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Dedicated to the 60th anniversary of the foundation of Dorodnicyn Computing Center of the Russian Academy of Sciences

Studies on the Zeros of Bessel Functions and Methods for Their Computation: 2. Monotonicity, Convexity, Concavity, and Other Properties

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Abstract—This work continues the study of real zeros of first- and second-kind Bessel functions and Bessel general functions with real variables and orders begun in the first part of this paper (see M.K. Kerimov, Comput. Math. Math. Phys. **54** (9), 1337–1388 (2014)). Some new results concerning such zeros are described and analyzed. Special attention is given to the monotonicity, convexity, and concavity of zeros with respect to their ranks and other parameters.

Keywords: Bessel functions of first and second kinds, general cylinder functions, real zeros, concavity and convexity of zeros, monotonicity of zeros, overview.

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INTRODUCTION

The Bessel functions and their zeros play an exceptional role in the solution of natural-science problems by applying exact or approximate methods. Accordingly, the necessary stage in solving real-world problems involving these functions and their zeros is the examination of their properties and the design of methods for their analysis and computation.

Some results concerning Bessel functions and their zeros can be found in numerous books dealing with differential equations, mathematical physics, and computational mathematics.

These issues are also covered in monographs.

An especially popular book, in fact, encyclopedia of the theory of Bessel functions is Watson's treatise [60], which contains important classical results on these functions published up to the 1920s. Many of these results are still valuable, so the book is widely quoted in the scientific literature.

Numerous results regarding special functions, in particular, Bessel functions can be found in handbooks and monographs, such as the handbook edited by Abramowitz and Stegun [1], the recently published handbook edited by Olver (see NIST Handbook [49]), and the monographs written by Olver [52], Kratzer and Franz [28], and Gray, Mathews, and MacRobert [19].

Various aspects of the theory of Bessel function zeros are covered in review articles, such as Muldoon's [47], where Bessel function zeros are examined as functions of their order; Laforgia and Natalini's work [33], which analyzes the monotonicity, convexity, and concavity of zeros as functions of the order; and Elbert's work [5], which overviews available (including most recent) publications concerning various properties of Bessel function zeros. A brief overview of Bessel function zeros can be found in (Finch [16]).

Worth noting are the articles dedicated to the memory of two prominent experts in special functions, where comprehensive bibliographies of their works on Bessel function zeros can be found: Laforgia, Muldoon, and Siafarikas' paper [32] dedicated to the memory of Hungarian mathematician Árpád Elbert (1939–2001) and Gautschi and Giordano's paper [17] dedicated to the memory of Italian mathematician Luigi Gatteschi (1923–2007).

Numerous results concerning Bessel function zeros and methods for their computation can be found in the author's papers (see Kerimov [23–25]).

Recently, I have made an attempt to write a series of overviews describing the theory of Bessel function zeros and computational methods for them (for all Bessel functions, i.e., real, imaginary, and complex). The aim of these overviews is to describe classical results in this area that are still of scientific value and to present numerous new results published in various papers. Many of these results have not yet been covered in monographs or handbooks of special functions.

While preparing these papers, I studied and analyzed a large number of works published in various (frequently hard-to-reach) sources (books, articles). Many of them deal with natural-science problems, which are frequently described in a specific language inherent in the considered application area. Accordingly, results concerning Bessel functions and their zeros had to be described in the usual mathematical language (of course, with improvements and corrections of found inaccuracies and errors).

The first paper in this series (see Kerimov [25]) covers primarily classical results that are still of importance. They are basically due to founders of the theory of special functions.

This second paper in the series overviews some new results regarding real zeros of Bessel functions.

Since the basic auxiliary results on Bessel functions and their zeros required for the exposition of new results were presented in [25], we sometimes refer to that work.

This paper consists of six sections. Section 1 is introductory. It presents the basic definitions and auxiliary results. Additionally, generalized definitions of zeros of Bessel functions and their derivatives are introduced in this section. New results concerning the monotonicity, convexity, and concavity of zeros and some of their applications are overviewed in Sections 2–6. Many assertions are given with proofs. A number of improvements of previously known results are made, and overlooked inaccuracies and misprints are corrected.

1. SOME DEFINITIONS AND AUXILIARY RESULTS CONCERNING POSITIVE ZEROS OF BESSEL FUNCTIONS AND THEIR DERIVATIVES

Let $j_{\nu,k}$, $j'_{\nu,k}$, $y_{\nu,k}$, $y'_{\nu,k}$, $c_{\nu,k}$, and $c'_{\nu,k}$ denote the positive zeros of the Bessel functions $J_\nu(x)$, $J'_\nu(x)$, $Y_\nu(x)$, $Y'_\nu(x)$, $C_\nu(x)$, and $C'_\nu(x)$, respectively, where $C_\nu(x) = J_\nu(x)\cos\alpha - Y_\nu(x)\sin\alpha$, $0 \le \alpha < \pi$, and α is a parameter. The zeros of these functions are interlaced according to the following laws (see Abramowitz and Stegun [1, p. 370, 9.51], Watson [60, Section 15.3], Kerimov [25], Olver [51, Section 1, Table 1]):

$$0 < j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < j_{\nu,3} < \dots, \tag{1.1}$$

$$0 < y_{\nu,1} < y_{\nu+1,1} < y_{\nu,2} < y_{\nu+1,2} < y_{\nu,3} < \dots, \tag{1.1'}$$

$$\nu \le j'_{\nu,1} < y_{\nu,1} < y'_{\nu,1} < j_{\nu,1} < j'_{\nu,2} < y_{\nu,2} < y'_{\nu,2} < j_{\nu,2} < j'_{\nu,3} < \dots, \tag{1.1''}$$

Moreover, equality in (1.1)" holds only at $\nu = 0$ and $j'_{0,1}$ is defined as the first zero of $J'_0(x)$.

Numerous issues concerning the properties of positive zeros satisfying inequalities (1.1)–(1.1)" were addressed in the first part of this paper (see Kerimov [25]).

Before describing new results on zeros, we discuss modified notation for Bessel function zeros, which is more universal than the conventional one when we address some properties of zeros.

For the first time, the modified notation was used by Hungarian mathematician Árpád Elbert. However, in a somewhat vague form, it goes back to Watson (see, e.g., [60, p. 508, formula (2)]). Elbert and his followers widely used this notation, which makes it possible to examine the properties of Bessel function zeros in the case of orders taking not only positive integer, but also negative and real values.

In [9, 10], for positive zeros $c_{\nu,k}$ of $C_\nu(x)$, Elbert and Laforgia introduced a modified (generalized) notation (referred to in [25] as the χ-notation) that covers negative ν, so that $c_{\nu,k}$ is a continuous function of ν and $c_{\nu,k} \to 0$ as $\nu \to \alpha/\pi - k$ and, on the interval

$$\frac{\alpha}{\pi} - k < \nu < \frac{\alpha}{\pi} - k + 1,$$

the function $c_{\nu,k}$ is the first positive zero of $C_\nu(x)$. Moreover, for χ such that $k-1<\chi<k$, where k is a positive integer, it is true that $j_{\nu,\chi}=c_{\nu,k}$, where $\alpha=(k-\chi)\pi$. Then there is a one-to-one correspondence between the values of $j_{\nu,\chi}$ and $c_{\nu,k}$. Moreover, the above limit relation for $c_{\nu,k}$ implies that

$$\lim_{\nu\to-\chi+0} j_{\nu,\chi}=0. \tag{1.2}$$

It is well known (see Watson [60, p. 508]) that the function $c_{\nu,k}$ satisfies the integrodifferential equation (referred to as Watson's equation)

$$\frac{d}{d\nu}c_{\nu,k}=2c_{\nu,k}\int_0^\infty K_0(2c_{\nu,k}\sinh t)e^{-2\nu t}dt, \tag{1.3}$$

where $K_0(\cdot)$ is a modified Bessel function of the second kind, which is positive on the interval $(0,\infty)$ and has the integral representation

$$K_0(u)=\int_0^\infty e^{-u\cosh z}dz. \tag{*}$$

Let $j_{\nu,\chi}$ be a solution of Eq. (1.3) for $\chi>0$ that satisfies boundary condition (1.2).

For $\chi=k=1,2,\ldots$, we obtain the zeros $j_{\nu,k}$ of $J_\nu(x)$, while, for $k-1<\chi<k$, we have $j_{\nu,\chi}=c_{\nu,k}$, where $\alpha=(k-\chi)\pi$. For example, in the new notation, for zeros $y_{\nu,k}$ of $Y_\nu(x)$, we obtain

$$j_{\nu,k-1/2}=y_{\nu,k},\quad k=1,2,\ldots.$$

The right-hand side of Eq. (1.3) satisfies the Lipschitz condition with respect to $j_{\nu,k}>0$. In the case $j_{\nu,\chi}=0$, we have

$$\lim_{\nu\to-\chi+0} j_{\nu,\chi}=0$$

for any $\chi>0$. Therefore, the condition

$$\lim_{\nu\to-\chi+0} j_{\nu,\chi'}=0$$

implies that $\chi'=\chi$; i.e., Eq. (1.3) with any initial condition (1.2) has a unique solution. Moreover, the uniqueness implies that, if $0<\chi'<\chi''$, then

$$j_{\nu,\chi'}<j_{\nu,\chi''},\quad \nu>-\chi', \tag{1.4}$$

which means that $j_{\nu,\chi}$ is strongly monotone as a function of χ. Interesting special cases of $J_\nu(x)$ and $Y_\nu(x)$ are the functions at $\nu=\pm\frac{1}{2}$. Then

$$\begin{gathered}J_{1/2}(x)=\sqrt{2/\pi x}\sin x,\quad Y_{1/2}(x)=-\sqrt{2/\pi x}\cos x,\\ c_{1/2}(x)=\sqrt{2/\pi x}\sin(x+\alpha),\quad 0\le\alpha<\pi.\end{gathered} \tag{1.5}$$

In special case (1.5), the zeros $c_{\nu,k}$ can be computed using the formula

$$c_{\frac{1}{2},k}=k\pi-\alpha, \tag{1.6}$$

or, in χ-notation,

$$j_{\frac{1}{2},\chi}=\chi\pi\quad\text{for}\quad\chi>0. \tag{1.7}$$

In the case $\nu=-\frac{1}{2}$, we obtain the formula

$$j_{-\frac{1}{2},\chi}=\left(\chi-\frac{1}{2}\right)\pi\quad\text{for}\quad\chi>\frac{1}{2}. \tag{1.8}$$

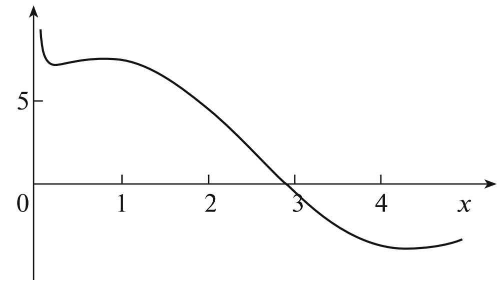

Fig. 1. Plot of the function $C_{1/2}(x, \alpha)$ for small α.

1.1. Generalized Definition of Zeros of the Derivative of $C_\nu(x)$

The generalized definition of the zeros $j_{\nu,\chi}$ of the function $C_\nu(x)$ was given above.

Now, in a similar (but more complicated manner), we introduce a generalized definition of the zeros $j'_{\nu,\chi}$ of the derivative $C'_\nu(x)$. Specifically, $j'_{\nu,\chi} = c'_{\nu,k}$ for $\nu > 0$, where $\chi = k - \frac{\alpha}{\pi}$, $k - 1 < \chi < k$, $k = 1, 2, \ldots$, and $c_{\nu,k}$ is the first zero of $C_\nu(x)$. Here, χ is a parameter. Now $j'_{\nu,\chi}$ solves a more complicated integrodifferential equation (see Watson [60, p. 510], Elbert and Laforgia [8]):

$$\frac{d}{d\nu} j'_{\nu,\chi} = \frac{2 j'_{\nu,\chi}}{j'^2_{\nu,\chi} - \nu^2} \int_0^\infty \left(j'^2_{\nu,\chi} \cosh 2t - \nu^2 \right) K_0\left(2 j'_{\nu,\chi} \sinh t \right) e^{-2\nu t} dt. \tag{1.9}$$

The right-hand side of Eq. (1.9) satisfies the Lipschitz condition with respect to $j'_{\nu,\chi}$, provided that $j'_{\nu,\chi} > 0$ and $j'_{\nu,\chi} \neq |\nu|$, and the solution of Eq. (1.9) satisfying certain conditions is unique at least in the domain where $j'_{\nu,\chi} > |\nu|$. This uniqueness is not extended to the case $j'_{\nu,\chi} = |\nu|$. The authors examine the behavior of $j'_{\nu,\chi}$ depending on the parameter α. The results supplement those of Lorch and Newman [39, p. 362] concerning the positive zeros $c_{\nu,k}$ of $C_\nu(x)$, namely, the kth positive zero $c_{\nu,k}(\theta)$ of $C_\nu(x, \theta)$ is greater than the kth positive zero $c_{\nu,k}(\chi)$ of $C_\nu(x, \chi)$ for $\nu \geq 0$ when $0 \leq \theta < \chi < \pi$.

Specifically, it is proved that the χth positive zero $j'_{\nu,\chi}$ is an increasing function of χ, $\alpha = (k - \chi)\pi$, and $0 < \alpha < \pi$ as long as $j'_{\nu,\chi} > |\nu|$.

In [7] Elbert, Kosik, and Laforgia consider only the case $\nu \geq 0$ and propose a modified (generalized) definition of $j'_{\nu,\chi}$ that is more convenient in the case under consideration.

Specifically, the following algorithm is used for this purpose.

Step A. For $\alpha = 0$, $C_\nu(x, 0) \equiv C_\nu(x) \equiv J_\nu(x)$. If $\nu > 0$, then the zeros of $J_\nu(x)$ are denoted as usual by 0, $j_{\nu,1}, j_{\nu,2}, \ldots$. By Rolle's theorem, the sequence $j'_{\nu,1}$, $j'_{\nu,2}$, ... of zeros of $J'_\nu(x)$ is such that

$$0 < j'_{\nu,1} < j_{\nu,1} < j'_{\nu,2} < j_{\nu,2} < \ldots . \tag{1.10}$$

Moreover, a stronger inequality holds (see Watson [60, p. 488]), namely,

$$\nu < j'_{\nu,1} < j_{\nu,1}. \tag{1.11}$$

Step B. Consider the case $0 < \alpha < \pi$. Since

$$\lim_{\nu \to -0+} Y_\nu(x) = -\infty, \tag{1.12}$$

we have

$$\lim_{\nu \to -0+} C_\nu(x) = +\infty. \tag{1.13}$$

Therefore, the inequality $C_\nu(x) > 0$ holds in a right neighborhood of the point $x = 0$.

Assume that the equation $C'_\nu = 0$ has a solution $c_{\nu,1}$ on the interval $0 < x < \nu$. Then, according to Muldoon and Spigler's work [48], this can happen only for $0 < \alpha < \pi/6$ (see Fig. 1).

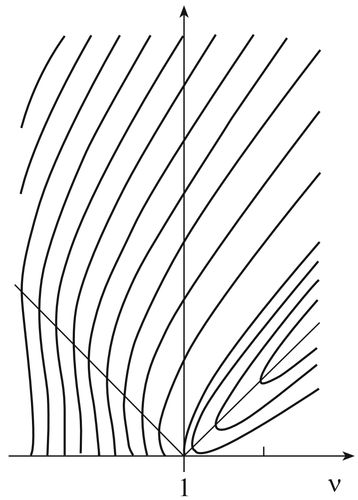

Fig. 2. Plot of the function $j'_{\nu,\chi}$ for $-2 \le \nu \le 2$, $0 \le j'_{\nu,\chi} \le 5$.

Another result from [48] is that the function $C_\nu(x)$ has a zero on the interval $0 < x \le \nu$ if and only if $5\pi/6 < \alpha < \pi$. Therefore, if $0 < c'_{\nu,1} \le \nu$, then $C_\nu(x) > 0$ on the interval $0 < x \le \nu$ and the first zero of $C_\nu(x)$ satisfies the inequality $c_{\nu,1} > \nu$ or, in the χ-notation, $-j_{\nu,\chi} > \nu$, where $\chi = 1 - \alpha/\pi$.

