hydrodynamic lifting $\mathbf{u}_\varepsilon$ satisfying bounds (3.6) holds in the case when $\mathbf{g}\in H^{1/2}_T(\Sigma)$, i.e., when $\mathbf{g}\cdot\mathbf{n}|_\Sigma=0$. To my knowledge, the issue of the existence of a solenoidal lifting $\mathbf{u}_\varepsilon$ satisfying bounds (3.6) for an arbitrary function $\mathbf{g}\in H^{1/2}(\partial\Omega)^3$ with $\int_{\partial\Omega}\mathbf{g}\cdot\mathbf{n}d\sigma=0$ remains open.

The role of the electric lifting can be played by any vector $\mathbf{E}_0\in H^0_{\Sigma_\tau}(\mathrm{curl},\mathbf{E})$ such that

$$\mathrm{curl}\mathbf{E}_0 = 0 \text{ in } \Omega, \quad \mathbf{E}_0\times\mathbf{n}|_{\Sigma_\tau} = \mathbf{k} \text{ (in } L^2_T(\Sigma_\tau)). \tag{3.7}$$

The existence of such a vector $\mathbf{E}_0$ follows from the condition on $\mathbf{k}$ in (iii), which is close to the necessary existence condition for the electric field appearing in model (1.1)–(1.3). Note that this lifting $\mathbf{E}_0$ satisfies the formula

$$(\mathbf{E}_0,\mathrm{curl}\,\mathbf{\Psi}) = (\mathbf{E}_0\times\mathbf{n}|_{\Sigma_\tau},\mathbf{\Psi}_T)_\Sigma = (\mathbf{k},\mathbf{\Psi}_T)_{\Sigma_\tau} \quad \forall\mathbf{\Psi}\in V_{\Sigma_\tau}(\Omega). \tag{3.8}$$

To derive (3.8), it is sufficient to apply Green's formula (2.4) to the pair $\mathbf{E}_0,\mathbf{\Psi}$, where $\mathbf{\Psi}\in V_{\Sigma_\tau}(\Omega)$, and use relations (3.7), the condition $\mathbf{\Psi}_T|_{\Sigma_\nu}=0$, and the equality

$$\langle\mathbf{E}_0\times\mathbf{n},\mathbf{\Psi}_T\rangle_\Sigma = (\mathbf{E}_0\times\mathbf{n}|_{\Sigma_\tau},\mathbf{\Psi}_T)_{\Sigma_\tau} + \langle\mathbf{E}_0\times\mathbf{n}|_{\Sigma_\nu},\mathbf{\Psi}_T\rangle_{\Sigma_\nu} \ \forall\mathbf{\Psi}\in H^1(\Omega)^3,$$

which is true due to the results of [19] for the function $\mathbf{E}_0\in H^0_{\Sigma_\tau}(\mathrm{curl},\Omega)$.

As for the magnetic lifting, its construction is considerably complicated by the specific features of the mixed boundary conditions (1.3) for the magnetic field $\mathbf{H}$. Typically, in the investigation of inhomogeneous mixed boundary value problems additional assumptions about the existence of at least one function satisfying the given boundary conditions are made. This is often used in the analysis of solvability of mixed boundary value problems for the fluid dynamics equations (e.g., see [3, 5]). In this paper, the role of this hypothesis is played by the following assumption that is a generalization of the corresponding assumption in [24]:

(iv) there exists a vector field $\mathbf{H}_0 \in L^3_{DC}(\Omega)$ such that

$$\operatorname{curl}\mathbf{H}_0 = 0, \quad \mathbf{H}_0 \cdot \mathbf{n}|_{\Sigma_\tau} = q, \quad \mathbf{H}_0 \times \mathbf{n}|_{\Sigma_\nu} = \mathbf{q}, \quad \|\mathbf{H}_0\|_{L^3(\Omega)^3} \le M_q \equiv C_\Sigma(\|q\|_{\Sigma_\tau} + \|\mathbf{q}\|_{\Sigma_\nu}). \tag{3.9}$$

Here, C_Σ is a constant independent of q and $\mathbf{q}$.

Note that condition (iv), which has the sense of the regularity and consistency of the boundary data q and $\mathbf{q}$ for the magnetic field $\mathbf{H}$, is satisfied under certain additional conditions on the decomposition of the boundary $\partial\Omega$ into the parts Σ_τ and Σ_ν. In particular, it holds when each part Σ_τ and Σ_ν consists of a finite number of connected components of $\partial\Omega$, i.e., in the case when

(v) $\partial\Omega = \bar{\Sigma}_\tau \cup \bar{\Sigma}_\nu$, $\bar{\Sigma}_\tau \cap \bar{\Sigma}_\nu = \varnothing$.

To justify this hypothesis, we use Theorem 4.2 in [25]. First, we introduce certain notation. Denote by $\hat{m}+1$ the number of connected components of Σ_ν; and by Σ_i ($i = 1, \ldots, s_0$, $s_0 \le m$), we denote the internal connected components of Σ contained in Σ_ν. It is clear that $\hat{m} \le m$, where m is defined in (i), and $\hat{m} = s_0$ in the case $\Sigma_0 \subset \Sigma_\nu$ and $\hat{m} = s_0 - 1$ if $\Sigma_0 \subset \Sigma_\tau$. Similarly we denote by n_1 (or n_2) the number of handles of Σ_ν (or Σ_τ). It is clear that $n_1 + n_2 = n$, where n is the number of handles of the boundary Σ defined in Section 2.

It is well known (e.g., see [25]) that, under conditions (i) and (v), the dimension of the space $\mathcal{H}_{\Sigma_\tau}(\Omega)$ is exactly $\hat{m} + n_2$, and the basis of the space $\mathcal{H}_{\Sigma_\tau}(\Omega)$ consists of the gradients $\nabla\hat{z}_j$ of the harmonic functions $\hat{z}_j \in H^2(\Omega)$ ($j = 1, \ldots, \hat{m}$) satisfying the boundary conditions $(\partial\hat{z}_j/\partial n)|_{\Sigma_\tau} = 0$, $\hat{z}_j|_{\Sigma_\nu\setminus\Sigma_j} = 0$, and $\hat{z}_j|_{\Sigma_j} = 1$ and the harmonic vector fields $\mathbf{y}_l \in \mathcal{H}_{\Sigma_\tau}(\Omega)$ ($l = 1, \ldots, n_2$) satisfying the condition $\int_{\zeta_k} \mathbf{y}_l \cdot \mathbf{n}\, d\sigma = \delta_{lk}$ for any cycle ζ_k ($k = 1, \ldots, n_2$) contained in Σ_τ and non homotopic to zero in Ω.