Recall that $C_\nu(x)$ is a solution of the Bessel differential equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0. \tag{1.14}$$

Therefore, if $c'_{\nu,1} = \nu$, then $C''_\nu(c'_{\nu,1}) = 0$ and $c'_{\nu,1} = \nu$ is a double zero of $C'_\nu(x)$. If $0 < c'_{\nu,1} < \nu$, then (1.14) implies that $C''_\nu(c'_{\nu,1}) > 0$; i.e., $C_\nu(x)$ has a local minimum at $x = c'_{\nu,1}$. On the other hand, the differential equation can be rewritten as

$$x(xy')' = (\nu^2 - x^2)y, \tag{1.15}$$

which implies that $xC'_\nu(x)$ is an increasing function of x on $0 < x \le \nu$, since $C_\nu(x) > 0$ on the interval $(0, \nu]$. Therefore, $C'_\nu(x) > 0$ and $C'_\nu(x)$ vanishes another time on the interval $(\nu, c_{\nu,1})$. However, it vanishes only once on the basis of (1.5); $xC'_\nu(x)$ is a strictly decreasing function on the interval $(\nu, c_{\nu,1})$.

Let $c'_{\nu,2}$ be the second zero of $C'_\nu(x)$. In χ-notation, the zero $c'_{\nu,2}$ is denoted by $j'_{\nu,\chi}$. Then $\nu < j'_{\nu,\chi} < j_{\nu,\chi}$, where $\chi = 1 - \alpha/\pi$. Proceeding in the same manner, we conclude that there is exactly one zero of $C'_\nu(x)$ between the consecutive zeros $j_{\nu,\chi}$ and $j_{\nu,\chi+1}$ of $C_\nu(x)$. This zero is denoted by $j'_{\nu,\chi+1}$. It satisfies the inequalities

$$j_{\nu,\chi} < j'_{\nu,\chi+1} < j_{\nu,\chi+1}, \quad \text{where} \quad \chi = k - \alpha/\pi, \quad k = 1, 2, \ldots.$$

Step C. Now consider a function $C_\nu(x)$ that decreases on the interval $(0, c_{\nu,1}]$. For example, it can be $C_\nu(x, \pi/2)$, i.e., $C_\nu(x, \pi/2) = -Y_\nu(x)$. Then the zeros of $C'_\nu(x)$ are $c'_{\nu,1}, c'_{\nu,2}, \ldots$, and they satisfy the inequalities $c_{\nu,1} < c'_{\nu,1} < c_{\nu,2} < c'_{\nu,2} < \ldots$.

Now we have $c'_{\nu,1} > \nu$. This can happen if $c_{\nu,1} > \nu$, while, relying on the above results of Muldoon and Spigler [48], the case $c_{\nu,1} \le \nu$ is possible only if $\alpha \in (5\pi/6, \pi)$, so $c'_{\nu,1}$ cannot belong to the interval $(0, \nu]$, since otherwise we would have $\alpha \in (0, \pi/6)$. Let the zero $j'_{\nu,\chi}$ be given by the relations

$$j'_{\nu,\chi} = c'_{\nu,k}, \quad \chi = k + 1 - \alpha/\pi, \quad k = 1, 2, \ldots.$$

Combining the above properties of $j'_{\nu,\chi}$, we obtain

$$j'_{\nu,\chi} > \nu, \tag{1.16}$$

$$j_{\nu,\chi-1} < j'_{\nu,\chi} < j_{\nu,\chi}, \tag{1.17}$$

assuming that $j_{n,\chi-1}$ exists.

An advantage of the modified zeros $j'_{\nu,\chi}$ is primarily that they satisfy inequalities (1.16) and (1.17), especially the latter, which is sometimes violated in the case of usual notation. This point will be discussed in more detail below.

Now we establish some properties of the modified zeros $j'_{\nu,\chi}$. They are stated as the following lemmas.

Lemma 1.1. *Let j' be a zero of the function*

$$C'_\nu(x) = \frac{d}{dx}C_\nu(x), \quad \nu \ge 0,$$

that satisfies the inequality $j' > \nu$. Then there exists a number $\chi > 0$ such that $j' = j'_{\nu,\chi}$.

Proof. It is sufficient to show that the above definition of $j'_{\nu,\chi}$ takes into account all possible cases. Assume that this is not the case. Recalling the definition, we find that the ignored function $C_\nu(x)$ must be monotone on $(0, \nu)$ but nonmonotone on $(0, c_{\nu,1}]$, i.e., nonmonotone on $[\nu, c_{\nu,1}]$. Then $C'_\nu(x)$ would have at least one zero c' such that $\nu < c' < c_{\nu,1}$. On the basis of (1.15), $xC'_\nu(x)$ is a decreasing function on $[\nu, c_{\nu,1}]$ and $C'(\nu) > 0$. According to our assumption, $C_\nu(x)$ is monotone on $(0, \nu)$ and $\operatorname{sign}C'_\nu(x) = \operatorname{sign}C'_\nu(\nu) = 1$ on this interval.

Among the functions $C_\nu(x)$ for which $0 \le \alpha < \pi$, only $J_\nu(x)$ is increasing on $(0, \nu]$. However, the zeros of $J'_\nu(x)$ were already defined as $j'_{\nu,1}, j'_{\nu,2}, \ldots$.

This contradiction shows that the above definition takes into account any function $C_\nu(x)$, which proves Lemma 1.1.

Now consider the function $\tilde\alpha = \tilde\alpha(\nu)$ defined by the formula

$$\tan\tilde\alpha(\nu) = J'_\nu(\nu)/Y'_\nu(\nu), \tag{1.18}$$

$$\bar\chi = \bar\chi(\nu) = 1 - \frac{\bar\alpha(\nu)}{\pi}. \tag{1.19}$$

The function $\bar\chi(\nu)$ decreases and satisfies the conditions $\bar\chi(0) = 1$ and $\bar\chi(\nu) > \lim_{\nu\to+\infty}\bar\chi(\nu) = 5/6$.

Lemma 1.2. *The function $j'_{\nu,\chi}$ is defined if $\chi > \bar\chi(\nu)$ and $\nu \ge 0$, where $\bar\chi$ is given by* (1.19).

Proof. The function $j'_{\nu,1}$ is defined for any $\nu > 0$. In the case $\chi > 1$, the values of $j_{\nu,\chi-1}$ and $j_{\nu,\chi}$ are consecutive zeros of $C_\nu(x)$. Therefore, by Rolle's theorem, $j'_{\nu,\chi}$ exists for all $\nu > 0$. Thus, we need to consider only the case $0 < \chi < 1$, where $\chi = 1 - \alpha/\pi$. This case was treated in Step B. Therefore, $j'_{\nu,\chi}$ is the second zero of $C'_\nu(x)$ and

$$\nu < j'_{\nu,\chi} < j_{\nu,\chi} = c_{\nu,1}.$$

On the basis of (1.15), $xC'_\nu(x)$ is a decreasing function on $[\nu, c_{\nu,1}]$; therefore,

$$C'_\nu(\nu) = J'_\nu(\nu)\cos\alpha - Y'_\nu(\nu)\sin\alpha > 0,$$

whence

$$\cot\alpha > Y'_\nu(\nu)/J'_\nu(\nu). \tag{1.20}$$

Comparing (1.20) with (1.18), we see that $\alpha < \bar\alpha(\nu)$ and, on the basis of (1.19),

$$\chi = 1 - \alpha/\pi > 1 - \bar\alpha(\nu)/\pi = \bar\chi(\nu),$$

which proves Lemma 1.2.

Now consider the special case $\nu = 1/2$. Then

$$C_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin(x + \alpha)$$

and $j'_{1/2,\chi}$ is a solution of the equation

$$\sin(j' + \alpha) - 2j'\cos(j' + \alpha) = 0. \tag{1.21}$$

In χ-notation, we have $\chi = k - \alpha/\pi$, $k = 1, 2, \ldots$. Therefore, $\alpha = k\pi - \chi\pi$ and Eq. (1.21) becomes

$$\sin(\chi\pi - j') + 2j'\cos(\chi\pi - j') = 0. \tag{1.21$'$}$$

According to [1], $j_{\frac{1}{2},1} = 1.1655\ldots$, so, setting

$$\delta_\chi = \chi\pi - j'_{\frac{1}{2},\chi}, \tag{1.22}$$

we obtain

$$\pi/2 < \delta_1 < 3\pi/4$$

and (1.21)' implies that

$$\pi/2 < \delta_\chi < \pi$$

and

$$\chi = -\frac{1}{2\pi}\tan\delta_\chi + \frac{1}{\pi}\delta_\chi. \tag{1.23}$$

The right-hand side of (1.23) is a convex function for $\pi/2 < \delta_\chi < \pi$ and has a minimum at $\delta_\chi = 3\pi/4$, where it takes the value $\chi_0 = 3/4 + 1/(2\pi)$, which belongs to the interval (0, 1). Thus, the function δ_χ is defined for $\chi > \chi_0$ and

$$\begin{aligned} &\frac{\pi}{2} < \delta_\chi < \frac{3\pi}{4} \quad \text{for} \quad \chi > \chi_0, \\ &\lim_{\chi\to\chi_0+0}\delta_\chi = \frac{3\pi}{4}, \quad \lim_{\chi\to+\infty}\delta_\chi = \frac{\pi}{2}. \end{aligned} \tag{1.24}$$

Combining what was said above about $j'_{1/2,\chi}$ yields the following result.

Lemma 1.3. *The function* $j'_{1/2,\chi}$ *is defined for* $\chi > \chi_0 > 3/4 + 1/2\pi = 0.90915\ldots$ *and satisfies the limit relations*

$$\lim_{\chi\to\chi_0+0} j'_{\frac{1}{2},\chi} = \frac{1}{2}, \quad \lim_{\chi\to+\infty} j'_{\frac{1}{2},\chi} = \infty.$$

Moreover, this function is strictly increasing and concave.

Proof. On the basis of (1.22) and (1.24), we have

$$j_{\frac{1}{2},\chi} = x\pi - \delta_\chi = \frac{1}{2}\tan\delta_\chi,$$

Therefore, according to (1.24), the limit relation for $j'_{1/2,\chi}$ holds. On the other hand, in view of (1.23), δ_χ is a decreasing and convex function of χ. Therefore, $j'_{1/2,\chi}$ is an increasing convex function. Lemma 1.3 is proved.

Remark 1.1. We see that $\delta_\chi = x\pi - j'_{1/2,\chi}$ is a decreasing and concave function satisfying the condition $\lim_{\chi\to+\infty}\delta_\chi = \frac{\pi}{2}$. Here, the following question arises. Let $\delta_{\nu,\chi} = j_{\nu,\chi} - j'_{\nu,\chi}$. For fixed $\nu \ge 0$, is the function $\delta_{\nu,\chi}$ decreasing and convex?

Remark 1.2. The authors conjecture that $j'_{\nu,\chi}$ is a concave function of χ for all $\nu > 0$. Although this conjecture is not proved, the following properties of zeros make it plausible, taking into account the complete monotonicity of the sequence

$$\{j'_{\nu,\chi+k+1} - j'_{\nu,\chi+k}\}_{k=0}^{\infty},$$

i.e., it decreases (see Lorch, Muldoon, and Szego [37]; Vosmanský [58, 59]).

Below is the basic theorem on the modified zeros $j'_{\nu,\chi}$.

Theorem 1.1. *The zero* $j'_{\nu,\chi}$ *is a continuous function of both* ν *and* χ *for* $\nu > 0$ *and* $\chi > \bar{\chi}(\nu)$, *where* $\bar{\chi}(\nu)$ *is defined by formula* (1.19). *Moreover*,

$$j'_{\nu,\chi'} > j'_{\nu,\chi} \quad \textit{for} \quad \chi' > \chi > \bar{\chi}(\nu), \quad \nu > 0, \tag{1.25}$$

and

$$\lim_{\chi \to \bar{\chi}(\nu)+0} j'_{\nu,\chi} = \nu.$$

Proof. First, consider the case of a fixed χ. Then the function $j'_{\nu,\chi}$ is the solution of the following nonlinear integrodifferential equation (see Watson [60, p 510]) or (1.9):

$$\frac{d}{d\nu} j' = 2j' \int_0^{\infty} \frac{j'^2 \cosh(2t) - \nu^2}{j'^2 - \nu^2} K_0(2j' \sinh t) e^{-2\nu t} dt, \tag{1.26}$$

where $j' = j'_{\nu,\chi}$.

The right-hand side of Eq. (1.26) satisfies the Lipschitz condition with respect to j', provided that $j' \neq \nu$ and $j' > 0$. Moreover, we assume that

$$j'_{\nu,\chi} > \nu > 0.$$

Therefore, the Cauchy problem for Eq. (1.26) with some initial value has a unique solution. Let $\chi > \chi_0$, where $\chi_0 \in (0, 1)$ is the same as in Lemma 1.3. Then a solution of Eq. (1.26) satisfying the initial condition

$$j'\big|_{\nu = \frac{1}{2}} = j'_{\frac{1}{2},\chi}$$

is the function $j'_{\nu,\chi}$, assuming that $j'_{\nu,\chi} > \nu > 0$.

Elbert and Laforgia [8] proved that

$$\frac{d}{d\nu} j'_{\nu,\chi} > 1 \quad \text{if} \quad j'_{\nu,\chi} > \nu. \tag{1.27}$$

Therefore, the inequality $j'_{\nu,\chi} > \nu$ is satisfied for $\nu \geq 1/2$. The continuous dependence of the solution on the initial condition guarantees that $j'_{\nu,\chi}$ is a continuous function of both ν and χ for $\chi > \bar{\chi}(\nu) > \chi_0$.

Additionally, we assume that

$$j'_{\nu,\chi'} > j'_{\nu,\chi} \quad \text{for} \quad \chi' > \chi > \chi_0 \tag{1.27'}$$

as long as $j'_{\nu,\chi} > \nu$.

To prove inequality (1.27), we first show that

$$j'_{\frac{1}{2},\chi'} > j'_{\frac{1}{2},\chi} > \frac{1}{2}.$$

This inequality holds, since, by Lemma 1.3, $j'_{1/2,\chi}$ is a strictly increasing function of χ. Thus, the uniqueness of the above initial value problem at $\nu = 1/2$ shows that $j'_{\bar{\nu},\chi'} = j'_{\bar{\nu},\chi}$ never holds if $j'_{\bar{\nu},\chi} > \bar{\nu} > 0$. Therefore, inequality (1.25) is satisfied.

Consider the case where χ_0 in (1.25) can be replaced by $\bar{\chi}(\nu)$. Since $j'_{\nu,1} > \nu$ (see Step A above), inequality (1.27) holds for $\chi' > \chi \geq 1$.

Now consider the case $\chi, \chi' < 1$. First, note that the first zero $y'_{\nu,1}$ of the function $Y'_\nu(x)$ is equal to $j'_{\nu,3/2}$. Indeed, since, for $\alpha = \pi/2$, we have $C_\nu(x, \pi/2) = -Y_\nu(x)$, the parameter χ can be set equal to 1/2, 3/2, 5/2, ...

The value 1/2 is not suitable, since, by Lemma 1.2, χ must be greater than $\bar{\chi}(\nu)$ and $\bar{\chi}$ must lie between 1 and 5/6. Therefore,

$$y'_{\nu,1} = j'_{\nu,3/2}.$$

Then, on the basis of (1.27), we obtain $j'_{\nu,3/2} > j'_{\nu,1}$ and

$$J'_\nu(x) > 0, \quad Y'_\nu(x) > 0, \quad 0 < x < j'_{\nu,1}. \tag{1.28}$$

For fixed $\nu > 0$, assume that $\overline{\chi}(\nu) < \chi < 1$.

In the definition of the zero $j'_{\nu,1}$, we saw that two zeros of the function $C'_\nu(x)$ on the interval $(0, j_{\nu,\chi})$ appear at Step B of the algorithm, namely, the zeros $c'_{\nu,1}$ and $c'_{\nu,2} = j'_{\nu,\chi}$ satisfying the relations

$$0 < c'_{\nu,1} < \nu < j'_{\nu,\chi} < j_{\nu,\chi};$$

moreover, $C'_\nu(x) < 0$ on the collection of intervals

$$(0, c'_{\nu,1}) \cup (j'_{\nu,\chi}, j_{\nu,\chi})$$

and $C'_\nu(x) > 0$ on the interval $(c'_{\nu,1}, j'_{\nu,\chi})$. Since

$$C_\nu(x) = J_\nu(x)\cos\alpha - Y_\nu(x)\sin\alpha, \quad \alpha = \pi - \chi\pi \in \left(0, \frac{\pi}{6}\right),$$

it follows from (1.28) that

$$\frac{\partial}{\partial x} C'_\nu(x) = \pi\left[J'_\nu(x)\sin\alpha + Y'_\nu(x)\cos\alpha\right] > 0 \tag{1.29}$$

for $0 < x < j'_{\nu,\chi}$. Therefore, $C'_\nu(x)$ is an increasing function of χ and ν for fixed x.