Similarly, the basis of the space $H_{\Sigma_\nu}(\Omega)$ consists of the gradients $\nabla\hat{z}_j$ of the harmonic functions $\hat{z}_j \in H^2(\Omega)$ ($j = \hat{m}+1, \ldots, m$) satisfying the boundary conditions $(\partial\hat{z}_j/\partial n)|_{\Sigma_\nu} = 0$, $\hat{z}_j|_{\Sigma_\tau\setminus\Sigma_j} = 0$, and $\hat{z}_j|_{\Sigma_j} = 1$, and the harmonic vector fields $\mathbf{y}_l \in \mathcal{H}_{\Sigma_\nu}(\Omega)$ ($l = n_2 + 1, \ldots, n$) satisfying the condition $\int_{\zeta_k} \mathbf{y}_l \cdot \mathbf{n}\, d\sigma = \delta_{lk}$ for any cycle ζ_k ($k = n_2 + 1, \ldots, n$) contained in Σ_ν Σ_τ and non homotopic to zero in Ω.

Consider the following mixed Dirichlet–Neumann homogeneous (with respect to the right-hand sides of the div–curl system) problem:

$$\operatorname{curl}\mathbf{u} = 0, \quad \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega, \quad \mathbf{u} \cdot \mathbf{n}|_{\Sigma_\tau} = q, \quad \mathbf{u} \times \mathbf{n}|_{\Sigma_\nu} = \mathbf{q}, \tag{3.10}$$

$$\left\langle (\mathbf{u} \cdot \mathbf{n})|_{\Sigma_i}, 1 \right\rangle_{\Sigma_i} = 0, \quad i = 1, 2, \ldots, \hat{m}, \quad \int_\Omega \mathbf{u} \cdot \mathbf{y}_l\, dx = 0, \quad l = 1, 2, \ldots, n_2. \tag{3.11}$$

Denote by $\operatorname{div}_{\Sigma_\nu}\mathbf{q}$ the surface divergence of the vector field $\mathbf{q}$ on the part Σ_ν of the boundary Σ (a detailed discussion of this concept see in [22, 25]). Theorem 4.2 proved in [25] and the result of [26] on the regularity of the solutions to the Maxwell equations in the case of L^2 boundary data imply the following result.

Lemma 3.3. *Let, in addition to conditions* (i) *and* (v)*, the conditions*

$$q \in L^2(\Sigma_\tau), \quad \mathbf{q} \in L^2_T(\Sigma_\nu), \quad \operatorname{div}_{\Sigma_\nu}\mathbf{q} = 0, \quad (\mathbf{q}, \mathbf{y}_l|_{\Sigma_\nu})_{\Sigma_\nu} = 0, \quad l = n_2 + 1, \ldots, n \tag{3.12}$$

be satisfied. Then, there exists a unique solution $\mathbf{u} \in L^3_{DC}(\Omega)$ *of problem* (3.10), (3.11), *which satisfies the bound* $\|\mathbf{u}\|_{L^3(\Omega)^3} \le C_\Sigma(\|q\|_{\Sigma_\tau} + \|\mathbf{q}\|_{\Sigma_\nu})$, *where the constant* C_Σ *is independent of* $\mathbf{u}$.

Set $\mathbf{H}_0 = \mathbf{u}$. This vector $\mathbf{H}_0$ satisfies all the conditions in (3.9); therefore, it is the desired magnetic lifting. Using the constructed liftings $\mathbf{u}_0$, $\mathbf{E}_0$, and $\mathbf{H}_0$, we now proceed to the investigation of the solvability of problem 1.

Assume that the quadruple $(\mathbf{u}, \mathbf{H}, p, \mathbf{E}) \in C^2(\bar{\Omega})^3 \times (C^1(\bar{\Omega})^3 \cap \mathcal{H}_{\Sigma_\tau}(\Omega)^\perp) \times C^1(\bar{\Omega}) \times C^1(\bar{\Omega})^3$ is a classical solution to problem 1. Multiply the first equation in (1.1) by the function $\mathbf{v} \in H_0^1(\Omega)^3$, the first equation in (1.2) by $\operatorname{curl}\mathbf{\Psi}$, where $\mathbf{\Psi} \in V_{\Sigma_\tau}(\Omega)$, integrate the result over Ω, and use Green's formulas (2.2)–(2.4) and relation (2.19). Since, due to the relations $\operatorname{curl}\mathbf{E} = 0$ in Ω, $\mathbf{E} \times \mathbf{n}|_{\Sigma_\tau} = \mathbf{k}$, $\mathbf{\Psi}_T|_{\Sigma_\nu} = 0$, and (3.7), we have

$$(\mathbf{E}, \operatorname{curl}\mathbf{\Psi}) = \int_{\Sigma_\tau} (\mathbf{E} \times \mathbf{n}) \cdot \mathbf{\Psi}_T d\sigma = (\mathbf{k}, \mathbf{\Psi}_T)_{\Sigma_\tau} = (\mathbf{E}_0, \operatorname{curl}\mathbf{\Psi}) \quad \forall \mathbf{\Psi} \in V_{\Sigma_\tau}(\Omega),$$

we obtain the identities

$$\nu(\nabla\mathbf{u}, \nabla\mathbf{v}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}) - \varkappa(\operatorname{curl}\mathbf{H} \times \mathbf{H}, \mathbf{v}) - (\operatorname{div}\mathbf{v}, p) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in H_0^1(\Omega)^3, \tag{3.13}$$

$$\nu_1(\operatorname{curl}\mathbf{H}, \operatorname{rot}\mathbf{\Psi}) + \varkappa(\operatorname{curl}\mathbf{\Psi} \times \mathbf{H}, \mathbf{u}) = \nu_1(\mathbf{j}, \operatorname{curl}\mathbf{\Psi}) + (\mathbf{k}, \mathbf{\Psi}_T)_{\Sigma_\tau} \quad \forall \mathbf{\Psi} \in V_{\Sigma_\tau}(\Omega). \tag{3.14}$$

Sum the restriction of identity (3.13) to the space $V \subset H_0^1(\Omega)^3$ with (3.14) to obtain the problem of determining the pair $(\mathbf{u}, \mathbf{H})$ from the equation $\operatorname{div}\mathbf{u} = 0$ in Ω, $\operatorname{div}\mathbf{H} = 0$ in Ω, and the relations

$$\begin{gathered}\nu(\nabla\mathbf{u}, \nabla\mathbf{v}) + \nu_1(\operatorname{curl}\mathbf{H}, \operatorname{curl}\mathbf{\Psi}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}) + \varkappa[(\operatorname{curl}\mathbf{\Psi} \times \mathbf{H}, \mathbf{u}) - (\operatorname{curl}\mathbf{H} \times \mathbf{H}, \mathbf{v})] \\ = \langle \mathbf{F}, (\mathbf{v}, \mathbf{\Psi}) \rangle \quad \forall (\mathbf{v}, \Psi) \in Z,\end{gathered} \tag{3.15}$$

$$\mathbf{u}|_\Sigma = \mathbf{g}, \quad \mathbf{H}\cdot\mathbf{n}|_{\Sigma_\tau} = q, \quad \mathbf{H} \times \mathbf{n}|_{\Sigma_\nu} = \mathbf{q}. \tag{3.16}$$