Consequently, the function $j'_{\nu,\chi}$ increases, while $c'_{\nu,1}$ decreases as χ grows. Taking into account (1.27), we conclude that (1.25) holds as long as $j'_{\nu,\chi}$ exists. In view of (1.28), we have $C'_\nu(x) < 0$ for $\alpha = \pi/2$ and $0 < x < j'_{\nu,1}$. Therefore, on the basis of (1.29), there exists a unique value $\alpha^* \in |0, \pi/2|$ or $\chi^* \in |1/2, 1|$ for $\alpha^* = \pi(1 - \chi^*)$ such that $x = j'_{\nu,\chi^*}$ is a double zero of the function $C'_\nu(x)$ at $\alpha = \alpha^*$. Therefore,

$$C'_\nu(j'_{\nu,\chi^*}) = C''_\nu(j'_{\nu,\chi^*}) = 0$$

and the Bessel equation (1.14) implies that $j'_{\nu,\chi^*} = \nu$. In view of (1.29) and (1.22), this can happen only if $\alpha^* = \overline{\alpha}(\nu)$ or $\chi^* = \overline{\chi}(\nu)$.

This proves the limit relation in Theorem 1.1. The proof of the theorem is complete.

Remark 1.3. Since the right-hand side of Eq. (1.26) satisfies the Lipschitz condition with respect to j' for $j' \neq \nu$ and $j' > 0$, it holds not only for $\nu > 0$, but also for $\nu \le 0$ and the domain of definition of the above relations can be continuously extended to $\nu \le 0$. The complete pattern of variations in the function $j'_{\nu,\chi}$ is demonstrated in the plot for $-2 \le \nu \le 2$ and $0 < j' \le 5$.

Remark 1.4. To confirm the necessity of introducing the modified definition of the zero $j'_{\nu,\chi}$, we consider a zero of the function $Y'_\nu(x)$:

$$y'_{\nu,1} = j'_{\nu,3/2}.$$

It is well known from [1] that $y'_{\frac{1}{7},1} = 2.975086$. The first two zeros of $J'_{1/2}(x)$ are $j'_{1/2,1} = 1.165561$ and $j'_{1/2,2} = 4.604217$. The monotonicity of $j'_{\nu,\chi}$ as a function of χ as given in (1.27)' can be violated if the first zero $y'_{1/2,1}$ of $Y'_{1/2}(x)$ is denoted by $y'_{1/2,1}$ in the old notation, which seems natural.

Now, we analyze the monotonicity of $j_{\nu,\chi}/j'_{\nu,\chi}$ as a function of ν for fixed $\chi' > \chi$. For this purpose, we consider the determinant

$$A = A(\delta) = \begin{vmatrix} j'_{\nu,\chi} & j'_{\nu+\delta,\chi} \\ j_{\nu,x'} & j_{\nu+\delta,x'} \end{vmatrix}. \tag{1.30}$$

It gives some information on the relation between the zeros of $C_\nu(x)$ and $C'_\nu(x)$.

Similar properties of the zeros $c'_{\nu,k}$ were examined by Laforgia [29, p. 30]. Specifically, it was proved that

$$\frac{c'_{\nu,k+2}}{c'_{\nu,k+1}} < \frac{c'_{\nu,k+1}}{c'_{\nu,k}}, \quad \nu \geq 0, \quad k = 1, 2, \ldots,$$

i.e., $c'_{\nu,k+m}/c'_{\nu,k} \downarrow$ as $k \to \infty$, $m = k + 1$.

Taking into account the asymptotic formula (see Watson [60, p. 507])

$$c'_{\nu,m} = \left(n + \frac{\nu}{2} + \frac{1}{4}\right)\pi - \alpha - O\left(n^{-1}\right), \quad \nu \geq 0,$$

where α is independent of n or ν, it is possible to prove the more accurate result

$$c'_{\nu,k+m}/c'_{\nu,k} \downarrow \quad \text{as} \quad k \to \infty, \quad m = k + 1.$$

Now consider the Turanian

$$T = \begin{vmatrix} c'_{\nu,k} & c'_{\nu,k+1} \\ c'_{\nu,k+1} & c'_{\nu,k+2} \end{vmatrix}$$

and the Wronskian

$$W(\gamma'_{\nu,k}, c'_{\nu,m}) = \begin{vmatrix} \gamma'_{\nu,k} & c'_{\nu,m} \\ \gamma'^{(1)}_{\nu,k} & c'^{(1)}_{\nu,m} \end{vmatrix},$$

where $\gamma'_{\nu,k}$ are the zeros of the derivative $C'_\nu(x)$, while $\gamma'^{(1)}_{\nu,k}$ and $c'^{(1)}_{\nu,m}$ denote the derivatives of the zeros $\gamma'_{\nu,k}$ and $c'_{\nu,m}$ with respect to ν (except for the case $c'_{\nu,1} > \nu > \gamma'_{\nu,1}$).

It is proved in the work that

$$T < 0, \quad W(\gamma'_{\nu,k}, c'_{\nu,m}) < 0, \quad \nu \geq 0.$$

The following result was proved in [7].

Theorem 1.2. *Let* $\nu_0 \geq 0$ *and* $\chi' > \chi > \overline{\chi}(\nu_0)$. *Then the determinant given by formula* (1.30) *is negative if* $\delta > 0$ *for all* $\nu > \nu_0$.

Proof. First, we note that, by Lemma 1.2, the function $j'_{\nu+\delta,\chi}$ is defined for all $\delta > 0$, since $\overline{\chi}(\nu)$ is a decreasing function. Therefore, $\overline{\chi}(\nu + \delta) < \overline{\chi}(\nu) < \chi$ and, on the basis of (1.16), we conclude that $j'_{\nu+\delta,\chi} > \nu + \delta$. Using Watson's formula and (1.26), we obtain

$$A'(\delta) = \frac{d}{d\delta}A(\delta) = 2j'_{\nu,\chi}j_{\nu+\delta,\chi'}\int_0^\infty K_0\left(2j_{\nu+\delta,\chi'}\sinh t\right)e^{-2(\nu+\delta)t}dt$$
$$- 2j_{\nu,\chi'}j'_{\nu+\chi}\int_0^\infty Q(j'_{\nu+\delta,\chi}, \nu + \delta, t)K_0(2j'_{\nu+\delta,\chi}\sinh t)e^{-2(\nu+\delta)t}dt,$$

where

$$Q(x, \nu, t) = \frac{x^2 \operatorname{c}\sinh(2t) - \nu^2}{x^2 - \nu^2}.$$

Applying formula (1.30) yields $A(0) = 0$; moreover,

$$A'(0) = 2j_{\nu,\chi'}j'_{\nu,\chi}\int_0^\infty\left[K_0\left(2j_{\nu,\chi}\sinh t\right) - K_0(j'_{\nu,\chi}\sinh t)Q(j'_{\nu,\chi}, \nu, t)\right]e^{-2\nu t}dt.$$

Let us show that $A'(0) < 0$. Since $\cosh(2t) > 1$ for $t > 0$ and $j'_{\nu,\chi} > \nu$, we have $Q(j'_{\nu,\chi'}, \nu, t) > 1$. Recalling that $K_0(u)$ is a decreasing function of u and using the inequality

$$j_{\nu,x'} > j_{\nu,\chi}, \quad \chi' > \chi > 0, \quad \nu > -\chi$$

and (1.17), we obtain $j_{\nu,x'} \geq j_{\nu,\chi} > j'_{\nu,\chi}$. Therefore,

$$K_0\left(2 j_{\nu,\chi'} \sinh t\right) < K_0(j'_{\nu,\chi} \sinh t),$$

whence $A'(0) < 0$.

Consequently, $A(\delta)$ is negative in a right neighborhood of the point $\delta = 0$. Now we show that $A(\delta) < 0$ for all $\delta > 0$.

Assume that this is not the case. Then there exists δ_1 such that $A(\delta_1) \geq 0$.

Define δ_0 as

$$\delta_0 = \min_{\delta \to 0} \{\delta : A(\delta) \geq 0\}.$$

Clearly, $\delta_0 > 0$, $A(\delta_0) = 0$, and $A(\delta) < 0$ for $0 < \delta < \delta_0$. Therefore, $A'(\delta_0) \geq 0$ and, on the basis of (1.30), we have

$$j_{\nu,\chi'} j'_{\nu+\delta_0,\chi} = j'_{\nu,\chi} j_{\nu+\delta_0,\chi'}.$$

Thus,

$$A'(\delta_0) = 2 j_{\nu,\chi'} j'_{\nu+\delta_0,\chi} \int_0^\infty \left[K_0\left(2 j_{\nu+\delta_0,\chi'} \sinh t\right) - K_0(2 j'_{\nu+\delta_0,\chi} \sinh t) Q(j'_{\nu+\delta_0,\chi}, \nu + \delta_0, t) \right] e^{-2(\nu+\delta_0)t} dt.$$

Now, $j'_{\nu+\delta_0,\chi} > \nu + \delta_0$, so $Q(j'_{\nu+\delta_0,\chi}, \nu + \delta_0, t) > 1$ for $t > 0$. Applying the inequality $j_{\nu+\delta_0,\chi'} > j'_{\nu+\delta_0,\chi}$, we obtain $K_0(2 j_{\nu+\delta_0,\chi'} \sinh t) < K_0(2 j'_{\nu+\delta_0,\chi} \sinh t)$, whence $A'(\delta_0) < 0$. However, this contradicts the inequality $A'(\delta_0) \geq 0$, which was mentioned above. Therefore, $A(\delta) < 0$ for all $\delta > 0$. The theorem is proved.

An equivalent form of Theorem 1.2 is stated as a corollary.

Corollary 1.1. Let χ and χ' be the same as in Theorem 1.2. Then $j_{\nu,\chi'}/j'_{\nu,\chi}$ is a decreasing function of ν and $\nu > \nu_0$.

In [39], which was briefly discussed above, Lorch and Newman propose some supplements to the Sturm theorem on the interlacing of zeros of solutions to second-order differential equations.

It is well known (see, e.g., Ince [21, p. 224]) that the Sturm theorem states that two linearly independent solutions of a homogeneous linear second-order differential equation have alternated internal zeros. Accordingly, the following problem arises. Given two linearly independent solutions of a homogeneous linear second-order differential equation with isolated zeros, determine which of the solutions has the largest internal zero of certain rank. A criterion for this is proved in Theorems 1–4 presented in the work. An internal zero of a solution is defined as a zero lying inside the considered interval consisting of usual points of the equation. Only internal zeros have prescribed ranks, and these zeros are counted in ascending order of magnitude. For example, for solutions of the Bessel equations (for Bessel functions), the strictly positive zeros are all internal, except for $x = 0$.

A related question arises, namely, whether, under certain conditions, there is a "zero-maximal solution," i.e., a solution having the extremal property and such that, among all nontrivial solutions with isolated zeros, there is one having the largest internal zero of prescribed rank. An affirmative answer to this question is given in Theorem 3.

The corresponding question for the smallest zero of any rank has a negative answer, since there always exists a nontrivial solution of the differential equation that vanishes at a prescribed point arbitrarily close to the lower end of an open interval consisting of usual points (see Ince [21, p. 73]).

The proofs of the indicated theorems are not presented here, but corollaries are stated that concern the Bessel equation, Bessel functions, and their zeros.

Consider the cylinder function $C_\nu(x, \theta)$. The following result is a consequence of Theorem 2 proved in the work.

Corollary 1.2. The *kth* positive zero $c_{\nu,k}$ of the function $C_\nu(x)$ is greater than a positive zero of the function $C_\nu(x, \psi)$ for $\nu \geq 0$, where $0 \leq \theta < \psi < \pi$.

Proof. Under the assumptions of Theorem 2, we have

$$\liminf_{z\to 0+} \frac{C_\nu(x,\theta)}{C_\nu(x,\psi)} = \frac{\sin\theta}{\sin\psi} < \frac{C_\nu(x,\theta)}{C_\nu(x,\psi)}$$

for all sufficiently small positive x.

As an application of Theorem 3 on a solution with the largest zero, the following result holds.

Corollary 1.3. The kth positive zero of the function $J_\nu(x)$ is greater than the kth positive zero of any other solution $C_\nu(x) \equiv J_\nu(x)$ of the Bessel equation for $\nu \ge 0$.

The proof follows from the limit relation

$$J_\nu(x)/C_\nu(x) \to 0 \quad \text{as} \quad x \to 0+.$$

The following result is a consequence of Theorem 4.

Corollary 1.4. For $\nu > 0$, $\nu \ne 1, 2, \ldots$, let $j_{-\nu,\,k}$ and $y_{\nu,\,k}$ be the kth positive zeros of the functions $J_{-\nu}(x)$ and $Y_\nu(x)$. Then

$$j_{\nu,k} > j_{-\nu,k} > y_{\nu,k} > \nu$$

for $\nu\pi$ located in quadrants I or III and

$$j_{-\nu,k} < y_{\nu,k} < j_{\nu,k}$$

for $\nu\pi$ located in quadrants II or IV, $k = 1, 2, \ldots$.

Proof. It was earlier shown that $J_\nu(x)$ is the maximum solution with respect to positive zeros. This establishes an upper bound everywhere, since $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent solutions of the Bessel differential equation when ν is not an integer.

To prove the remaining inequality, we consider the identity (see Erdelyi, et al. [15, p. 4, formula 4])

$$J_{-s}(x) = J_\nu(x)\cos\nu\pi - Y_\nu(x)\sin\nu\pi.$$

To apply the general Theorem 4, we set $z_0 = 0$, $w_1(x) = J_\nu(x)$, and $w_2(x) = -Y_\nu(x)$.

In quadrant I, let $y_1(x) = J_{-\nu}(x)$ and $y_2(x) = -Y_\nu(x)$. Then $\alpha\delta - \beta\gamma = \cos(\nu\pi) > 0$ and the middle inequality is valid ($y_{\nu,\,1} > \nu$ is known to hold for $\nu > 0$).

In quadrant III, let $y_1(x) = -J_{-\nu}(x)$. Then $y_1(x)$ is positive for x close to the point x_0 (since $J_\nu(0) = 0$, $-Y_\nu(0) = 2$) and $y_2(x) = -Y_\nu(x)$. Then $\alpha\delta - \beta\gamma = \cos(\nu\pi) > 0$, which is the case considered earlier.

In quadrant II, let $y_1(x) = -Y_\nu(x)$ and $y_2(x) = J_{-\nu}(x)$.

In quadrant IV, let $y_1(x) = -Y_\nu(x)$ and $y_2(x) = -J_{-\nu}(x)$.

Now let us analyze in more detail Muldoon and Spigler's work [48], which deals with zeros of the functions $C_\nu(x, \theta)$ and $C'_\nu(x, \theta)$ with respect to x on the interval $0 < x \le \nu$.

The authors prove that the function $C_\nu(x, \theta)$ has no such zeros if $0 \le \theta \le 5\pi/6$, while $C'_\nu(x, \theta)$ has no such zeros if $\pi/6 \le \theta \le \pi$. This assertion in implicit form is contained in Olver's work (see [50, p. 707, footnote]).

In Lorch and Newman's work [39, the corollary to Theorem 2], the Sturm method and the monotonicity of $Y_\nu(x)/J_\nu(x)$ as a function of x were used to prove that the x-zeros of $C_\nu(x, \theta)$ decrease with increasing θ for $0 \le \theta < \pi$. It is also well known that the first positive zero of $Y_\nu(x)$ is greater than ν (see Watson [60, p. 487]). Therefore, there is a value $\theta_0(\nu)$ with $\pi/2 < \theta_0(\nu) < \pi$ such that the function $C_\nu(x, \nu)$ has no x-zeros on the interval $0 < x < \nu$ for $0 < \theta < \theta_0(\nu)$.