Here, $\mathbf{F} : X \to R$ is the linear functional acting by the formula

$$\langle \mathbf{F}, (\mathbf{v}, \mathbf{\Psi}) \rangle = \langle \mathbf{f}, \mathbf{v} \rangle + \nu_1(\mathbf{j}, \operatorname{curl}\mathbf{\Psi}) + (\mathbf{k}, \mathbf{\Psi}_T)_{\Sigma_\tau}. \tag{3.17}$$

Conditions (iii) imply that $\mathbf{F} \in X^*$, and, due to (2.8) and (3.1), it holds that

$$\|\mathbf{F}\|_{X^*} \le M := \|\mathbf{f}\|_{-1,\Omega} + \varkappa^{-1/2}(C_1\nu_1\|\mathbf{j}\|_\Omega + C_0\|\mathbf{k}\|_{\Sigma_\tau}). \tag{3.18}$$

The simple analysis shows that all the terms in (3.15) are well defined and $\mathbf{u}$ and $\mathbf{H}$ satisfy the solenoidality conditions $\operatorname{div}\mathbf{u} = 0$, $\operatorname{div}\mathbf{H} = 0$ in Ω if $\mathbf{u} \in H_T^1(\Omega, \operatorname{div})$ and $\mathbf{H} \in L_{DC}^3(\Omega)$. Hence, we define the weak solution to problem 1 as a pair $(\mathbf{u}, \mathbf{H}) \in H_T^1(\Omega, \operatorname{div}) \times L_{DC}^3(\Omega)$ satisfying (3.15) and (3.16). In addition to the definition of the weak solution, we also define the electric component of the solution to problem 1—this is any vector $\mathbf{E} \in H^0(\operatorname{curl}, \Omega)$ satisfying, together with the weak solution $(\mathbf{u}, \mathbf{H})$, the first equation in (1.2) and the identity

$$\langle \mathbf{E} \times \mathbf{n}, \mathbf{\Psi}_T \rangle_\Sigma = (\mathbf{k}, \mathbf{\Psi}_T)_{\Sigma_\tau} \quad \forall \mathbf{\Psi} \in V_{\Sigma_\tau}(\Omega). \tag{3.19}$$

Identity (3.19) contains useful information about the boundary condition $\mathbf{E} \times \mathbf{n}|_{\Sigma_\tau} = \mathbf{k}$ in (1.3). Moreover, if we succeed in proving the inclusion $\mathbf{E} \times \mathbf{n}|_{\Sigma_\tau} \in L_T^2(\Sigma_\tau)$ for the electric component $\mathbf{E}$, then (3.19) will take the familiar form $\mathbf{E} \times \mathbf{n}|_{\Sigma_\tau} = \mathbf{k}$ in $L_T^2(\Sigma_\tau)$.

Note that (3.15) does not include the pair $(p, \mathbf{E})$. However, the pair $(p, \mathbf{E})$ can be uniquely reconstructed from the pair $(\mathbf{u}, \mathbf{H}) \in H_T^1(\Omega, \operatorname{div}) \times L_{DC}^3(\Omega)$, which satisfies (3.15), so that the identities (3.13), (3.14), (3.19), and all the relations in (1.2) hold true. This is a consequence of the following lemma, which proves the feasibility of the weak solution to problem 1.

Lemma 3.4. *Suppose that, under conditions* (i)–(iii), $(\mathbf{u}, \mathbf{H}) \in H_T^1(\Omega, \operatorname{div}) \times L_{DC}^3(\Omega)$ *is a solution to problem* (3.15), (3.16). *Then, there exist a unique pair* $(p, \mathbf{E}) \in L_0^2(\Omega) \times H^0(\operatorname{curl}, \Omega)$ *such that the triple* $(\mathbf{u}, \mathbf{H}, p)$ *satisfies identities* (3.13), (3.14), *and* $\mathbf{E}$ *is the electric component satisfying identity* (3.19) *and, together with* $(\mathbf{u}, \mathbf{H})$, *relations* (1.2). *Moreover, the following bounds on the norms* $\|p\|_\Omega$ *and* $\|\mathbf{E}\|_\Omega$ *in terms of the norms* $\|\mathbf{u}\|_{1,\Omega}$ *and* $\|\mathbf{H}\|_{L_{DC}^3(\Omega)}$ *hold true*:

$$\|p\|_\Omega \le \beta^{-1}(\nu\|\mathbf{u}\|_{1,\Omega} + \gamma_0\|\mathbf{u}\|_{1,\Omega}^2 + \gamma_2\varkappa\|\mathbf{H}\|_{L_{DC}^3(\Omega)}^2 + \|\mathbf{f}\|_{-1,\Omega}), \tag{3.20}$$

$$\|\mathbf{E}\|_\Omega \le \sqrt{3}(\nu_1 + \gamma_2\varkappa\|\mathbf{u}\|_{1,\Omega})\|\mathbf{H}\|_{L^3_{DC}(\Omega)} + \nu_1\|\mathbf{j}\|_\Omega. \tag{3.21}$$

Here β, γ_0, *and* $\gamma_2 = C_6$ *are the constants defined in Lemma* 2.2.

Proof. Assume that the pair $(\mathbf{u},\mathbf{H}) \in H^1_T(\Omega,\mathrm{div}) \times L^3_{DC}(\Omega)$ is a solution to problem (3.15), (3.16). As usual, the pressure $p \in L^2_0(\Omega)$ is reconstructed (so as to satisfy identity (3.13)) using the de Rham theorem and the inf–sup condition (2.17) (details can be found in [22, 23]), while identity (3.14) is obtained from (3.15) if we set $\mathbf{v} = 0$ in (3.15).

To prove the existence of the electric component $\mathbf{E} \in H^0(\mathrm{curl}, \Omega)$ satisfying, together with the pair $(\mathbf{u},\mathbf{H})$, all the relations in (1.2) and identity (3.19), consider identity (3.14). Taking into account (2.10), (2.19), and (3.8), we rewrite it in the form

$$(\nu_1\mathrm{curl}\mathbf{H} + \varkappa\mathbf{H}\times\mathbf{u} - \nu_1\mathbf{j} - \mathbf{E}_0, \mathrm{curl}\boldsymbol{\Psi}) = 0 \quad \forall\boldsymbol{\Psi} \in H_{DC\Sigma_\tau}(\Omega). \tag{3.22}$$

(3.22) means that the vector $\mathbf{A} = (\nu_1\mathrm{curl}\mathbf{H} + \varkappa\mathbf{H}\times\mathbf{u} - \nu_1\mathbf{j} - \mathbf{E}_0) \in L^2(\Omega)^3$ is orthogonal to the vector $\mathrm{curl}\boldsymbol{\Psi}$, where $\boldsymbol{\Psi} \in H_{DC\Sigma_\tau}$ is an arbitrary function. Due to (2.9), this can be true if and only if

$$\nu_1\mathrm{curl}\mathbf{H} + \varkappa\mathbf{H}\times\mathbf{u} - \nu_1\mathbf{j} = \mathbf{E} \equiv \mathbf{E}_0 + \nabla\varphi + \mathbf{e}. \tag{3.23}$$

Here $\varphi \in H^1(\Omega, \Sigma_\tau)$ is the scalar potential, $\mathbf{e} \in \mathscr{H}_{\Sigma_\nu}(\Omega)$ is a vector (harmonic vector potential), and the pair $(\varphi, \mathbf{e})$ is uniquely determined by the vector $\mathbf{A}$. Since $\mathbf{e} \in \mathscr{H}_{\Sigma_\nu}(\Omega)$ and $\varphi \in H^1(\Omega,\Sigma_\tau)$, we have $\mathrm{curl}(\mathbf{e} + \nabla\varphi) = 0$ and, therefore, $\mathrm{curl}\mathbf{E} = 0$ in Ω. This implies that $\mathbf{E} \in H^0(\mathrm{curl},\Omega)$. Now multiply (3.23) by $\mathrm{curl}\boldsymbol{\Psi}$, integrate over Ω, and apply Green's formula (2.4) to the term $(\mathbf{E}, \mathrm{curl}\,\boldsymbol{\Psi})$. This yields the identity

$$\nu_1(\mathrm{curl}\mathbf{H}, \mathrm{curl}\boldsymbol{\Psi}) + \varkappa(\mathrm{curl}\boldsymbol{\Psi}\times\mathbf{H}, \mathbf{u}) - \langle\mathbf{E}\times\mathbf{n}, \boldsymbol{\Psi}_T\rangle_\Sigma = \nu_1(\mathbf{j}, \mathrm{curl}\boldsymbol{\Psi}) \quad \forall\boldsymbol{\Psi} \in V_{\Sigma_\tau}(\Omega). \tag{3.24}$$

Subtract (3.24) from (3.14) to obtain (3.19). Together with (3.23), this implies that $\mathbf{E}$ is the desired electric component.

It remains to derive bounds (3.20) and (3.21) for the pair $(p, \mathbf{E})$. In order to prove (3.20), we use the inf–sup condition (2.17) due to which, for the function $p \in L^2_0(\Omega)$ indicated above and any number $\delta > 0$, there exists a function $\mathbf{v}_0 \in \mathbf{H}^1_0(\Omega)$ ($\mathbf{v}_0 \ne 0$) such that the inequality $-(\mathrm{div}\,\mathbf{v}_0, p) \ge \beta_0\|\mathbf{v}_0\|_{1,\Omega}\|p\|_\Omega$ holds with the constant $\beta_0 = \beta - \delta > 0$. Set $\mathbf{v} = \mathbf{v}_0$ in (3.13). Then, we obtain

$$\nu(\nabla\mathbf{u}, \nabla\mathbf{v}_0) + ((\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}_0) - \varkappa(\mathrm{curl}\mathbf{H}\times\mathbf{H}, \mathbf{v}_0) - (\mathrm{div}\,\mathbf{v}_0, p) = \langle\mathbf{f}, \mathbf{v}_0\rangle. \tag{3.25}$$

Using the preceding bound on $-(\mathrm{div}\,\mathbf{v}_0, p)$, (2.12), (2.14), and (2.16), we conclude from (3.25) that

$$\begin{aligned}\beta_0\|\mathbf{v}_0\|_{1,\Omega}\|p\|_\Omega &\le -(\mathrm{div}\,\mathbf{v}_0, p) \le \nu\|\mathbf{v}_0\|_{1,\Omega}\|\mathbf{u}\|_{1,\Omega} + \gamma_0\|\mathbf{v}_0\|_{1,\Omega}\|\mathbf{u}\|^2_{1,\Omega} \\ &+ \gamma_2\varkappa\|\mathrm{curl}\mathbf{H}\|_\Omega\|\mathbf{H}\|_{L^3(\Omega)}\|\mathbf{v}_0\|_{1,\Omega} + \|\mathbf{f}\|_{-1,\Omega}\|\mathbf{v}_0\|_{1,\Omega}.\end{aligned} \tag{3.26}$$

Reducing this inequality by $\|\mathbf{v}_0\|_{1,\Omega} \ne 0$ and applying the bounds

$$\|\mathrm{curl}\mathbf{H}\|_\Omega \le \|\mathbf{H}\|_{L^3_{DC}(\Omega)}, \quad \|\mathbf{H}\|_{L^3(\Omega)^3} \le \|\mathbf{H}\|_{L^3_{DC}(\Omega)} \quad \forall\mathbf{H} \in L^3_{DC}(\Omega), \tag{3.27}$$

which follow from the definition of the norm in $L^3_{DC}(\Omega)$ in (2.11), we use (3.26) to derive the bound

$$|p|_\Omega \le \beta_0^{-1}(\nu\|\mathbf{u}\|_{1,\Omega} + \gamma_0\|\mathbf{u}\|^2_{1,\Omega} + \gamma_2\varkappa\|\mathbf{H}\|^2_{L^3_{DC}(\Omega)} + \|\mathbf{f}\|_{-1,\Omega}). \tag{3.28}$$

Since $\delta > 0$ is an arbitrary number, (3.28) implies bound (3.20) for p.