To derive a result independent of ν, we need to examine the monotonicity of the ratio $Y_\nu(x)/J_\nu(x)$ and its limit as $\nu \to \infty$.

For this purpose, for $\nu \ge 0$ and fixed $x > 0$, let $\theta_1(x, \nu)$ and $\theta_2(x, \nu)$ denote unique numbers satisfying the inequality $0 \le \theta_{1,\,2}(x, \nu) < \pi$ and such that the functions $C_\nu(x, \theta)$ and $C'_\nu(x, \theta)$ vanish, respectively, at $\theta = \theta_1, \theta_2$. Consider the functions

$$\theta_1(x,\nu) = \frac{\pi}{2} - \arctan[Y_\nu(x)/J_\nu(x)],$$

$$\theta_2(x,\nu) = \frac{\pi}{2} - \arctan[Y_\nu'(x)/J_\nu'(x)].$$

The following assertions are proved in the work.

(1) The function $\theta_1(\nu, \nu)$ decreases from π to $5\pi/6$ as ν increases on the interval $(0, \infty)$.

(2) The function $\theta_2(\nu, \nu)$ increases from 0 to $\pi/6$ as ν increases on the interval $(0, \infty)$.

(3) If $0 \le \theta \le 5\pi/6$, then $C_\nu(x, \theta)$ has no x-zero on the interval $0 < x \le \nu$.

(4) If $\pi/6 \le \theta \le \pi$, then $C_\nu'(x, \theta)$ has no x-zero on the interval $0 < x \le \pi$.

(5) $\sqrt{3}J_\nu(x) + Y_\nu(x) < 0$ for $0 < \nu < \infty$.

(6) $\sqrt{3}J_\nu^{'(x)} - Y_\nu'(x) < 0$ for $0 < \nu < \infty$.

Remark. In fact, assertion (1) was proved in (Watson [60, p. 515]), while the values at the endpoints follow from (Spigler [54], formulas (1.9), (1.10)).

The quantities $\theta_{1,2}(\nu, \nu)$ are the values of θ such that $c_{\nu,1}(\theta) = \nu$ and $c_{\nu,1}'(\theta) = \nu$, respectively. These values have been tabulated and can be found in (Spigler [54, pp. 81–82]) for $\nu = 0.1(0.1)10$.

The proof is omitted here. Note only that the above results imply that, for every fixed positive ν, the function $C_\nu(x, \theta)$ has exactly one x-zero on the interval $(0, \nu)$ if and only if $\theta_1(\nu, \nu) \le \theta < \pi$, while the function $C_\nu'(x, \theta)$ has exactly one x-zero on $(0, \nu)$ if and only if $0 \le \theta < \theta_2(\nu, \nu)$.

2. CONVEXITY, CONCAVITY, AND MONOTONICITY OF ZEROS OF BESSEL FUNCTIONS

Now we describe some convexity, concavity, and monotonicity properties of zeros of Bessel functions and their derivatives. These properties are important and interesting in applications.

Let us begin with McCann's work [43]. The main result of [43] is the assertion that $j_{\nu,k}/\nu$ and $j_{\nu,k}'/\nu$ are strictly decreasing functions of ν. The proof is based on a variational technique applied to the eigenvalue problem

$$-(xy')' + x^{-1}y = \lambda x^{2\nu-1}y, \quad \nu > 0, \tag{2.1}$$

$$y(a) = y(1) = 0, \quad 0 < a < 1. \tag{2.2}$$

The general solution of problem (2.1), (2.2) is the function

$$y(x) = C_1 J_{1/p}\left(\lambda^{1/2}x^p/p\right) + C_2 Y_{1/p}\left(\lambda^{1/2}x^p/p\right),$$

where C_1 and C_2 are constants, while the eigenvalues λ_k of the problem are the positive roots of the transcendental equation

$$J_{1/p}\left(\lambda^{1/2}/p\right) - \frac{J_{1/p}\left(\lambda^{1/2}a^p/p\right)}{Y_{1/p}\left(\lambda^{1/2}a^p/p\right)} Y_{1/p}\left(\lambda^{1/2}/p\right) = 0. \tag{2.3}$$

Specifically, the nth eigenvalue of problem (2.1), (2.2) is the nth positive root of Eq. (2.3).

Lemma 2.1. *For* $q > 0$ *and* $0 < a < 1$, *the following assertions hold for the function*

$$f_{a,q}(x) = J_q(x) - \frac{J_q\left(a^{1/q}x\right)}{Y_q\left(a^{1/q}x\right)} Y_q(x). \tag{2.3'}$$

(i) $f_{a,q} \to J_q$ *uniformly as* $a \to 0+$ *on any interval of the form* $[\alpha, \beta]$, *where* $0 < \alpha < \beta \le 1$.

(ii) $f_{a,q}' \to J_q'$ *uniformly as* $a \to 0+$ *on any interval of the form* $[\alpha, \beta]$, *where* $0 < \alpha < \beta \le 1$.

(iii) *There are numbers* $\varepsilon, \delta > 0$ *such that* $f_{a,q}(x) > 0$ *and* $J_q(x) > 0$ *for* $a \in (0, \delta)$ *and* $x \in (0, \varepsilon)$.

The proof relies on asymptotic formulas for $J_\nu(z)$, $Y_\nu(z)$, and their derivatives as $z \to 0+$ (see Temme [56, p. 208]):

$$J_q(z) \sim \left(2^q\Gamma(q+1)\right)^{-1} z^q, \quad Y_q(z) \sim -\frac{2^q\Gamma(q)}{\pi} z^{-q},$$

$$J_q'(z) \sim q\left(2^q\Gamma(q+1)\right)^{-1} z^{q-1}, \quad Y_q'(z) \sim -\frac{q2^q\Gamma(q)}{\pi} z^{-q-1}.$$

Assertion (i) follows from $J_q(0) = 0$ and $\left|Y_q(a^{1/q}x)\right| \to \infty$ as $a \to 0+$ for any $x > 0$. By using the above asymptotic expansions, it is easy to show that

$$f_{a,q}'(x) = J_q'(x) - \frac{a^{1/q}J_q'\left(a^{1/q}x\right)}{Y_q\left(a^{1/q}x\right)}Y_q(x) - \frac{Y_q'(x)Y_q\left(a^{1/q}x\right) - a^{1/q}Y_q'\left(a^{1/q}x\right)Y_q(x)}{\left[Y_q\left(a^{1/q}x\right)\right]^2}J_q\left(a^{1/q}x\right) \to J_q'(x)$$

uniformly on an interval of the form $[\alpha, \beta]$, where $0 < \alpha < \beta \le 1$. For sufficiently small a, we have

$$f_{a,q}(x) \sim \left(2^q\Gamma(q+1)\right)^{-1} x^q\left(1-a^2\right) > \frac{1}{2}\left(2^q\Gamma(q+1)\right)^{-1} x^q \sim \frac{1}{2}J_q(x).$$

Thus, there are $\varepsilon, \delta > 0$ such that $f_{a,\,q}(x) > 0$ and $J_q(x) > 0$ if $a \in (0, \delta)$ and $x \in (0, \varepsilon)$.

This proves assertion (iii).

Lemma 2.2. *Let $z_n(a, q)$ denote the nth positive zero of the function $f_{q,\,q}$, and let $j_{q,\,n}$ denote the nth positive zero of the function J_q. Then*

$$z_n(a,q) \to j_{q,n} \quad as \quad a \to 0+ \quad if \quad q > 0.$$

The complete proof of the lemma is omitted. Note only that it is based on induction and Lemma 2.1.

The study of the eigenvalues of problem (2.1), (2.2) is based on a variational technique. For this purpose, consider the Rayleigh quotient

$$R[p,y] = \int_a^1\left(-(xy')' + x^{-1}y\right)y\,dx\Big/\int_a^1 x^{2p-1}y^2dx.$$

It is well known (see, e.g., Mikhlin [46, Sections 31, 35]) that if V is the linear set of all functions from $C^2(a, 1)$ that satisfy boundary conditions (2.2), then

$$\lambda_1(p) = \min R[p,y], \quad y \in V, \quad y \ne 0.$$

Let $y_1, y_2, \ldots, y_n$ be functions from V; A be the subset of V spanned by the collection $y_1, y_2, \ldots, y_n$; and A_1 be the orthogonal complement of A with respect to V. Then

$$\lambda_{n+1}(p) = \max\min R[p,y], \quad A \quad y \in A^1, \quad y \ne 0,$$

where the maximum is taken over all sets of n functions from V. Here,

$$\lambda_n(p) = \left[pz_n(a,1/p)\right]^2,$$

where $z_n(a, 1/p)$ is the nth positive zero of $f_{n,1/p}$. Since $x^{2p-1} \le x^{2q-1}$ for $x \in [a, 1]$ and $p \ge q$, we have $R[p, y] \ge R[q, y]$.

Moreover, $\lambda_n(p) \ge \lambda_q(q)$ for $p \ge q$, which is equivalent to the inequality

$$pz_n(a,1/p) \ge qz_n(a,1/p) \quad \text{for} \quad p \ge q.$$

Setting $t = q^{-1}$ and $s = p^{-1}$ yields

$$z_n(a,s)/s \ge z_n(a,t)/t \quad \text{for} \quad t \ge s, \tag{2.4}$$

from which, letting $a \to 0+$ and applying Lemma 2.2, we obtain

$$\frac{t}{s}j_{s,n} \ge j_{t,n} \quad \text{for} \quad t \ge s > 0. \tag{2.5}$$

Theorem 2.1. *For positive zeros $j_{q,\,n}$ of the Bessel function $J_q(x)$, it is true that*

$$\frac{t}{s}j_{s,n} > j_{t,n} > j_{s,n} \quad for \quad t > s > 0.$$

Proof. The second inequality is well known. Its proof with the use of the Sturm theorem can be found in (Bôcher [3]). An alternative proof is given in (Watson [60, p. 508]). It follows from (2.5) that $s^{-1}j_{s,\,n} \ge t^{-1}j_{t,\,n}$.

Assume that there exist numbers t and s such that $0 < s < t$ and $s^{-1}j_{s,n} \geq t^{-1}j_{t,n}$. Then

$$\frac{j_{s,n}}{s} = \frac{j_{p,n}}{p}, \quad s \leq p \leq t.$$

To simplify the presentation, we assume that $s^{-1}j_{s,n} = k > 0$. Then $j_{p,n} = kp$ for $p \in [s, t]$ and we obtain

$$0 = J_p(j_{p,n}) = J_p(kp) \quad \text{for} \quad p \in [s,t]. \tag{2.6}$$

It is well known (see Watson [60, p. 44]) that $J_p(z)$ is an analytic function of z for all z (except for, possibly, $z = 0$) and it is an analytic function of p for all p. Moreover, the series for $J_p(z)$ converges absolutely and uniformly in any closed domain of z (the origin does not belong to this domain if $\mathrm{Re}(p) < 0$) and in any bounded domain of p. It follows that $J_p(kp)$ is an analytic function of p on any compact interval not containing the point $p = 0$. Therefore, $J_p(kp) \equiv 0$ on any compact interval not containing $p = 0$.

It is also well known (see Watson [60, p. 508]) that $j_{t,1}$ is an increasing function of t and $j_{t,n} \geq j_{t,1} > j_{0,1} > 2.4$ for $t > 0$. If $p = 1/k$, then $J_{1/k}(1) = 0$. Therefore, $j_{1/k,n} = 1$ for some n, which is not possible. This contradiction proves that $s^{-1}j_{s,n} \neq t^{-1}j_{t,n}$ if $s \neq t$, which implies the desired inequality.

Theorem 2.1 states that $j_{t,n}/t$ is a strictly decreasing function of t, which contrasts with the well-known result that $j_{t,n}$ is a strictly increasing function of t for $t > 0$ (see Watson [60, p. 508]).

Tricomi's asymptotic formula (see Tricomi [57])

$$j_{t,n} \sim t + c_{1,k}t^{1/3} + c_{2,n}t^{-1/3} + \ldots, \quad t \gg 1, \quad n = 1, 2, \ldots,$$

where $c_{i,k}$ are constants independent of t, implies that

$$\lim_{t\to\infty} t^{-1}J_{t,n} = 1 \quad \text{for} \quad n = 1, 2, \ldots.$$

Corollary 2.1. For every n, the function $j_{t,n}$ uniformly satisfies the first-order Lipschitz condition on any interval $0 < a \leq t < \infty$.

Indeed, let $t, s \in [a, \infty)$ and $t \leq s$. Then

$$0 \leq j_{t,n} - j_{s,n} \leq \frac{t}{s} j_{s,n} - j_{s,n} = \frac{t-s}{s} j_{s,n} \quad \text{and} \quad j_{s,n} < (s/a)\, j_{a,n},$$

Therefore,

$$0 \leq j_{t,n} - j_{s,n} \leq \frac{j_{a,n}}{a}(t-s),$$

which implies the desired result.

By applying Theorem 2.1, we can obtain various inequalities for $j_{t,n}$. The best available upper bound for $j_{t,1}$ (see Watson [60, p. 485], note that the text contains a misprint) is

$$j_{t,1} < \left(2(t+1)(t+3)\right)^{1/2}.$$

By using tabulated numerical data from [51], it can be shown that $(11.5)^{-1}j_{11.5,1} < \sqrt{2}$. Therefore,

$$j_{t,1} < (11.5)^{-1}\, j_{11.5,1} < \sqrt{2}t < \left(2(t+1)(t+3)\right)^{1/2}$$

if $t > 11.5$. This illustrates the fact that the estimate for $j_{t,n}$ obtained in the theorem is better than a previously known inequality.

If the boundary conditions in problem (2.1), (2.2) are replaced by

$$y'(a) = y'(0) = 0, \quad 0 < a < 1,$$

then the above procedure can be modified in order to prove that $j'_{t,n}/t$ is a strictly decreasing function of t, where $j'_{t,n}$ is the nth zero of the function $J'_t(x)$.

Indeed, consider the function

$$g_{a,q}(x) = J'_q(x) - \frac{J'_q\left(a^{1/q}x\right)}{Y'_q\left(a^{1/q}x\right)} Y'_q(x).$$

Applying Lemmas 2.1 and 2.2 to it yields

$$z_n(a,q) \to j'_{q,n} \quad \text{as} \quad a \to 0+.$$

Relation (2.4) remains valid if (2.2) is replaced by (2.2)', and we have

$$\frac{t}{s} j'_{s,n} \geq j'_{t,n} \quad \text{for} \quad t \geq s > 0.$$

Proceeding as in the proof of Theorem 2.1, we see that the following result holds.

Theorem 2.2. *The zeros* $j'_{t,n}$ *of the function* $J'_t(x)$ *satisfy the inequality*

$$\frac{t}{s} j'_{s,n} \geq j'_{t,n} \quad \textit{for} \quad t > s > 0.$$

Corollary 2.2. For every n, the function $j'_{p,n}$ uniformly satisfies a first-order Lipschitz condition on any interval $0 < a \leq t < \infty$.

The proof is similar to that presented above for the zero $j_{t,\,n}$.

Corollary 2.3. It is true that

$$\lim_{t \to \infty} t^{-1} j'_{t,n} = 1.$$

As proof, we note that $t < j'_{t,1} < j_{t,\,1}$ (see Watson [60, pp. 485, 487]).

Between any two consecutive zeros of the function $J_t(x)$, there is only one zero of $J'_t(x)$ (a consequence of the mean value theorem). Moreover, it holds that

$$j'_{t,n} < j_{t,n},$$

so

$$1 < t^{-1} j'_{t,1} < t^{-1} j_{t,1} \to 1 \quad \text{as} \quad t \to \infty.$$

The monotonicity of zeros $j_{\nu,\,k}$ of the function $J_\nu(x)$ was also examined by McCann and Love [45].

The main result proved in [45] is stated as follows.

Theorem 2.3. *Let* $j_{\nu,\,1}$ *be the first positive zero of the function* $J_\nu(x)$. *Then the ratio* $j_{\nu,\,1}/(\nu + \alpha)$ *is a strictly decreasing function of* ν *for fixed* $\alpha \geq 0$ *and sufficiently large* ν *if the following conditions are satisfied*:

(i) $\alpha \leq 1.1$ *and* $3.5 \leq \nu$;

(ii) $\alpha \leq 2.411$ *and* $20.5 \leq \nu$;

(iii) $\nu > \alpha$ *and* $K(\nu) > 2\alpha\nu/(\nu - \alpha)$.

Moreover, $j_{\nu,\,1}/(\nu + \alpha) \to 1$ *as* $\nu \to \infty$. *Here*, K *denotes the smallest positive integer* m *such that* $j_{\nu,\,1} < j'_{\nu+m,1}$, *where* $j'_{\nu,1}(x)$ *is the first zero of* $J'_\nu(x)$.

In other words, for any $\alpha > 0$, *the function* $j_{\nu,\,1}/(\nu + \alpha)$ *is strictly decreasing when* ν *takes sufficiently large values. For every* α, *a lower bound for* ν *is indicated that ensures the monotonicity of* $j_{\nu,\,1}/(\nu + \alpha)$.

Proof. The theorem is proved by applying the variational method to a boundary eigenvalue problem of the form

$$-(xy')' + A^2 x^{-1} y = \lambda x^{2p-1} y, \quad a < x < 1, \tag{2.7}$$

$$y(a) = y(1) = 0, \tag{2.8}$$

where $0 < a < 1$, $0 < p$, and $A \equiv A(p) = 1 - \alpha p$ on the interval $(0, \alpha^{-1})$, $\alpha > 0$.

The general solution of problem (2.7), (2.8) is given by

$$y(x,p) = C_1 J_{A/p}\left(\lambda^{1/2} x^p / p\right) + C_2 Y_{A/p}\left(\lambda^{1/2} x^p / p\right),$$

and the eigenvalues $\lambda_n(p, a)$ coincide with the positive roots of the transcendental equation

$$g(p,a,\lambda) = J_{A/p}\left(\lambda^{1/2}/p\right) - \frac{J_{A/p}\left(\lambda^{1/2} a^p / p\right)}{Y_{A/p}\left(\lambda^{1/2} a^p / p\right)} Y_{A/p}\left(\lambda^{1/2}/p\right).$$

Consider the Rayleigh quotient

$$R[p,y,a] = \frac{\int_a^1 \left\{(-xy')' + A^2x^{-1}y\right\} + y dx}{\int_a^1 x^{2p-1}y^2 dx}.$$

It is well known (see Mikhlin [46, Sections 31 and 35]) that the eigenvalues $\{\lambda_n(p, a)\}$ of problem (2.7), (2.8) can be obtained from the Rayleigh quotient.

Let $V(a)$ be the linear space of all functions from $C^2([0, 1])$ that satisfy boundary conditions (2.8). Then

$$\lambda_1(p,a) = \min R[p,y,a], \quad y \in V(a), \quad y \neq 0.$$

If y_1 is the eigenfunction of problem (2.7), (2.8) corresponding to the eigenvalue $\lambda_1(p, a)$, then

$$\lambda_1(p,a) = R[p,y_1,a].$$

The eigenfunction y_1 is given by

$$y_1(p,a,x) = J_{A/p}\left(\lambda^{1/2}x^p/p\right) - \frac{J_{A/p}\left(\lambda_1^{1/2}a^p/p\right)}{Y_{A/p}\left(\lambda_1^{1/2}a^p/p\right)} Y_{A/p}\left(\lambda_1^{1/2}x^p/p\right),$$

where $\lambda_1 \equiv \lambda_1(p, a)$.

The subsequent results are based on 14 lemmas given with complete proofs. We omit the proofs of these lemmas and the theorem based on them, but present only the following lemmas.

Lemma 2.3. *For* $\nu > 0$, *let* $j_{\nu,1}$ *and* $j'_{\nu,1}$ *be the smallest positive zeros of the functions* $J_\nu(x)$ *and* $J'_\nu(x)$, *respectively. Let* $K(\nu)$ *be the smallest positive integer satisfying* $j_{\nu,1} < j'_{\nu+m,1}$. *Then there exists* $K(\nu)$ *such that*

(i) *for fixed* $\lambda \geq \frac{1}{4}$ *such that* $K(\lambda) \geq 3$, *it is true that* $K(\nu) \geq 3$ *for all* $\nu \geq \lambda$;

(ii) *for fixed* $\nu \geq 5$ *such that* $K(\lambda) \geq 4$, *it is true that* $K(\nu) \geq 3$ *for all* $\nu \geq \lambda$.

The existence of $K(\nu)$ for all $\nu > 0$ follows from the fact that $j_{\nu,1} > j'_{\nu,1}$ and $j'_{\mu,1} > \mu$ for all $\mu > 0$ (see Watson [60, p. 485]).

Lemma 2.4. *Let* $K(\nu)$ *be the same as in Lemma* 2.3.

Then the following assertions hold:

(i) $K(\nu) \to \infty$ *as* $\nu \to \infty$.

(ii) *If m is a positive integer, then* $J'_{\nu+m}(j_{\nu,1}) > 0$ *if and only if* $K(\nu) \leq m$.

(iii) $K(\nu) \geq 2$ *if* $\nu > 0$.

(iv) $K(\nu) \geq 3$ *if* $\nu \geq 3.5$.

(v) $K(\nu) \geq 4$ *if* $\nu \geq 18.5$.

Theorem 2.3 implies an upper bound for the zero $j_{\nu,1}$. For example, using assertion (ii) from the theorem, we obtain

$$\begin{aligned} j_{\nu,1} &< j_{20.5,1}(\nu + 2.411)(22.911)^{-1} < 1.133(\nu + 2.411) \\ &< 1.133\nu + 2.732 \quad \text{for} \quad 20.5 < \nu, \end{aligned} \tag{2.9}$$

where $j_{20,5,1}$ is taken from (Watson [60]).

The best previously known upper bound for $j_{\nu,1}$ is

$$j_{\nu,1} < \left\{\frac{4}{3}(\nu+1)(\nu+5)\right\}^{1/2} \tag{2.10}$$

(see Watson [60, p. 487]).

Computations show that the upper bound in (2.9) is better than (2.10) for $20.5 < \nu$. The theorem can also be used to obtain a lower bound for $j_{\nu,1}$. Since $j_{\nu,1} < 2.405$ and $j_{\nu,1}(\nu+\alpha)^{-1} > 1$ if $\alpha \le 2.44$ and $20.5 \le \nu$, we have $j_{\nu,1} > \nu + j_{\nu,1}$ for $20.5 \le \nu$. Using tables from [51, 19, 29], we find that

$$j_{\nu,1} > 20.5 + j_{0,1} \ge \nu + j_{0,1} \quad \text{for} \quad \nu \in [18, 20.5],$$
$$j_{\nu,1} > 18 + j_{0,1} \ge \nu + j_{0,1} \quad \text{for} \quad \nu \in [15.5, 18],$$
$$j_{\nu,1} > 15.5 + j_{0,1} \ge \nu + j_{0,1} \quad \text{for} \quad \nu \in [13.5, 15.5],$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$j_{\nu,1} > 0.075 + j_{0,1} \ge \nu + j_{0,1} \quad \text{for} \quad \nu \in [0.05, 0.035].$$

Therefore, for $0.05 < \nu$,

$$j_{\nu,1} > \nu + j_{0,1}.$$

The following result was also proved in the work.

Theorem 2.4. *If $j_{\nu,1}$ is the smallest positive zero of the function $J_\nu(x)$, then*

$$j_{\nu,1} > \nu + j_{\nu,1} \quad \textit{for} \quad \nu > 0. \tag{2.11}$$

Proof. It is well known (see Watson [60, p. 508]) that, for $\nu > 0$,

$$\frac{dj}{d\nu} = \frac{2\nu}{jJ_{\nu+1}^2(j)}\int_0^j s^{-1}J_\nu^2(s)\,ds, \quad j = j_{\nu,1}.$$

Using the relation

$$2\nu\int_0^s s^{-1}J_\nu^2(s)\,ds = J_\nu^2(z) + 2\sum_{m=1}^{\infty} J_{\nu+m}^2(z)$$

(see Watson [60, p. 152, formula (5)]), we obtain

$$\frac{dj}{d\nu} > \frac{2}{j}\left\{1 + \frac{J_{\nu+2}^2(j)}{J_{\nu+1}^2(j)}\right\}.$$

The recurrence formula

$$J_{t-1}(x) + J_{t+1}(x) = 2tx^{-1}J_t(x), \quad t = \nu + 1,$$

yields

$$J_{\nu+2}(j) = 2(\nu+1)\,j^{-1}J_{\nu+1}(j).$$

Therefore,

$$dj_{\nu,1}/d\nu > 2j_{\nu,1}^{-1}\left\{1 + 4(\nu+1)^2\, j_{\nu,1}^{-2}\right\}.$$

From the table in (Grey, Mathews, MacRobert [19, p. 317]), we find that $j_{\nu,1} < 2.4817$ on the interval [0, 005]. Therefore, on this interval,

$$\frac{dj_{\nu,1}}{d\nu} > 2(2.4817)^{-1}\left\{1 + 4(2.4817)^{-2}\right\} > 1,$$

which implies that $j_{\nu,1} > \nu + j_{\nu,0}$ for $\nu \in (0, 005]$.

Combining what was said above, we complete the proof of the theorem. The best known lower bound for $j_{\nu,1}$ is

$$j_{\nu,1} > \{\nu(\nu+2)\}^{1/2}$$

(see Watson [60, p. 486]).

Obviously, $\nu + j_{\nu,0} > \{\nu(\nu+2)\}^{1/2}$ for all $\nu > 0$.

It was proved in (McCann [43]) that $j_{\nu,1} \ge \left(\nu^2 + j_{\nu,0}^2\right)^{1/2}$. Clearly, for $\nu > 0$, we obtain $\nu + j_{\nu,1} > (\nu^2 + j_{\nu,0}^2)^{1/2}$.

In (Hethcote, [20]) it was proved that

$$j_{\nu,1} \ge \nu\pi/2 + 3\pi/4, \quad \nu \in [0, 1/2]. \tag{2.12}$$

The strict inequality (2.11) becomes equality at $\nu = 0$, while (2.12) holds as equality at $\nu = 1/2$. Therefore, for $\nu \in [0, 1/2]$, neither (2.12) nor (2.11) follow from one another. Inequality (2.11) holds for all $\nu > 0$, while (2.12) is valid only for $\nu \in [0, 1/2]$. Inequality (2.12) can physically be interpreted as follows. Consider a membrane in the form of a circular sector $0 \le r \le 1$, $0 \le \theta \le \alpha$, where $\alpha < 2\pi$, with fixed boundaries. The fundamental frequency of its transverse vibrations is proportional to the Bessel function zero $j_{\pi/\alpha,1}$. The theorem implies that the fundamental frequency is a superlinear function of α^{-1}.

A task of interest is to examine the behavior of the function

$$f(\nu) = j_{\nu,1}/(\nu + j_{0,1}).$$

Using the tables from (Olver [51]) for $\nu = k/2$, $k = 0, 1, \ldots, 41$, we see that the function $f(k/2$ increases from $f(0) = 1.0000$ to $f(4.5) = 1.1850$ and then decreases to $f(20.5) = 1.1332$.

Theorem 2.3 shows that $f(\nu)$ then decreases to 1. The behavior of this function is illustrated in Theorem 2.4.

3. CONCAVITY OF ZEROS $j_{\nu,k}$ OF THE BESSEL FUNCTION $J_\nu(x)$

Now we analyze Elbert's work [4], where the concavity of zeros $j_{\nu,k}$ of the Bessel function $J_\nu(x)$ with respect to ν is examined.

For $\nu \ge 0$, consider the kth positive zero $j_{\nu,k}$ of the Bessel function $J_\nu(x)$. The behavior of $j_{\nu,k}$ as a function of ν for fixed k was investigated by Watson [60, p. 373], Erdelyi, et al. [15], Lewis and Muldoon [34], Tricomi [57], Makai [42], and McCann [44].

In his seminal work [4] on Bessel function zeros, Elbert studied the concavity of the zero $j_{\nu,k}$ of the Bessel function $J_\nu(x)$ with respect to ν.

Before describing this work, we recall Watson's well-known formula (1.3) (see Watson [60, p. 508]) with $c_{\nu,k}$ replaced by $j_{\nu,k}$ for any real ν, which implies that $j_{\nu,k}$ is an increasing function of ν. Recall also Tricomi's asymptotic formula (see [57]) for the kth zero $j_{\nu,k}$:

$$j_{\nu,k} \sim \nu + c_{1,k}\nu^{1/3} + c_{2,k}\nu^{-1/3} + \ldots \quad \text{for} \quad \nu \gg 1, \quad k = 1, 2, \ldots,$$

where $c_{i,k}$ are constants independent of ν or k.

The definition of $j_{\nu,k}$ implies that the number sequence $c_{1,1}, c_{1,2}, c_{1,3}, \ldots$ is nondecreasing.

The behavior of $j_{\nu,k}/\nu$ was examined independently by Lewis and Muldoon [34], Makai [42], and McCann [44]. Specifically, it was shown that this ratio is a strictly increasing function. Combining this result with Tricomi's formula yields

$$0 < \frac{\nu}{j_{\nu,k}} < 1 \quad \text{for} \quad \nu > 0, \quad k = 1, 2, \ldots.$$

Elbert was the first to extend the definition of $j_{\nu,k}$ from the domain $0 \le \nu < \infty$ to $-k < \nu < \infty$ for $k = 1, 2, \ldots$. For this purpose, we consider the following series for $J_\nu(x)$:

$$J_\nu(x) = \sum_{i=0}^{\infty} (-1)^i \frac{(x/2)^{2i+\nu}}{i!\,\Gamma(i+\nu+1)}.$$

Since $J_\nu(x)$ is an analytic function of both x and ν, except at $x = 0$, the zeros of $J_\nu(x)$ can be continued analytically as long as they are positive. First, consider a neighborhood of $\nu = -l$ and $x = 0$ in the (ν, x) plane, where $l = 1, 2, \ldots$.

The function $f(\nu, x)$ defined by the formula

$$f(\nu, x) = (\nu + l)\Gamma(\nu + 1)\left(\frac{x}{2}\right)^{-\nu} J_\nu(x) = (\nu + l)\left|1 - \frac{(x/2)^2}{1!(\nu + 1)} + \dots\right.$$
$$\left.+ (-1)^{l-1}\frac{(x/2)^{2l-2}}{(l-1)!(\nu+1)\dots(\nu+l-1)}\right| + (-1)^l \frac{(x/2)^{2l}}{l!(\nu+1)\dots(\nu+l-1)}\left|1 - \frac{(x/2)^2}{(l+1)\dots(\nu+l+1)} + \dots\right|$$

is analytic in the indicated neighborhood, provided that the latter is sufficiently small. Since $f(-l, 0) = 0$ and $f'_\nu(-l, 0) = 1 \neq 0$, the equation $f(\nu, x) = 0$ has a unique analytical solution of the form

$$\nu + l = \sum_{m=1}^{\infty} a_m x^m$$

for sufficiently small x (see, e.g., Bieberbach [2, p. 192]). Moreover, $a_1 = \dots = a_{2l-1} = 0$, $a_{2l} = 2^{-l}/[l!(l-1)!]$, and a_m are real numbers. This function has an inverse $x = j(\nu)$ (see Bieberbach [2, p. 190]):

$$j(\nu) = \sum_{m=1}^{\infty} b_m \left|(\nu + l)^{1/(2l)}\right|^m, \tag{3.1}$$

where $b_l = 2|[l!(l-1)!]^{1/(2l)}| > 0$ and the coefficients b_m are all real. The function $w = z^{1/(2l)}$ is $2l$-valent. We choose the branch that maps real values of z to positive values of w. Then the function $j(\nu)$ is positive for sufficiently small positive $\nu + l$. If we choose a different branch of $w = z^{1/(2l)}$, then the function $j(\nu)$ is neither real nor positive for small positive $\nu + 1$. Therefore, on the basis of formula (3.1), the function $j(\nu)$ is only a real and positive zero of $J_\nu(x)$ for such ν. Consequently, $j(\nu)$ coincides with one of the positive functions $\{j_{\nu,k}\}_{k=1}^{\infty}$ in a right neighborhood of $\nu = -l$. Since $j(\nu)$ is unique in the above-indicated sense, there is no other function $j_{\nu,k}$ that vanishes at $\nu = -l$. Then there is no negative noninteger ν at which any function from $\{j_{\nu,k}\}_{k=1}^{\infty}$ vanishes. Since these functions are increasing in ν for fixed k, the function $j_{\nu,1}$ vanishes at $\nu = -1$ and exists only for $-1 < \nu < \infty$. Then $j_{\nu,2}$ must vanish at $\nu = -2$ and so on. Thus, only the functions $j_{\nu,k+1}, j_{\nu,k+2}, \dots$ are defined on the interval $-k - 1 < \nu < -k$, $k = 0, 1, 2, \dots$, and $j_{\nu,k+1}$ is the first positive real zero of the function $J_\nu(x)$. It should be noted that, by the Hurwitz well-known theorem on zeros of $J_\nu(x)$ (see Watson [60, p. 483], Kerimov [23, 24]), the function $J_\nu(x)$ with $-k - 1 < \nu < -k$, $k = 0, 1, 2, \dots$, has only $2k$ complex zeros; all real zeros are $\pm j_{\nu,k+1}$, $\pm j_{\nu,k+2} \dots$; and $J_\nu(x)$ has only real zeros if $\nu > -1$.