Next, using the first equality in (3.23) and Hölder's inequality, we have

$$\|\mathbf{E}\|^2_\Omega = \int_\Omega |\,\nu_1\mathrm{curl}\mathbf{H} + \varkappa\mathbf{H}\times\mathbf{u} - \nu_1\mathbf{j}^2|^2 dx$$

$$\le 3\left(\int_\Omega |\nu_1\mathrm{curl}\mathbf{H}|^2 dx + \int_\Omega |\varkappa\mathbf{H}\times\mathbf{u}|^2 dx + \int_\Omega |\nu_1\mathbf{j}|^2 dx)\right)$$

$$\le 3\left[\nu_1^2\|\mathrm{curl}\mathbf{H}\|_\Omega^2 + \varkappa^2\left(\int_\Omega |\mathbf{H}|^3 dx\right)^{2/3}\left(\int_\Omega |\mathbf{u}|^6 dx\right)^{1/3} + \nu_1^2\int_\Omega |\mathbf{j}|^2 dx\right]$$

$$= 3(\nu_1^2\|\mathrm{rot}\,\mathbf{H}\|_\Omega^2 + \varkappa^2\|\mathbf{H}\|_{L^3(\Omega)^3}^2\|\mathbf{u}\|_{L^6(\Omega)^3}^2 + \nu_1^2\|\mathbf{j}\|_\Omega^2).$$

Using the inequality $\sqrt{a^2+b^2+c^2} \le a+b+c$ for the numbers $a, b, c \ge 0$, we hence obtain

$$\|\mathbf{E}\|_\Omega \le \sqrt{3}(\nu_1\|\mathrm{curl}\mathbf{H}\|_\Omega + \varkappa\|\mathbf{H}\|_{L^3(\Omega)^3}\|\mathbf{u}\|_{L^6(\Omega)^3} + \nu_1\|\mathbf{j}\|_\Omega). \tag{3.29}$$

Finally, using (2.12) for $r = 6$ and (3.27), we derive from (3.29) bound (3.21) for $\mathbf{E}$.

Proceed to problem (3.15), (3.16). To prove that it has a solution $(\mathbf{u},\mathbf{H}) \in H_T^1(\Omega,\mathrm{div}) \times L_{DC}^3(\Omega)$, set the number ε mentioned in Lemma 3.2 equal to ε_0, where

$$\varepsilon_0 = \nu_*\min(1/\gamma_0', 1/\gamma_1')/2\|\mathbf{g}\|_{1/2,\Sigma}, \quad \nu_* = \min(\delta_0\nu, \delta_1\nu_m). \tag{3.30}$$

Here γ_0' and γ_1' are the constants introduced in (2.14) and (2.15). (We assume that $\|\mathbf{g}\|_{1/2,\Sigma} > 0$.) Set

$$M_\mathbf{g} = C_{\varepsilon_0}\|\mathbf{g}\|_{1/2,\Sigma}, \quad \tilde{M} = M + \nu M_\mathbf{g} + \gamma_0 M_\mathbf{g}^2 + C_1\gamma_2\sqrt{\varkappa}M_\mathbf{g}M_q. \tag{3.31}$$

Here C_{ε_0} is the constant corresponding to the number ε_0 in (3.30), which was introduced in Lemma 3.1, and M and M_q are the constants introduced in (3.18) and (3.9). We now prove the following result.

Theorem 3.1. *Under conditions* (i)–(iv)*, there exists a weak solution* $(\mathbf{u},\mathbf{H}) \in H_T^1(\Omega,\mathrm{div}) \times L_{DC}^3(\Omega)$ *of problem* 1, *and the following bounds hold:*

$$\|\mathbf{u}\|_{1,\Omega} \le M_\mathbf{u} \equiv \frac{2}{\nu_*}(M + \nu M_\mathbf{g} + \gamma_0 M_\mathbf{g}^2 + C_1\gamma_2\sqrt{\varkappa}M_\mathbf{g}M_q) + C_{\varepsilon_0}\|\mathbf{g}\|_{1/2,\Sigma}, \tag{3.32}$$

$$\|\mathbf{H}\|_{L_{DC}^3(\Omega)} \le M_\mathbf{H} \equiv \frac{2(C_1+C_3)}{\nu_*\sqrt{\varkappa}}(M + \nu M_\mathbf{g} + \gamma_0 M_\mathbf{g}^2 + C_1\gamma_2\sqrt{\varkappa}M_\mathbf{g}M_q) + C_\Sigma(\|q\|_{\Sigma_\tau} + \|\mathbf{q}\|_{\Sigma_\nu}). \tag{3.33}$$

Proof. Based on the boundary conditions in (3.16) for the velocity $\mathbf{u}$ and the magnetic field $\mathbf{H}$, we will seek the desired solution $(\mathbf{u},\mathbf{H})$ to problem (3.15), (3.16) in the form

$$\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}, \quad \mathbf{H} = \mathbf{H}_0 + \tilde{\mathbf{H}}. \tag{3.34}$$

Here $\mathbf{u}_0 \equiv \mathbf{u}_{\varepsilon_0} \in H_T^1(\Omega,\mathrm{div})$ and $\mathbf{H}_0 \in L_{DC}^3(\Omega)$ are the liftings of the boundary functions $\mathbf{g}$, q, and $\mathbf{q}$, appearing in Lemma 3.2 and in (3.9), while $\tilde{\mathbf{u}} \in V$ and $\tilde{\mathbf{H}} \in V_{\Sigma_\tau}(\Omega)$ are new unknown functions. Substitute (3.34) into (3.15) and use notation (3.2) to obtain the identity

$$\begin{gathered} a((\tilde{\mathbf{u}},\tilde{\mathbf{H}}),(\mathbf{v},\mathbf{\Psi})) + ((\tilde{\mathbf{u}}\cdot\nabla)\mathbf{u}_0,\mathbf{v}) + ((\tilde{\mathbf{u}}_0\cdot\nabla)\tilde{\mathbf{u}},\mathbf{v}) + ((\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}},\mathbf{v}) \\ - \varkappa[(\mathrm{curl}\tilde{\mathbf{H}}\times\mathbf{H}_0,\mathbf{v}) + (\mathrm{curl}\tilde{\mathbf{H}}\times\tilde{\mathbf{H}},\mathbf{v})] + \varkappa[(\mathrm{curl}\mathbf{\Psi}\times\mathbf{H}_0,\tilde{\mathbf{u}}) + (\mathrm{curl}\mathbf{\Psi}\times\tilde{\mathbf{H}},\mathbf{u}_0) \\ + (\mathrm{curl}\mathbf{\Psi}\times\tilde{\mathbf{H}},\tilde{\mathbf{u}})] = \langle\mathbf{F}_1,(\mathbf{v},\mathbf{\Psi})\rangle \quad \forall(\mathbf{v},\mathbf{\Psi}) \in Z. \end{gathered} \tag{3.35}$$

Here $\mathbf{F}_1 : X \to \mathbb{R}$ is the linear functional defined by the formula

$$\langle\mathbf{F}_1,(\mathbf{v},\mathbf{\Psi})\rangle = \langle\mathbf{F},(\mathbf{v},\mathbf{\Psi})\rangle - \nu(\nabla\mathbf{u}_0,\nabla\mathbf{v}) - ((\mathbf{u}_0\cdot\nabla)\mathbf{u}_0,\mathbf{v}) - \varkappa(\mathrm{curl}\mathbf{\Psi}\times\mathbf{H}_0,\mathbf{u}_0). \tag{3.36}$$

It is clear that $F_1 \in X^*$ because, due to (3.36), (2.8), (2.14), (2.16), (3.9), and (3.1), we have

$$\begin{gathered} |\langle\mathbf{F}_1,(\mathbf{v},\mathbf{\Psi})\rangle \le M\|(\mathbf{v},\mathbf{\Psi})\|_X + (C_{\varepsilon_0}\nu\|\mathbf{g}\|_{1/2,\Sigma} + \gamma_0 C_{\varepsilon_0}^2\|\mathbf{g}\|_{1/2,\Sigma}^2)\|\mathbf{v}\|_{1,\Omega} \\ + C_1\gamma_2\varkappa\|\mathbf{u}_0\|_{1,\Omega}\|\mathbf{H}_0\|_{L^3(\Omega)^3}\|\mathbf{\Psi}\|_{1,\Omega} \le \tilde{M}\|(\mathbf{v},\mathbf{\Psi})\|_X \quad \forall(\mathbf{v},\mathbf{\Psi}) \in X. \end{gathered} \tag{3.37}$$

Here M and $\tilde{M}$ are the constants defined in (3.18) and (3.31), respectively.