To prove the main result of the paper under discussion, namely, the fact that $j_{\nu,k}$ is a concave function of ν, we need the following assertion.

Lemma 3.1. *Let* $j = j_{\nu,k}$ *for* $k = 1, 2, \dots$. *Then*

$$(\nu + k)\frac{dj_{\nu,k}}{d\nu} \le j_{\nu,k} \quad \text{for} \quad -k < \nu \le 0, \tag{3.2}$$

$$\left(\nu + \frac{1}{2}\right)\frac{dj_{\nu,k}}{d\nu} \le j_{\nu,k} \quad \text{for} \quad \nu \ge 0. \tag{3.3}$$

Remark 3.1. It follows from (3.1) and (3.2) that the ratio $j_{\nu,k}/(\nu + 1/2)$ is a strictly decreasing function for $-k \le \nu < \infty$. This strengthens the above-described results on the behavior of $j_{\nu,k}/\nu$. A decrease in the ratio $j_{\nu,k}/(\nu + k)$ for $-k < \nu < \infty$ is not considered by the author. This issue was considered in part above in the analysis of McCann and Love's work [45] on the first zero $j_{\nu,1}$.

The proof of Lemma 3.1 is rather complicated, so it is given only in part.

The proof of the lemma in the case $-k < \nu \le 0$ is omitted. Consider the case $\nu > 0$. Since $\sinh t > t$ for $t > 0$ and $K_0(x)$ is a strictly decreasing function of x, Watson's formula (1.3) implies that

$$j' < 2j\int_0^{\infty} K_0(2jt)e^{-2\nu t}dt = \int_0^{\infty} K_0(u)e^{-\frac{\nu}{j}u}du = \frac{\arccos\frac{\nu}{j}}{\sqrt{1 - \nu^2/j^2}}, \tag{3.4}$$

where $u = jt$ and $j = j_{\nu,\,k}$. The integral on the right-hand side is evaluated using a well-known formula (see Watson [60, p. 388]). Consider the function $\alpha = \alpha(\nu)$ defined as

$$\sin\alpha = \nu/j. \tag{3.5}$$

On the basis of (3.4), $\alpha(\nu)$ is defined for all $\nu \geq 0$. Since j/ν is a strictly decreasing function, $\alpha(\nu)$ is a strictly increasing one. Therefore, combining (3.5) with the definition of $\alpha(\nu)$ yields

$$\alpha(0) = 0, \quad \lim_{\nu\to\infty}\alpha(\nu) = \pi/2,$$

and, on the basis of formula (3.1), we obtain

$$j' < (\pi/2 - \alpha)/\cos\alpha. \tag{3.6}$$

The function $(\pi/2 - \alpha)/\cos\alpha$ takes its maximum value $\pi/2$ at $\alpha = 0$. Therefore, on the basis of (3.4), we have $j' < \pi/2$. In the case $0 \leq \nu \leq 1/2$,

$$(\nu + 1/2)\frac{j'}{j} < \frac{\pi/2}{j_{\nu,k}} \leq \frac{\pi/2}{j_{\nu,1}} \leq \frac{\pi/2}{j_{0,1}} < 1,$$

since $j_{0,\,1} = 2.4048\ldots$. Thus, the lemma is valid for these values of ν.

The proof of the lemma for $1/2 < \nu < \infty$ is omitted.

Theorem 3.1. *The zeros $j_{\nu,\,k}$ of $J_\nu(x)$ are concave functions of ν on the interval $1/2 < \nu < \infty$ for $k = 1, 2, \ldots$.*

Proof. Setting $u = j_{\nu,\,k}\,t$ in (1.3) and differentiating it with respect to ν, we obtain

$$j' = 2\int_0^\infty K_0'\left(2j\sinh\frac{u}{j}\right)e^{-\frac{2\nu}{j}u}\frac{d}{d\nu}\left(2j\sinh\frac{u}{j}\right)du - 2\int_0^\infty K_0\left(2j\sinh\frac{u}{j}\right)e^{-\frac{2\nu}{j}u}\frac{d}{d\nu}\left(2j\sinh\frac{u}{j}\right)du. \tag{3.7}$$

Integration by parts of the first term on the right-hand side of the formula gives

$$\int_0^\infty K_0'\left(2j\sinh\frac{u}{j}\right)2\cosh\frac{u}{j}\,\frac{e^{-\frac{2\nu}{j}u}\dfrac{d}{d\nu}\left(2j\sinh\dfrac{u}{j}\right)}{2\cosh\dfrac{u}{j}}du = \left[K_0\left(2j\sinh\frac{u}{j}\right)\frac{e^{-\frac{2\nu}{j}u}\dfrac{d}{d\nu}\left(2j\sinh\dfrac{u}{j}\right)}{2\cosh\dfrac{u}{j}}\right]_0^\infty$$

$$-\int_0^\infty K_0\left(2j\sinh\frac{u}{j}\right)\frac{d}{du}\left[\frac{e^{-\frac{2\nu}{j}u}\dfrac{d}{d\nu}\left(2j\sinh\dfrac{u}{j}\right)}{2\cosh\dfrac{u}{j}}\right]du.$$

The first term on the right-hand side is zero, since

$$\frac{d}{d\nu}\left(2j\sinh\frac{u}{j}\right) = 2j'\sinh\frac{u}{j} - 2u\frac{j'}{j}\cosh\frac{u}{j}.$$

The function $K_0(x)$ is given by the asymptotic formula

$$K_0(x) = \begin{cases} O\left(\log\dfrac{1}{x}\right), & x > 0, \quad x \sim 0, \\ O\left(e^{-x}\right), & x \gg 1. \end{cases}$$

(see Watson [60, p. 202]).

After some computations, we obtain

$$j'' = -2\int_0^\infty K_0\left(2j\sinh\frac{u}{j}\right)e^{-\frac{2\nu}{j}u}\left\{-2\nu\frac{j'}{j}\tanh\frac{u}{j} - \frac{j'}{j}\tanh^2\frac{u}{j} + 2\frac{u}{j}\right\}du. \tag{3.8}$$

Let the expression in curly brackets be denoted by I, i.e.,

$$I = 2t - 2\nu\frac{j'}{j}\tanh t - \frac{j'}{j}\tanh^2 t, \quad \text{where} \quad t = u/j.$$

If we prove that $I > 0$, then, on the basis of formula (3.8), the theorem will be proved. Since $j' > 0$ and $\tanh t < 1$, we have

$$I > 2t - 2\nu\frac{j'}{j}\tanh t - \frac{j'}{j}\tanh t = 2t - (2\nu + 1)\frac{j'}{j}\tanh t.$$

It follows that, if $\nu \le -1/2$, then $I > 2t > 0$. If $-1/2 < \nu < \infty$, then, by Lemma 3.1, it is true that $(\nu + 1/2)j' < j$. Using the inequality $\tanh t < t$ for $t > 0$, we obtain

$$I > 2t\left[1 - \left(\nu + \frac{1}{2}\right)\frac{j'}{j}\right] > 0,$$

which completes the proof of the theorem.

4. MONOTONICITY AND CONCAVITY OF ZEROS $c_{\nu,k}$ OF THE BESSEL FUNCTION $C_\nu(x)$

In [30] Laforgia and Muldoon examined the monotonicity and concavity of Bessel function zeros $j_{\nu,k}$, $y_{\nu,k}$, and $c_{\nu,k}$ regarded as functions of ν. Specifically, Elbert's results [8] concerning the zeros $j_{\nu,k}$ were extended to the zeros $c_{\nu,k}$ of the function $C_\nu(x)$. It was proved that $c_{\nu,k}$ ($k = 2, 3, \ldots$) is a concave function of ν on the interval $0 < \nu < \infty$ for any α. For the first zero $c_{\nu,1}$, the same result was proved for $0 \le \alpha \le \pi/2$. This case also covers the important special case of the zero $y_{\nu,1}$.

The main idea behind the proof is to use the inequality

$$\frac{(\nu + 1/2)\,dc_{\nu,k}}{d\nu} < c_{\nu,k}, \quad 0 < \nu < \infty, \tag{4.1}$$

whence $c_{\nu,k}/(\nu + 1/2)$ is a decreasing function of ν. Combining this with Watson's formula for $dc_{\nu,k}/d\nu$, namely,

$$\frac{dc_{\nu,k}}{d\nu} = 2c_{\nu,k}\int_0^\infty K_0(2c_{\nu,k}\sinh t)e^{-2\nu t}dt,$$

we can prove that $c_{\nu,k}$ is a concave function of ν.

The concavity property can be used to prove various inequalities for $c_{\nu,k}$ that are stronger than (4.1). In the case of the interval $0 < \nu < \infty$, some of the proved results improve Elbert's ones for the special case of zeros $j_{\nu,k}$, but, in contrast to Elbert's, they cannot be extended to $\nu < 0$. For these values, Elbert used an indirect version of the Sturm comparison theorem and arguments based on Watson's formula. Only some of these arguments can be extended to the zeros $c_{\nu,\chi}$.

The following general theorem is proved in the work.

Theorem 4.1. *For $k = 1, 2, \ldots$, let α be fixed so that $0 \le \alpha < \pi$, and let $c_{\nu,k} = c_{\nu,k}(\mathrm{a})$ be the kth zero of the function $C_\nu(x, \alpha)$. Let f be a positive, nondecreasing, and differentiable function on (a, b), where $a \ge 0$. Assume that*

$$c_{\nu,k} > \nu + \frac{\pi}{2}f(\nu), \quad a < \nu < b. \tag{4.2}$$

Then

$$[\nu + f(\nu)]\,dc_{\nu,k}/d\nu < c_{\nu,k}. \quad a < \nu < b.$$

Proof. In view of Watson's formula for $dc_{\nu,k}/d\nu$, it suffices to prove that

$$2[\nu + f(\nu)]\int_0^\infty K_0(2c_{\nu,k}\sinh t)e^{-2\nu t}dt < 1, \quad a < \nu < b. \tag{4.3}$$

Combining $\sinh t > t$, $t > 0$, with inequality (4.2) and the fact that $K_0(x)$ is a decreasing function, we conclude that the integral on the left-hand side of (4.3) is less than the function

$$I(\nu) = \int_0^\infty K_0[\{2\nu + \pi f(\nu)\}t]e^{-2\nu t}dt. \tag{4.4}$$

Making a substitution of variables gives

$$I(\nu) = y[2\nu]^{-1}\int_0^\infty K_0(u)e^{-yu}du\ ,$$

where $y = 2\nu[2\nu + \pi f(\nu)]^{-1}$ is a number lying between 0 and 1. The last infinite integral is calculated according to (Watson [60, p. 388]), so we obtain

$$I(\nu) = y[2\nu]^{-1}\left(1 - y^2\right)^{-1/2}\arccos y.$$

Therefore, inequality (4.3) holds if we show that

$$[\nu + f(\nu)]\left[\nu + \frac{\pi}{2}f(\nu)\right]^{-1}x < \sin x, \quad 0 < x < \frac{\pi}{2}, \tag{4.5}$$

where $x = \arccos y$. It follows from (4.5) that

$$2\nu = \nu f(\nu)\cos x(1 - \cos x)^{-1},$$

so (4.5) is reduced to the form

$$\{\pi\cos x + 2(1 - \cos x)\}x < \pi\sin x, \quad 0 < x < \pi/2. \tag{4.6}$$

To prove (4.6), we write

$$g(x) = \pi\sin x - x\{\pi\cos x + 2(1 - \cos x)\}.$$

Then $g(0) = g(\pi/2) = 0$ and

$$g'(x) = -2\sin x\{\tanh(x/2) - (\pi - 2)x/2\}.$$

It follows that there exists a number x_0, $0 < x_0 < \pi/2$, such that $g'(x) > 0$ for $0 < x < x_0$ and $g'(x) < 0$ for $x_0 < x < \pi/2$. Therefore, $g(x) > 0$ for $0 < x < \pi/2$ and relation (4.6) holds. Theorem 4.1 is proved.

4.1. Linear Comparison Functions

Now we consider the special case of the theorem when $f(\nu) = 1/2$.

Theorem 4.2. *Suppose that the positive zero $c_{\nu,k} = c_{\nu,k}(\alpha)$ of the cylinder function $C_\nu(x, \alpha)$ satisfies the inequality*

$$c_{\nu,k} > \nu + \pi/4, \quad a < \nu < \infty, \tag{4.7}$$

where a is a fixed number such that $a \geq 0$. Then

$$(\nu + 1/2)\,dc_{\nu,k}/d\nu < c_{\nu,k}, \quad a < \nu < \infty, \tag{4.8}$$

and

$$d^2c_{\nu,k}/d\nu^2 < 0, \quad 0 < \nu < \infty. \tag{4.9}$$

Specifically, inequalities (4.8) *and* (4.9) *are satisfied with $a = 0$ for $k = 2, 3, \ldots$ and any α. If $k = 1$, then* (4.8) *and* (4.9) *hold with $a = 0$ for α satisfying $0 \leq \alpha \leq \pi/2$.*

Corollary 4.1. For every $k = 1, 2, \ldots$,

$$d^2y_{\nu,k}/d\nu^2 < 0, \quad 0 < \nu < \infty.$$

Proof of Theorem 4.2. The validity of (4.5) follows from Theorem 4.1. To prove inequality (4.6), we use the method applied in (Elbert [4]) for the case $c_{\nu,k} = j_{\nu,k}$. In this situation, all of Elbert's arguments hold and they lead to the formula

$$\frac{d^2}{d\nu^2}c_{\nu,k} = -2c_{\nu,k}\int_0^\infty f(\nu,t)\,K_0(2c_{\nu,k}\sinh t)\,e^{-2\nu t}dt,$$

where

$$f(\nu,t) = 2t - c_{\nu,k}^{-1}(dc_{\nu,k}/d\nu)\tanh t(2\nu + \tanh t).$$

For $t > 0$ and $\nu > a$, it follows from (4.5) that $f(\nu, t) > 0$. Therefore, (4.6) is satisfied. For $k \geq 2$, we have

$$c_{\nu,k} \geq c_{\nu,2} \geq j_{\nu,1} > \nu + \pi/4, \quad 0 < \nu < \infty.$$

The last inequality holds in view of the strict inequality

$$j_{\nu,1} > \nu + j_{0,1} > \nu + 2.404\ldots, \quad 0 < \nu < \infty \tag{4.10}$$

(see Laforgia and Muldoon [31, formula (2.4)]). Therefore, inequalities (4.8) and (4.9) hold for $a = 0$. Without imposing additional constraints on α, the inequality

$$c_{\nu,1} > \nu + \pi/4 \tag{4.11}$$

is not satisfied for $0 < \nu < \infty$. However, we can show that (4.11) holds on some interval $a(\alpha) < \nu < \infty$ irrespective of the value of α. Assume that $0 \leq \alpha \leq \pi/2$. Then $c_{\nu,1} \geq y_{\nu,1}$, which can be seen by inspecting the graph of $J_\nu(x)/Y_\nu(x)$. Therefore, to complete the proof, we need to show that

$$y_{\nu,1} > \nu + \pi/4, \quad 0 < \nu < \infty. \tag{4.12}$$

The remark following the lemma in [8, Section 2] implies that

$$dy_{\nu,1}/d\nu > 1, \quad \nu \geq 0,$$

if $y_{0,1} > 1/4$. Combining this result with the fact that $y_{0,1} = 0.89$, we conclude the validity of (4.12). Theorem 4.2 is proved.

Proof of Corollary 4.1. This result follows from Theorem 4.2.

Now we use (4.9) to sharpen (4.8) and well-known inequalities for zeros $c_{\nu,k}$.

Theorem 4.3. *Let $c_{\nu,k} = c_{\nu,k}(\alpha)$ be a positive zero of the cylinder function $C_\nu(x, \alpha)$, where α takes arbitrary values for $k = 2, 3, \ldots$, but $0 \leq \alpha \leq \pi/2$ for $k = 1$. Then*

$$c_{\nu,k} > \nu + c_{0,k}, \quad 0 < \nu < \infty \tag{4.13}$$

and

$$(\nu + \alpha_k)\,dc_{\nu,k}/d\nu < c_{\nu,k}, \quad 0 < \nu < \infty, \tag{4.14}$$

where α_k is a positive number defined by the formula

$$\alpha_k^{-1} = c_{0,k}^{-1}[dc_{\nu,k}/d\nu]_{\nu=0}. \tag{4.14'}$$

Proof. Under the assumptions made, Theorem 4.2 implies that $c_{\nu,k}$ is a concave function of ν. Combining this result with the limit relation $c_{\nu,k} \sim \nu$ as $\nu \to \infty$ yields (4.13). This inequality also follows from (Elbert and Laforgia [8, Remark 2.1]). The slope of the tangent at $(\nu, c_{\nu,k})$ is less than the slope of the chord joining the point $(0, c_{0,k})$ and $(\nu, c_{\nu,k})$, i.e.,

$$dc_{\nu,k}/d\nu < (c_{\nu,k} - c_{0,k})/\nu, \quad 0 < \nu < \infty. \tag{4.15}$$

To prove (4.14), we need the inequality

$$dc_{\nu,k}/d\nu < c_{\nu,k}(\nu + \alpha_k)^{-1}, \quad 0 < \nu < \infty,$$

which follows from (4.15) if we prove that

$$c_{\nu,k} < c_{0,k}(1 + \nu\alpha_k)^{-1}, \quad 0 < \nu < \infty.$$

This inequality follows from the concavity of the zero $c_{\nu,k}$. It expresses the fact that the graph of $c_{\nu,k}$ lies below its tangent at the point $(0, c_{0,k})$. This completes the proof of Theorem 4.3, which can be used to prove the following result.

Theorem 4.4. *Under the conditions of Theorem* 4.3*,*

$$\left[\nu + \frac{2}{\pi}c_{\nu,k}\right]dc_{\nu,k}/d\nu < c_{\nu,k}, \quad 0 < \nu < \infty.$$

Proof. This inequality can be proved using two methods. On the one hand,

$$\left[\frac{dc_{\nu,k}}{d\nu}\right]_{\nu=0} = 2c_{0,k}\int_0^\infty K_0(2c_{0,k}\sinh t)\,dt < 2c_{0,k}\int_0^\infty K_0(2c_{0,k})\,dt = \frac{\pi}{2},$$

where the value of the integral over the infinite interval is known (see Watson [60, p. 388, formula (8)]. Thus, a_k defined by formula (4.14)' satisfies the inequality $a_k > \frac{2}{\pi}c_{0,k}$ and the desired result follows from (4.14). On the other hand, this result can be obtained by applying Theorem 4.1 with $f(\nu) = \frac{2}{\pi}c_{0,k}$, $a = 0$, and $b = \infty$ and using formula (4.13).

In view of (4.8) and (4.14), which give lower bounds of the form ν + const for $c_{\nu,k}|dc_{\nu,k}/d\nu|^{-1}$, we may assume that there are upper bounds of the same form for this function. However, this is not the case, which follows from the asymptotic formula for zeros for large ν (see Olver [50], Tricomi [57]).

4.2. The Case of a Nonlinear Comparison Function

The asymptotic formula (see Tricomi [57])

$$j_{\nu,k} = \nu + a_k\nu^{1/3} + b_k\nu^{-1/3} + O\left(\nu^{-1}\right), \quad \nu \to \infty,$$

can be used to examine the monotonicity of the zero $j_{\nu,k}|\nu + c\nu^{1/3}|^{-1}$, where c is a constant. The following result holds for the function $j_{\nu,1}$.

Theorem 4.5. *The function*

$$j_{\nu,1}\left|\nu + \frac{2}{\pi}a_1\nu^{1/3}\right|^{-1}$$

decreases with increasing ν, $0 < \nu < \infty$, *where*

$$a_1 = \lim_{\nu\to\infty}\nu^{-1/3}(j_{\nu,1} - \nu) = 1.855757\dots.$$

Proof. To prove this result, we use Theorem 4.1 with $c_{\nu,k} = j_{\nu,1}$, $f(\nu) = \frac{2}{\pi}a_1\nu^{1/3}$, $a = 0$, and $b = \infty$. Before applying this theorem, we need to show that

$$j_{\nu,1} > \nu + a_1\nu^{1/3}, \quad 0 < \nu < \infty.$$

For $\nu \geq 1$, this inequality was proved in (Hethcote [20]), while, for $0 < \nu < 1$, it follows from (4.10), since $j_{0,1} > a\nu^{1/3}$ for $0 < \nu < 1$.

Note that the monotonicity proved in Theorem 4.5 is stronger for large ν and weaker for small ν than the properties based on inequality (4.8) or (4.9).

5. CONVEXITY OF ZEROS $c_{\nu,k}$ OF THE BESSEL FUNCTION $C_\nu(x)$

Elbert's results [4] concerning the convexity of zeros $j_{\nu,k}$ $(k = 1, 2, \dots)$ of the Bessel function $J_\nu(x)$ were described above.

Applying the methods used in [4], Laforgia and Muldoon [31] proved that, for $k \geq 2$, the zeros $c_{\nu,k}(\alpha)$ of $C_\nu(x)$ are concave functions of ν on the interval $(0, \infty)$ and $c_{\nu,1}(\alpha)$ is a concave function on $(0, \infty)$ at least for $0 \leq \alpha \leq \pi/2$.

In [10] Elbert and Laforgia proved that this result on the behavior of $c_{\nu,k}$ is not true when α takes values close to or smaller than π.

More specifically, it was shown that there exists a number ε (= 0.336697...) such that, if $\pi - \varepsilon < \alpha < \pi$, then $c_{\nu,1}(\alpha) \le \nu + \varepsilon - 1/2$ and $c_{\nu,1}(\alpha)$ becomes a convex function of ν on the interval $[1/2, \infty)$.

To prove the main result, we have to recollect some well-known formulas and assertions. All of them are valid in both k-notation and generalized χ-notation. As was noted above, Elbert [4] proved that the zeros $j_{\nu,k}$, $k = 1, 2, \ldots$, are concave functions in their entire domain of definition, and Laforgia and Muldoon [31] extended this result to the zeros $c_{\nu,k}$ or, in χ-notation, to $j_{\nu,\chi}$. It was shown that $j_{\nu,\chi}$ is a concave function for $\chi \ge 1/2$ and $\nu \ge 0$. Simultaneously, a question arose as to whether $j_{\nu,\chi}$ is a concave function for any $\chi > 0$.

In [10] Elbert and Laforgia answered this question in the negative and showed that, for sufficiently small χ, the function $j_{\nu,\chi}$ is convex at least for $\nu \ge 1/2$.

To prove the main result, we recall some well-known formulas.

Specifically, it is well known that

$$\int_0^\infty K_0(x)e^{-x}dx = 1 \tag{5.1}$$

(see Watson [60, p. 388]) and

$$j'' = 2j\int_0^\infty K_0(2j\sinh t)e^{-2\nu t}I(t)\,dt, \tag{5.2}$$

$$I(t) = 2\nu\frac{j'}{j}\tanh t + \frac{j'}{j}\tanh^2 t - 2t \tag{5.2'}$$

(see Elbert [4]). Although, in the indicated work, these formulas were obtained for zeros $j_{\nu,k}$ with $k = 1, 2, \ldots$, they are also valid for zeros $j_{\nu,\chi}$ for all $\chi > 0$.

Recall the formula

$$K_0(x) = \int_0^\infty e^{-x\cosh t}dt. \tag{5.3}$$

It follows from (5.3) that $K_0(x)$ is a decreasing function of x and $e^x K_0(x)$ is strictly decreasing on the interval $0 < x < \infty$. The last property is expressed by the inequality

$$e^x K_0(x) < e^y K_0(y) \quad \text{if} \quad 0 < y < x. \tag{5.4}$$

First, we prove the following lemmas.

Lemma 5.1. *There exists* $\varepsilon_0 > 0$ *such that*

$$e^{2\nu(\sinh t - t) + 2\varepsilon\sinh t} > \cosh t$$

for $\varepsilon > \varepsilon_0$ *and* $t > 0$. *The value of* ε_0 *lies between* 0.16330286 *and* 0.16330298.

Proof. Consider the function

$$f(t,\varepsilon) = 4\varepsilon\sinh t - 2\varepsilon t - \log\cosh t.$$

Find ε_0 such that $f(t,\varepsilon) > 0$ for all $t > 0$ and $\varepsilon > \varepsilon_0$. Since $2\sinh t - t > 0$ for $t > 0$, we obtain

$$\varepsilon > \frac{1}{2}\frac{\log\cosh t}{2\sinh t - t} = g(t)$$

for $t > 0$. The function $g(t)$ satisfies the conditions $\lim_{t\to+0} g(t) = 0$ and $\lim_{t\to+\infty} g(t) = 0$. Moreover, $g(t) > 0$ for $t > 0$. Therefore, $g(t)$ is bounded above and $\varepsilon_0 = \max_{t>0} g(t) = g(t_0) > 0$ for some $t_0 \in (0, \infty)$. Consequently, $f(t_0, \varepsilon_0) = 0$ and, due to the constraint $f(t_0, \varepsilon_0) \ge 0$, we have $\left(\frac{\partial}{\partial t}\right) f(t_0, \varepsilon_0) = 0$.

To determine t_0 and ε_0, we note that the function

$$\frac{\partial}{\partial t} f(t,\varepsilon) = 4\varepsilon\cosh t - 2\varepsilon - \tanh t$$

is convex for $0 \le t < \infty$, $(\partial/\partial t)f(0, \varepsilon) = 2\varepsilon > 0$, and

$$\lim_{t\to\infty} (\partial/\partial t)\, f(t,\varepsilon) = \infty$$

for any $\varepsilon > 0$.

Let us show that the function $(\partial/\partial t)f(t, \varepsilon_0)$ has exactly two simple zeros. Suppose that this is not the case. Then, since $(\partial/\partial t)f(t, \varepsilon_0)$ is convex, there is a double zero at $t = t_0$. Therefore, $(\partial/\partial t)f(t, \varepsilon_0)$ is nonnegative and $f(t, \varepsilon_0)$ is nondecreasing. However, $f(0, \varepsilon_0) = f(t_0, \varepsilon_0) = 0$. Therefore, $f(t, \varepsilon_0) \equiv 0$ on the interval $0 \le t \le t_0$, which contradicts $(\partial/\partial t)f(0, \varepsilon_0) = 2\varepsilon_0 > 0$.

Assume that ε is in a neighborhood of ε_0. Let $t_1(\varepsilon)$ and $t_2(\varepsilon)$ be zeros of $(\partial/\partial t)f(t, \varepsilon)$ such that $t_1(\varepsilon) < t_2(\varepsilon)$. Then $f(t, \varepsilon)$ has a local maximum at $t = t_1(\varepsilon)$ and a local minimum at $t = t_2(\varepsilon)$. At $t = t_2(\varepsilon)$, the second derivative $(\partial^2/\partial t^2)f(t, \varepsilon)$ must be positive. Thus, t_0 and ε_0 can be computed according to the following algorithm. Define $t_2(\varepsilon)$ as $t_2 > 0$. Then $\varepsilon = \varepsilon(t_2) = g(t_2)$. Verify the inequality

$$\left(\partial^2/\partial t^2\right) f(t,\varepsilon)\Big|_{t=t_2,\varepsilon=\varepsilon(t_2)} > 0.$$

Computations show that it holds if $t_2 \ge 1$. Let $F(t_2) = f(t_2\varepsilon(t_2))$. Then we solve the equation $F(t) = 0$. For $t^{(1)} = 1.65513$ and $t^{(2)} = 1.165514$, we find that $F(t^{(1)}) = 1.5 \times 10^{-7}$ and $F(t^{(2)}) = -2.8 \times 10^{-7}$. Therefore, the desired value t_0 lies between $t^{(1)}$ and $t^{(2)}$, so

$$\varepsilon\left(t^{(2)}\right) = 0.16330286 < \varepsilon_0 < \varepsilon\left(t^{(2)}\right) = 0.16330298,$$

as required.

The following lemmas characterize the local behavior of the function $j_{\nu,\chi}$.

Lemma 5.2. *Given an arbitrary real* $\chi > 0$ *and a fixed* $\nu_0 > 0$, *if*

$$\frac{\nu_0}{j_{\nu_0,\chi}}\frac{dj_{\nu_{0,\chi}}}{d\nu} \ge 1,$$

then

$$\frac{d^2}{d\nu^2}\, j_{\nu,\chi}\Big|_{\nu=\nu_0} > 0.$$

Proof. We introduce the following notation:

$$j_0 = j_{\nu_{0,\chi}}, \quad j_0' = \frac{d}{d\nu}\, j_{\nu,\chi}\Big|_{\nu=\nu_0} \quad \text{and} \quad j_0' = \frac{d^2}{d\nu^2}\, j_{\nu,\chi}\Big|_{\nu=\nu_0}.$$

It follows from (5.2) that

$$I(0) = 0, \quad I'(t) = \frac{2\nu_0 j_0'}{j_0}\frac{1}{\cosh^2 t} + \frac{2j_0'}{j_0}\frac{\tanh t}{\cosh^2(t)} - 2.$$

Therefore, we have

$$\lim_{t\to\infty} I'(t) = -2, \quad I'(0) = 2\left(\frac{\nu_0 j_0'}{j_0} - 1\right) > 0.$$

On the other hand,

$$(j_0/2j_0')\cosh^4 t I''(t) = -2\nu_0 \sinh t \cosh t + 1 - 2\sinh^2 t.$$

Since $\nu_0 > 0$, the function on the right-hand side decreases from 1 to $-\infty$ as t increases from 0 to ∞. Therefore, $I(t)$ is convex for small t and concave for larger t. This property of $I(t)$ shows that there is t_0 such that $I(t) > 0$ for $0 < t < t_0$ and $I(t) < 0$ for $t > t_0$. Now we use the property that $K_0(x)$ decreases for $x > 0$. It follows from (3.2) that

$$\frac{j_0''}{2j_0} = \int_0^\infty K_0(2j_0 \sinh t) I(t) e^{-2\nu_0 t} dt > \int_0^\infty K_0(2j_0 \sinh t_0) I(t) e^{-2\nu_0 t} dt = K_0(2j_0 \sinh t_0)(I_1 + I_2),$$

where

$$I_1 = \int_0^\infty \frac{j_0'}{j_0} \tanh^2 t e^{-2\nu_0 t} dt, \quad I_2 = \int_0^\infty \left(\frac{2\nu_0 j_0'}{j_0} \tanh t - 2t\right) e^{-2\nu_0 t} dt.$$

Integration by parts in I_2 yields

$$I_2 = -\left[\left(\frac{2\nu_0 j_0'}{j_0} \tanh t - 2t\right) \frac{e^{-2\nu_0 t}}{2\nu_0}\right]_0^\infty + \frac{1}{2\nu_0}\left(\frac{2\nu_0 j_0'}{j_0} \frac{1}{\cosh^2 t} - 2\right) e^{-2\nu_0 t} dt.$$

For $\nu_0 > 0$, the first term on the right-hand side is zero, so

$$\frac{j_0''}{2j_0} > K_0(2j_0 \sinh t_0) \int_0^\infty \left(\frac{j_0'}{j_0} - \frac{1}{\nu_0}\right) e^{-2\nu_0 t} dt \geq 0;$$

i.e., $j_0'' > 0$ and Lemma 5.2 is proved.