To prove the existence of the solution $(\tilde{\mathbf{u}},\tilde{\mathbf{H}})\in Z\equiv V\times V_{\Sigma_\tau}(\Omega)$ to problem (3.35), we use Schauder's fixed point theorem. To this end, define the mapping $G: Z\to Z$ acting by the formula $G(\mathbf{w},\mathbf{h})=(\tilde{\mathbf{u}},\tilde{\mathbf{H}})$, where the pair $(\tilde{\mathbf{u}},\tilde{\mathbf{H}})\in Z$ is a solution to the linear problem

$$a((\tilde{\mathbf{u}},\tilde{\mathbf{H}}),(\mathbf{v},\boldsymbol{\Psi}))+a_{\mathbf{w},\mathbf{h}}((\tilde{\mathbf{u}},\mathbf{H}),(\mathbf{v},\boldsymbol{\Psi}))=\langle\mathbf{F}_1,(\mathbf{v},\boldsymbol{\Psi})\rangle\quad\forall(\mathbf{v},\boldsymbol{\Psi})\in Z \tag{3.38}$$

obtained by the linearization of problem (3.35). Here, the bilinear form $a_{\mathbf{w},\mathbf{h}}: X\times X\to\mathbb{R}$ is defined by

$$\begin{gathered}a_{\mathbf{w},\mathbf{h}}((\tilde{\mathbf{u}},\mathbf{H}),(\mathbf{v},\boldsymbol{\Psi}))=((\tilde{\mathbf{u}}\cdot\nabla)\mathbf{u}_0,\mathbf{v})+((\mathbf{u}_0\cdot\nabla)\tilde{\mathbf{u}},\mathbf{v})+((\mathbf{w}\cdot\nabla)\tilde{\mathbf{u}},\mathbf{v})-\varkappa[(\operatorname{curl}\tilde{\mathbf{H}}\times\mathbf{H}_0,\mathbf{v})\\ +(\operatorname{curl}\tilde{\mathbf{H}}\times\mathbf{h},\mathbf{v})]+\varkappa[(\operatorname{curl}\boldsymbol{\Psi}\times\mathbf{H}_0,\tilde{\mathbf{u}})+(\operatorname{curl}\boldsymbol{\Psi}\times\tilde{\mathbf{H}},\mathbf{u}_0)+(\operatorname{curl}\boldsymbol{\Psi}\times\mathbf{h},\tilde{\mathbf{u}})].\end{gathered} \tag{3.39}$$

It is easy to verify that the mapping G is well defined. Indeed, the results of Section 2 imply that the form a is continuous and coercive on X, and, for the bilinear form $a_{\mathbf{w},\mathbf{h}}$ defined in (3.39), we have due to (2.18), (2.14), (2.15), and (3.30)

$$|a_{\mathbf{w},\mathbf{h}}((\mathbf{v},\boldsymbol{\Psi}),(\mathbf{v},\boldsymbol{\Psi}))|=|(\mathbf{v}\cdot\nabla)\mathbf{v},\mathbf{u}_0)+\varkappa(\operatorname{curl}\boldsymbol{\Psi}\times\boldsymbol{\Psi},\mathbf{u}_0)|$$

$$=|((\mathbf{v}\cdot\nabla)\mathbf{u}_0,\mathbf{v})+\varkappa(\operatorname{curl}\boldsymbol{\Psi}\times\boldsymbol{\Psi},\mathbf{u}_0)|\le(\gamma_0'\|\mathbf{v}\|_{1,\Omega}^2+\gamma_1'\varkappa\|\boldsymbol{\Psi}\|_{1,\Omega}^2)\|\mathbf{u}_0\|_{L^4(\Omega)^3}$$

$$\le\gamma_0'\varepsilon_0\|\mathbf{g}\|_{1/2,\Sigma}\|\mathbf{v}\|_{1,\Omega}^2+\gamma_1'\varepsilon_0\|\mathbf{g}\|_{1/2,\Sigma}\varkappa\|\boldsymbol{\Psi}\|_{1,\Omega}^2\le(\nu_*/2)(\|\mathbf{v}\|_{1,\Omega}^2+\varkappa\|\boldsymbol{\Psi}\|_{1,\Omega}^2)\quad\forall(\mathbf{v},\boldsymbol{\Psi})\in Z.$$

Since $\mathbf{F}_1\in X^*$ due to (3.37), Lemma 3.1 applied to problem (3.38) implies that, for each pair $(\mathbf{w},\mathbf{h})\in Z$, there exists a unique solution $(\tilde{\mathbf{u}},\tilde{\mathbf{H}})\in Z$ to problem (3.38), and it satisfies the independent of $(\mathbf{w},\mathbf{h})$ bound

$$\|(\tilde{\mathbf{u}},\tilde{\mathbf{H}})\|_X\equiv\left(\|\tilde{\mathbf{u}}\|_{1,\Omega}^2+\varkappa\|\tilde{\mathbf{H}}\|_{1,\Omega}^2\right)^{1/2}\le r=(2/\nu_*)\tilde{M}. \tag{3.40}$$

Consider the ball $B_r=\{(\mathbf{v},\boldsymbol{\Psi})\in Z:\|(\mathbf{v},\boldsymbol{\Psi})\|_X\le r\}$ in the space Z. It follows from (3.40) that the operator G defined above maps this ball B_r into itself. Reasoning as in [11], it can be shown that G is compact and continuous on B_r. Then, Schauder's theorem implies that G has a fixed point $(\tilde{\mathbf{u}},\tilde{\mathbf{H}})=G(\tilde{\mathbf{u}},\tilde{\mathbf{H}})\in B_r$. This fixed point $(\tilde{\mathbf{u}},\tilde{\mathbf{H}})\in Z$ is the desired solution to problem (3.35), and it satisfies bounds (3.40). Therefore, the pair $(\mathbf{u},\mathbf{H})$, where $\mathbf{u}=\mathbf{u}_0+\tilde{\mathbf{u}}$ and $\mathbf{H}=\mathbf{H}_0+\tilde{\mathbf{H}}$, is the desired solution to problem (3.15), (3.16), and bound (3.32) for $\mathbf{u}$ is an obvious consequence of bound (3.40) and the first bound in (3.6), where we should set $\varepsilon=\varepsilon_0$. Next, using (2.11), (2.8), (2.12), (3.9), and (3.34), we conclude that

$$\|\mathbf{H}\|_{L^3_{DC}(\Omega)}\le\|\operatorname{curl}\tilde{\mathbf{H}}\|_\Omega+\|\mathbf{H}_0+\tilde{\mathbf{H}}\|_{L^3(\Omega)^3}\le(C_1+C_3)\|\tilde{\mathbf{H}}\|_{1,\Omega}+M_q. \tag{3.41}$$

Now, (3.41) and (3.40) imply bound (3.33) for $\mathbf{H}$.

Let $(\mathbf{u},\mathbf{H})$ be a weak solution to problem 1 the existence of which follows from Theorem 3.1. Recall that, due to Lemma 3.4, there exists a pair the pressure $p\in L_0^2(\Omega)$, the electric field $\mathbf{E}\in H^0(\operatorname{curl},\Omega)$, which is uniquely determined by the pair $(\mathbf{u},\mathbf{H})$, and the pair $(p,\mathbf{E})$ satisfies bounds (3.20), (3.21). These bounds and bounds (3.32), (3.33) for $\|\mathbf{u}\|_{1,\Omega}$ and $\|\mathbf{H}\|_{L^3_{DC}(\Omega)}$ imply the following bounds on $\|p\|_\Omega$ and $\|\mathbf{E}\|_\Omega$ in terms of the norms of the data of problem 1:

$$\|p\|_\Omega\le M_p\equiv\beta^{-1}[(\nu+\gamma_0M_{\mathbf{u}})M_{\mathbf{u}}+\gamma_2\varkappa M_{\mathbf{H}}^2+\|\mathbf{f}\|_{-1,\Omega}], \tag{3.42}$$

$$\|\mathbf{E}\|_\Omega\le M_E\equiv\sqrt{3}[(\nu_1+\gamma_2\varkappa M_{\mathbf{u}})M_{\mathbf{H}}+\nu_1\|\mathbf{j}\|_\Omega]. \tag{3.43}$$

Now we formulate sufficient conditions for the data of problem 1 that ensure the uniqueness of its solution.

Theorem 3.2. *Let, in addition to the conditions of Theorem* 3.1*, the functions* $\mathbf{f}$, $\mathbf{j}$, $\mathbf{g}$, $\mathbf{q}$, q, *and* $\mathbf{k}$ *be small or the "viscosities"* ν *and* ν_m *be large in the sense that*

$$\gamma_0M_{\mathbf{u}}+\gamma_1\left(\frac{\sqrt{\varkappa}}{2}\right)M_{\mathbf{H}}<\delta_0\nu,\quad\gamma_1M_{\mathbf{u}}+\gamma_1\left(\frac{\sqrt{\varkappa}}{2}\right)M_{\mathbf{H}}<\delta_1\nu_m. \tag{3.44}$$

Then, the weak solution $(\mathbf{u},\mathbf{H})$ *of problem* 1 *is unique in the class of functions for which the magnetic component* $\mathbf{H}$ *can be represented as* $\mathbf{H}=\mathbf{H}_0+\tilde{\mathbf{H}}$*, where* $\mathbf{H}_0$ *is the magnetic lifting satisfying* (3.9) *and* $\tilde{\mathbf{H}}\in V_{\Sigma_\tau}(\Omega)$ *is a function.*

Proof. Let $(\mathbf{u}_1, \mathbf{H}_1)$, where $\mathbf{H}_1 = \mathbf{H}_0 + \tilde{\mathbf{H}}_1$ for a certain function $\tilde{\mathbf{H}}_1 \in V_{\Sigma_\tau}(\Omega)$, be a weak solution to problem 1 the existence of which was proved above. Denote by $(\mathbf{u}_2, \mathbf{H}_2)$ another weak solution to problem 1 for which $\mathbf{H}_2 = \mathbf{H}_0 + \tilde{\mathbf{H}}_2$ for a certain function $\tilde{\mathbf{H}}_2 \in V_{\Sigma_\tau}(\Omega)$. It is clear that $\mathbf{u} \equiv \mathbf{u}_1 - \mathbf{u}_2 \in V$, $\mathbf{H} \equiv \mathbf{H}_1 - \mathbf{H}_2 = \tilde{\mathbf{H}}_1 - \tilde{\mathbf{H}}_2 \in V_{\Sigma_\tau}(\Omega)$, and the pair $(\mathbf{u}, \mathbf{H})$ satisfies the equation

$$\nu(\nabla\mathbf{u}, \nabla\mathbf{u}) + \nu_1(\mathrm{curl}\,\mathbf{H}, \mathrm{curl}\,\mathbf{H}) + ((\mathbf{u}\cdot\nabla)\mathbf{u}_1, \mathbf{u}) - \varkappa(\mathrm{curl}\,\mathbf{H}_1 \times \mathbf{H}, \mathbf{u}) + \varkappa(\mathrm{curl}\,\mathbf{H} \times \mathbf{H}, \mathbf{u}_1) = 0. \quad (3.45)$$

Due to (2.14), (2.15) and bounds (3.32), (3.33), which hold for the pair $(\mathbf{u}_1, \mathbf{H}_1)$, we have

$$|((\mathbf{u}\cdot\nabla)\mathbf{u}_1, \mathbf{u})| \le \gamma_0 \|\mathbf{u}_1\|_{1,\Omega} \|\mathbf{u}\|^2_{1,\Omega} \le \gamma_0 M_{\mathbf{u}} \|\mathbf{u}\|^2_{1,\Omega},$$

$$\varkappa|(\mathrm{curl}\,\mathbf{H} \times \mathbf{H}, \mathbf{u}_1)| \le \gamma_1\varkappa \|\mathbf{u}_1\|_{1,\Omega} \|\mathbf{H}\|^2_{1,\Omega} \le \gamma_1\varkappa M_{\mathbf{u}} \|\mathbf{H}\|^2_{1,\Omega},$$

$$\varkappa|(\mathrm{curl}\,\mathbf{H}_1 \times \mathbf{H}, \mathbf{u})| \le \gamma_1\varkappa \|\mathbf{H}_1\|_{1,\Omega} \|\mathbf{H}\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} \le (\gamma_1\sqrt{\varkappa}/2) M_{\mathbf{H}} \left(\|\mathbf{u}\|^2_{1,\Omega} + \varkappa\|\mathbf{H}\|^2_{1,\Omega}\right).$$

Using these bounds, (2.7), and (2.13), we derive from (3.45) the inequality

$$[\delta_0\nu - \gamma_0 M_{\mathbf{u}} - \gamma_1(\sqrt{\varkappa}/2) M_{\mathbf{H}}]\|\mathbf{u}\|^2_{1,\Omega} + [\delta_1\nu_m - \gamma_1 M_{\mathbf{u}} - \gamma_1(\sqrt{\varkappa}/2) M_{\mathbf{H}}]\varkappa\|\mathbf{H}\|^2_{1,\Omega} \le 0. \quad (3.46)$$

Under conditions (3.44), inequality (3.46) implies that $\mathbf{u} = 0$, $\mathbf{H} = 0$. Therefore $\mathbf{u}_1 = \mathbf{u}_2$ and $\mathbf{H}_1 = \mathbf{H}_2$, which completes the proof of the theorem.