Lemma 5.3. *For* $\nu_0 < \varepsilon_0$ *and* $\chi > 0$, *let* $0 < j_{\nu_0,\chi} < \nu_0 - \varepsilon_0$, *where* ε_0 *is defined in Lemma* 5.1. *Then*

$$\frac{\nu_0}{j_{\nu_0,\chi}} \frac{d}{d\nu} j_{\nu,\chi}\bigg|_{j=j_0,\nu=\nu_0} > 1.$$

Proof. Using inequality (5.4), setting $x = 2\nu_0 \sinh t$ and $y = 2j_0 \sinh t$, and applying Watson's formula, we obtain

$$\frac{j_0'}{j_0} = \int_0^\infty K_0(2j_0 \sinh t) e^{-2\nu_0 t} dt > 2\int_0^\infty K_0(2\nu_0 \sinh t) e^{2\nu_0(\sinh t - t) + 2(\nu_0 - j_0)\sinh t - 2\nu_0 \sinh t} dt.$$

By Lemma 5.1,

$$e^{2\nu_0(\sinh t - t) + 2\varepsilon_0 \sinh t} \geq \cosh t,$$

so, substituting $x = 2\nu_0 \sinh t$ and taking into account (5.1), we obtain

$$\frac{j_0'}{j_0} > \int_0^\infty K_0(2\nu_0 \sinh t) e^{-2\nu_0 \sinh t} \cosh t dt = \frac{1}{\nu_0} \int_0^\infty K_0(x) e^{-x} dx = \frac{1}{\nu_0},$$

which proves Lemma 5.3.

Lemma 5.4. *Given an arbitrary* $\chi > 0$ *and* $\nu_0 > 0$, *if* $0 < j_{\nu_0,\chi} \leq \nu_0$, *then*

$$\frac{d}{d\nu} j_{\nu,\chi \nu=\nu_0} < 1.$$

Proof. Using Watson's formula and (5.1) yields

$$j_0' = 2j_0 \int_0^\infty K_0(2j_0 \sinh t) e^{-2\nu_0 \sinh t} dt < 2j_0 \int_0^\infty K_0(2j_0 t) e^{-2j_0 t} dt = 1,$$

as required.

Lemma 5.4 is proved.

Now we present the main result of [10].

6. MONOTONICITY AND CONCAVITY OF ZEROS $c'_{\nu,k}$ OF THE DERIVATIVE $C'_\nu(x)$ OF THE BESSEL FUNCTION $C_\nu(x)$

In the preceding sections, we described results concerning the monotonicity, convexity, and concavity of positive zeros $c_{\nu,k}$ of the Bessel function $C_\nu(x)$. The basic tool was Watson's formula for the derivative $dc_{\nu,k}/d\nu$ (see (1.9)). However, in the study of the properties of zeros $c'_{\nu,k}$, instead of Watson's formula (1.9), we have to use the more complicated integrodifferential equation

$$\frac{d}{d\nu}c'_{\nu,k} = \frac{2c'_{\nu,k}}{c'^2_{\nu,k}-\nu^2}\int_0^\infty\left(c'^2_{\nu,k}\cosh(2t)-\nu^2\right)K_0\left(2c'_{\nu,k}\sinh t\right)e^{-2\nu t}dt. \tag{6.1}$$

In the subsequent study, we apply the well-known formulas

$$\int_0^\infty K_0(x)e^{-ax}dx = A(a) = \begin{cases}\dfrac{\arccos a}{\sqrt{1-a^2}}, & |a|<1,\\ 1, & a=1,\\ \dfrac{\operatorname{arccosh} a}{\sqrt{a^2-1}}, & a=1,\end{cases} \tag{6.2}$$

$$\int_0^\infty K_0(x)x\,dx = 1$$

(see Watson [60, p. 388]) and the inequality

$$j'_{\nu,1} > \sqrt{\nu(\nu+2)}, \quad \nu > 0 \tag{6.3}$$

(see Watson [60, p. 487]). Additionally, the following result is used.

Lemma 6.1. *For* a > 1, *it is true that*

$$\frac{\operatorname{arccosh} a}{\sqrt{a^2-1}} < \frac{2^3}{15}-\frac{3}{5}a+\frac{2}{15}a^2. \tag{6.4}$$

Proof. Since $\operatorname{arccosh} a = \log\left(a+\sqrt{a^2+1}\right)$, we need to show that the function

$$f(a) = \sqrt{a^2-1}\left(\frac{2^2}{15}-\frac{3}{5}a+\frac{2}{15}a^2\right)-\log\left(a+\sqrt{a^2-1}\right)$$

is positive. Since $f(1) = 0$, it suffices to prove that $f'(a) > 0$ for $a > 1$. We obtain $\sqrt{a^2-1}f(a) = \frac{2}{5}(a-1)^3$, which is positive for $a > 1$.

Passing to χ-notation and setting $\chi = k - \alpha/\pi$, we consider the function $j'_{\nu,\chi}$ defined by the formula

$$j'_{\nu,\chi} \equiv c'_{\nu,k}, \quad \nu > 0,$$

where χ is a parameter. Then $j'_{\nu,\chi}$ solves integrodifferential equation (6.1). It can be shown that the right-hand side of this equation satisfies the Lipschitz condition with respect to $c'_{\nu,k}$, provided that $c'_{\nu,k} > 0$ and $c'_{\nu,k} \neq |\nu|$. Moreover, the Cauchy problem for Eq. (6.1) with a certain initial condition has a unique solution in the domain $c > |\nu|$. This uniqueness does not hold for $c = |\nu|$, and the case $0 < c' < |\nu|$ requires a special consideration. What was said allows the authors to examine various properties of the zeros $c'_{\nu,k}$ with respect to α.

The results obtained supplement similar ones in [39] concerning positive zeros $c_{\nu,k}$ of the function $C_\nu(x)$. Specifically, it is shown that $j_{\nu,\chi}$ is an increasing function of χ ($\alpha = (k-\chi)\pi$).

For this purpose, we first consider the case $\nu = 1/2$. Then

$$C_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\sin(x+\alpha)$$

and the function $\gamma = \gamma(\chi) = j'_{1/2,\chi}$ satisfies the relation

$$\tan\Gamma = 2\gamma, \tag{6.5}$$

where $\Gamma = \Gamma(\chi) = \gamma + (k - \chi)\pi$. If $\gamma = 1/2$, then

$$\chi = \chi_0 = \frac{3}{4} + \frac{1}{2\pi} = 0.90915\dots.$$

Differentiating Eq. (6.5) with respect to χ, we obtain $\gamma'(1 - 2\cos^2\Gamma) = \pi$.

Consider only the case $\gamma(k) > 1/2$. Then it follows from (6.5) that $\Gamma > 1$. Therefore, $\cos^2\Gamma < 1.2$ and $\gamma' > 0$ if $\chi > \chi_0$. This means that $j'_{1/2,\chi}$ is a strictly increasing function of χ.

Now consider the solution $\sigma = \sigma(\nu)$ of the integrodifferential equation

$$\frac{d}{d\nu}\sigma = \frac{2\sigma}{\sigma^2 - \nu^2}\int_0^\infty \left(\sigma^2\cosh 2t - \nu^2\right) K_0(2\sigma\sinh t)e^{-2\nu t}dt \tag{6.6}$$

with initial condition

$$\sigma(1/2) = \gamma(\chi) \quad \text{for} \quad \chi > \chi_0. \tag{6.7}$$

If $\gamma(\chi) > 1/2$, then the uniqueness of the solution to the initial value problem (6.6), (6.7) implies that $\sigma(\nu) = j'_{\nu,\chi}$ and

$$j'_{\nu,\chi_1} < j'_{\nu,\chi_2} \quad \text{if} \quad \chi_0 < \chi_1 < \chi_2 \quad \text{and} \quad j'_{\nu,\chi_1} > |\nu|. \tag{6.8}$$

Since $\alpha = (k - \chi)\pi$, the kth zero $c'_{\nu,k}$ of $C'_\nu(x)$ is a decreasing function of α for $0 < \alpha < \pi$ as long as $j'_{\nu,\chi} > |\nu|$. This proves the assertion in the general case.

It is proved in the work that $j'_{\nu,\chi}$ is a concave function of ν if $j'_{\nu,\chi} > |\nu| > 0$. It follows that $j'_{\nu,\chi}$ is a concave function for any $\chi = 2, 3, \dots$. In the case of zeros $j'_{\nu,\chi}$ of $\frac{d}{dx}J_\nu(x)$, this property also holds at $k = 1$ for any $\nu \geq 0$. Thus, the main result of this important work is stated as follows.

Theorem 6.1. *If*

$$\sigma(\nu) > \begin{cases} \sqrt{2\nu}, & 0 < \nu \leq 1, \\ \nu + 1/2, & \nu \geq 1/2, \end{cases} \tag{6.9}$$

then

$$d^2\sigma/d\nu^2 < 0, \tag{6.10}$$

i.e., if $j_{\nu,\chi} > |\nu|$ *for* $\chi \geq 1$, *then* $j'_{\nu,\chi}$ *is a concave function of* ν.

The complete proof (which is lengthy) is not presented here. It requires a separate analysis of the cases (a) $0 < \nu \leq 1/2$ and (b) $\nu > 1/2$ and is based on the following lemmas, which are of interest on their own.

First, formula (6.6) is rewritten as

$$\frac{d\sigma}{d\nu} = 2\int_0^\infty f(t,\sigma,\nu)K_0(2\sigma\sinh t)dt,$$

where

$$f(t,\sigma,\nu) = \frac{\sigma}{\sigma^2 - \nu^2}\left(\sigma^2\cosh 2t - \nu^2\right)e^{-2\nu t}.$$

Differentiation with respect to ν gives the expression

$$\frac{d^2\sigma}{d\nu^2} = 2\int_0^\infty [f_\sigma\sigma' + f_\nu]K_0(2\sigma\sinh t)dt + 2\int_0^\infty fK'_0(2\sigma\sinh t)2\sigma'\sinh t\,dt, \tag{6.11}$$

where $f_\sigma = \partial/f\partial\sigma$, $f_\nu = \partial f/\partial\nu$, and $\sigma' = d\sigma/d\nu$.

Integrating the second integral in (6.11) by parts, we see that it is equivalent to the relation

$$\int_0^\infty f \frac{\sigma'}{\sigma} \frac{\sinh t}{\cosh t} K_0'(2\sigma \sinh t) 2\sigma \cosh t dt = \left[f \frac{\sigma'}{\sigma} \tanh t K_0(2\sigma \sinh t) \right]_0^\infty - \int_0^\infty K_0(2\sigma \sinh t) \frac{d}{dt}\left\{ f \frac{\sigma'}{\sigma} \tanh t \right\} dt, \tag{6.12}$$

where the term in square brackets vanishes due to the asymptotic formula for $K_0(x)$.

For the integral on the right-hand side of (6.12), we obtain

$$\frac{d}{dt}\{f \tanh t\} = f_t \tanh t + f \frac{1}{\cosh^2 t},$$

so

$$\frac{d^2\sigma}{d\nu^2} = 2\int_0^\infty \left(f_\sigma \sigma' + f_\nu - \frac{\sigma'}{\sigma} f_t \tanh t - \frac{\sigma'}{\sigma} f \frac{1}{\cosh^2 t} \right) K_0(2\sigma \sinh t) dt.$$

Further simplifications yield the following formula the second derivative of $\sigma(\nu)$:

$$\frac{d^2\sigma}{d\nu^2} = 2\int_0^\infty \left[\frac{\sigma'}{\sigma} \tanh^2 t + 2\nu \frac{\sigma'}{\sigma} \tanh t - 2t + \frac{4\sigma^2 \tanh^2 t}{(\sigma^2 - \nu^2)(\sigma^2 \cosh 2t - \nu^2)} (\nu - \sigma\sigma') \right] f K_0(2\sigma \sinh t) dt, \quad \sigma \neq |\nu|. \tag{6.13}$$

Now we prove the following lemmas.

Lemma 6.2. *Let* $\sigma(\nu)$ *be the solution of integrodifferential equation* (6.6) *in a neighborhood of* $\nu = \nu_0$ *and* $\sigma(\nu_0) > |\nu_0|$. *Then*

$$\left.\frac{d}{dt}\sigma(\nu)\right|_{\nu=\nu_0} > 1. \tag{6.14}$$

Proof. It is well known (see Watson [60, p. 388]) that

$$\int_0^\infty K_0(x) e^{-x} dx = 1.$$

Substituting $x = 2\sigma \sinh t$ into this integral, we obtain

$$\int_0^\infty K_0(2\sigma \sinh t) e^{-2\sigma \sinh t} 2\sigma \cosh t dt = 1.$$

To prove inequality (6.14) with the use of (6.6), it is sufficient to show that

$$(\sigma^2 \cosh 2t - \nu^2) e^{-2\sigma \sinh t} > (\sigma^2 - \nu^2) e^{-2\nu t},$$

which holds since $\sigma > |\nu|$.

Corollary 6.1. Let $j'_{\nu_0,\chi} > |\nu_0|$. Then $j'_{\nu,\chi} > j'_{\nu_0,\chi} + \nu - \nu_0$, $\nu > \nu_0$.

Proof. This result follows from the fact that $j'_{\nu,\chi} - \nu$ is an increasing function of ν; i.e., by Lemma 6.2, where $\sigma(\nu) = j'_{\nu,\chi}$, we obtain

$$\frac{d}{dt} j'_{\nu,\chi} - 1 > 0.$$

Lemma 6.3. *If the function* $\sigma(\nu)$ *satisfies the same conditions as in Lemma* 6.2 *and the additional condition*

$$\nu_0 \left.\frac{d}{dt}\sigma(\nu)\right|_{\nu=\nu_0} < \sigma(\nu_0),$$

then

$$\left.\frac{d^2}{d\nu^2}\sigma(\nu)\right|_{\nu=\nu_0} < 0.$$

Proof. Under the assumptions made, we have

$$2\nu_0\frac{\sigma'}{\sigma}\tanh t - 2t < 2|\nu_0|\frac{\sigma'}{\sigma} - 2t < 0.$$

Then, in view of (6.13), it suffices to show that

$$g(t) = \frac{\sigma'}{\sigma}\tanh^2 t + \frac{4\sigma^2 st^2 t}{\left(\sigma^2 - \nu^2\right)\left(\sigma^2\cosh 2t - \nu^2\right)}(\nu - \sigma\sigma') < 0.$$

Since $\sigma' > 1$, $\nu - \sigma\sigma' < 0$, and

$$\frac{\sigma^2\cosh^2 t}{\left(\sigma^2\cosh 2t - \nu^2\right)} > \frac{1}{2}, \quad t \geq 0,$$

we obtain

$$\frac{g(t)}{\tanh^2 t} < \frac{\sigma'}{\sigma} + \frac{4}{\sigma^2 - \nu^2}\frac{1}{2}(\nu - \sigma\sigma') = \frac{2\nu\sigma - \sigma'\left(\sigma^2 + \nu^2\right)}{\sigma\left(\sigma^2 - \nu^2\right)} < \frac{2\nu\sigma - \left(\sigma^2 + \nu^2\right)}{\sigma\left(\sigma^2 - \nu^2\right)} < 0,$$

which proves Lemma 6.3.

According to Lemmas 6.2 and 6.3, the function $j'_{\nu,\chi}$ is concave if $\nu \leq 0$. To prove this fact in the case $\nu > 0$, we need to impose an additional constraint on $j'_{\nu,\chi}$ in order to satisfy the condition $\nu\sigma' < \sigma$ in Lemma 6.3. This constraint is formulated in the second condition in (6.9) in Theorem 6.1. The proof of the theorem is dropped because of its cumbersomeness.

Corollary to Theorem 6.1. For $x \geq 1$ and $\nu \geq 0$, $j'_{\nu,\chi}$ is a concave function of ν. The same holds for $\chi \geq 1$ and $\nu < 0$ under the additional constraint $j'_{\nu,\chi} > |\nu|$.

Proof. If $\nu \geq 0$, then the well-known inequality

$$j'_{\nu,1} > \sqrt{\nu(\nu + 2)}, \quad \nu > 0$$

(see Watson [60, p. 487]) implies that $j'_{\nu,1}$ satisfies the conditions of Theorem 6.1, since $j'_{\nu,1}$ is a concave function. The same holds for $j'_{\nu,\chi}$ if $x > 1$, since $j_{\nu,\chi} > j_{\nu,1}$.

In the case $\nu \geq 0$, the assertion of the corollary follows from the lemmas.

CONCLUSIONS

Numerous new results concerning the properties of positive real zeros of first and second kind Bessel functions, Bessel general cylinder functions, and their derivatives were overviewed. The monotonicity, convexity, and concavity of zeros with respect to the order were analyzed in detail. The overview covers nearly all important works having been published to date in the literature. Many of the results are given with detailed proofs.

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