We have proved the global solvability of the mixed boundary value problem (1.1)–(1.3) for the steady-state MHD equations and proved the local uniqueness of this solution. To analyze the solvability of problem (1.1)–(1.3), an approach that made it possible to considerably weaken the differential properties of the functions involved in the boundary conditions for the magnetic field $\mathbf{H}$ in (1.3) by choosing them from L^2 spaces was used. This fact will play an important role in the solution of applied problems and, in particular, boundary control problems for the MHD model under consideration. The investigation of these problems and the analysis of unsteady-state analogues of problem (1.1)–(1.3) will be carried out in future works.

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REFERENCES

1. V. A. Solonninkov and V. E. Shchadilov, "On a boundary value problem for the steady-state Navier–Stokes system," Tr. Mat. Inst. im. V.A. Steklova, Akad. Nauk SSSR **125**, 196–210 (1973).
2. O. Pironneau, "Conditions aux limites sur la pression pour les equation de Stokes et de Navier–Stokes," Comptes Rendus Acad. Sci., Ser I–Math. **303**, 403–406 (1986).
3. C. Conca, F. Murat, and O. Pironneau, "The Stokes and Navier–Stokes equations with boundary conditions involving the pressure," Japan. J. Math. **20**, 279–318 (1994).
4. A. A. Illarionov and A. Yu. Chebotarev, "Solvability of a mixed boundary value problem for the stationary Navier–Stokes equations," Differ. Equations **37**, 724–731 (2001).
5. T. Kim and D. Cao, "Some properties on the surfaces of vector fields and its application to the Stokes and Navier–Stokes problem with mixed boundary conditions," Nonlinear Anal. **113**, 94–114 (2015).
6. O. A. Ladyzhenskaya and V. A. Solonninkov, "Solution of some unsteady-state problems of magnetohydrodynamics for viscous incompressible fluid," Tr. Mat. Inst. im. V.A. Steklova, Akad. Nauk SSSR **59**, 115–174 (1960).
7. V. A. Solonninkov, "On steady-state boundary value problems in magnetohydrodynamics," Tr. Mat. Inst. im. V.A. Steklova, Akad. Nauk SSSR **59**, 174–187 (1960).
8. M. D. Gunzburger, A. J. Meir, and J. S. Peterson, "On the existence, uniqueness, and finite element approximation of solution of the equation of stationary, incompressible magnetohydrodynamics," Math. Comp. **56**, 523–563 (1991).
9. G. V. Alekseev, "Control problems for stationary equations of magnetohydrodynamics of a viscous incompressible fluid," Prikl. Mekh. Tekh. Fiz. **44** (6), 170–179 (2003).
10. G. V. Alekseev, "Control Problems for Stationary Equations of Magnetic Hydrodynamics," Dokl. Math. **69**, 310–313 (2004).

11. G. V. Alekseev, "Solvability of control problems for stationary equations of magnetohydrodynamics of a viscous fluid," Sib. Math. J. **45**, 197–213 (2004).
12. D. Schotzau, "Mixed finite element methods for stationary incompressible magneto-hydrodynamics," Numer. Math. **96**, 771–800 (2004).
13. G. V. Alekseev and R. V. Brizitskii, "Solvability of the mixed boundary value problem for the stationary equations of magnetohydronamic equations of a viscous fluid," Dalnevost. Mat. Zh. **3** (2), 285–301 (2002).
14. R. V. Brizitskii and D. A. Tereshko, "On the solvability of boundary value problems for the stationary magnetohydrodynamic equations with inhomogeneous mixed boundary conditions," Differ. Equations **43**, 246–258 (2007).
15. A. J. Meir, "The equation of stationary, incompressible magnetohydrodynamics with mixed boundary conditions," Comp. Math. Appl. **25**, 13–29 (1993).
16. E. J. Villamizar-Roa, H. Lamos-Diaz, and G. Arenas-Diaz, "Very weak solutions for the magnetohydrodynamic type equations," Discrete Cont. Dynamic. Syst., Ser. B **10**, 957–972 (2008).
17. A. J. Meir and P. G. Schmidt, "Variational methods for stationary MHD flow under natural interface conditions," Nonlinear Anal. **26**, 659–689 (1996).
18. G. Alekseev and R. Brizitskii, "Solvability of the boundary value problem for stationary magnetohydrodynamic equations under mixed boundary conditions for the magnetic field," Appl. Math. Lett. **32**, 13–18 (2014).
19. P. Fernandes and G. Gilardi, "Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions," Math. Mod. Meth. Appl. Sci. **7**, (957–991 (1997).
20. G. Auchmuty, "The main inequality of vector field theory," Math. Mod. Meth. Appl. Sci. **14**, 79–103 (2004).
21. G. Auchmuty and J. S. Alexander, "Finite energy solutions of mixed 3D div-curl systems," Quart. Appl. Math. **64**, 335–357 (2006).
22. G. V. Alekseev, *Optimization in Steady-State Problems of Heat and Mass Transfer and Magnetohydrodynamics* (Nauchnyi mir, Moscow, 2010) [in Russian].
23. V. Girault and P. A. Raviart, *Finite Element Methods for Navier–Stokes Equations. Theory and Algorithms* (Springer, Berlin, 1986).
24. G. V. Alekseev, R. V. Brizitskii, and V. V. Pukhnachev, "Solvability of the inhomogeneous mixed boundary value problem for stationary magnetohydrodynamic equations," Dokl. Phys. **59**, 467–471 (2014).
25. A. Alonso and A. Valli, "Some remarks on the characterization of the space of tangential traces of $H(\mathrm{curl};\Omega)$ and the construction of an extension operator," Manuscript Math. **89**, 159–178 (1996).
26. M. Costabel, "A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains," Math. Meth. Appl. Sci. **12**, 365–368 (1990).

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