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On a Class of Optimal Control Problems with Distributed and Lumped Parameters

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Abstract—The optimal control of moving sources governed by a parabolic equation and a system of ordinary differential equations with initial and boundary conditions is considered. For this problem, an existence and uniqueness theorem is proved, sufficient conditions for the Fréchet differentiability of the cost functional are established, an expression for its gradient is derived, and necessary optimality conditions in the form of pointwise and integral maximum principles are obtained.

Keywords: moving sources, integral identity, maximum principle, Hamilton–Pontryagin function, necessary optimality conditions, control problem for a parabolic equation.

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INTRODUCTION

Processes concerning the influence of moving sources on various physical media have long been a subject of interest in physics, engineering, and mathematics. Practical examples of moving action sources include electron, laser, and ion rays; electric arcs; and currents induced by moving inductors. Additional types of moving action sources are ions, protons, and other high-speed particles; electric currents (electric arcs, conduction and induced currents); plasma flows; chemical and nuclear reactions; substance sources (in diffusion); oscillation sources (mechanical, acoustic, and electromagnetic); and sources of pressure. These sources are used in many processes, such as metal melting and refining in metallurgy; heat treatment, welding, and microprocessing in mechanical engineering and instrument making; the manufacturing of semiconductors and resistors in microelectronics engineering; and activation, exposure, and drying in biology, medicine, and agriculture.

Classical results concerning the optimal control of distributed systems can be found in [1–5]. Although problems with controlled moving sources are of applied importance, they have received the least attention thus far (see [2, 5–7]). For the first time, the problem of optimal control of moving sources for distributed parameter systems was theoretically formulated in [2, 5], where numerous examples of systems with moving sources of various natures were given and the basic features of such systems that prevent addressing them with well-known methods were indicated.

A major feature of control systems for moving sources is that they are nonlinear with respect to the control determining the law of source motion. This is especially clear when the control problem is formulated in terms of moments. The moment problem becomes nonlinear. Accordingly, the method of moments, which is widely used to find optimal controls in linear systems with distributed and lumped parameters, becomes unsuitable for systems with controlled moving sources.

Note that only distributed parameter systems were addressed in the indicated practical examples. At the same time, numerous dynamic systems involve auxiliary elements with lumped parameters that are essential to control processes. The behavior of such systems is described by a set of ordinary and partial differential equations with initial and boundary conditions.

In [5] the optimal control of point sources was studied assuming that the control functions are only the intensities of fixed sources. A variational method for the optimal control of moving sources in the case of systems described only by the heat equation was considered in [6]. The optimal control of source motion for systems governed by the heat equation and a system of ordinary differential equations was addressed in [7]. Specifically, the problem of the most accurate heating was studied; i.e., the task was to find admissible controls $\overline{\vartheta} = (p(t), \vartheta(t))$ that, over a fixed time T, drive the system from its initial state $u(x,0) = \varphi(x)$ to one that deviates least (in a certain sense) from the desired state $u(x,T) = y(x)$. The right-hand side of the system of differential equations involved only $s_k(t)$.

In this paper, a variational method is applied to the problem of optimal control of moving sources for systems governed by a parabolic equation, in conjunction with sets of ordinary differential equations. The rms deviation of the system's state from its desired state at any moment of time is used as an optimality criterion. For this problem, we prove an existence and uniqueness theorem, establish sufficient conditions for the Fréchet differentiability of the cost functional, find an expression for its gradient, and obtain necessary optimality conditions in the form of pointwise and integral maximum principles.

1. FORMULATION OF THE PROBLEM

Let $l > 0$ and $T > 0$ be given numbers, $0 \le x \le l$, $0 \le t \le T$, $\Omega_t = (0,l)\times(0,t)$, and $\Omega = \Omega_T$. In what follows, we will need the function spaces $W_2^{1,0}(\Omega)$, $W_2^{1,1}(\Omega)$, $V_2(\Omega)$, and $V_2^{1,0}(\Omega)$, which are introduced, for example, in [5].

Let the state of a controlled process be described by functions $u(x,t)$ and $s(t)$. Assume that, inside the domain Ω, the function $u(x,t)$ satisfies the parabolic equation

$$\frac{\partial u(x,t)}{\partial t} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \sum_{k=1}^{n} p_k(t)\delta(x - s_k(t)), \tag{1.1}$$

with initial and boundary conditions

$$u(x,0) = \varphi(x), \quad 0 \le x \le l, \tag{1.2}$$

$$\frac{\partial u(0,t)}{\partial x} = 0, \quad \frac{\partial u(l,t)}{\partial x} = 0, \quad 0 < t \le T, \tag{1.3}$$

where $a > 0$ is a given number, $\varphi(x) \in L_2(0,l)$ is a given function, $\delta(\cdot)$ is the delta function, and $p(t) = (p_1(t), p_2(t), \ldots, p_n(t)) \in L_2(0,T;R^n)$ is the control function.

Assume also that the functions $s_k(t) \in C[0,T]$, $k = \overline{1,n}$, are the solution of the Cauchy problem

$$\frac{ds_k(t)}{dt} = f_k(s(t), \vartheta(t), t), \quad 0 < t \le T, \quad s_k(o) = s_{k0}, \quad k = \overline{1,n}, \tag{1.4}$$

where $s_{k0} \in [0,l]$ is a given number; the functions $f_k = f_k(s,\vartheta,t)$ $(k = \overline{1,n})$ are assumed to be given; $\vartheta = \vartheta(t) = (\vartheta_1(t), \vartheta_2(t), \ldots, \vartheta_r(t)) \in L_2(0,T;R^r)$ is a control function such that the following constraint on the position of the moving action is satisfied: $s_k(t) = s_k(t;\vartheta)$: $0 \le s_k(t) \le l$, $k = \overline{1,n}$.

The pair of functions $\overline{\vartheta} = (p(t), \vartheta(t))$ is called a *control*. For brevity, let $H = L_2(0,T;R^n)\times L_2(0,T;R^r)$ denote the Hilbert space of pairs $\overline{\vartheta} = (p(t), \vartheta(t))$ with inner product

$$\left\langle \overline{\vartheta}^1, \overline{\vartheta}^2 \right\rangle_H = \int_0^T \left[\left(p^1(t), p^2(t)\right) + \left(\vartheta^1(t), \vartheta^2(t)\right)\right] dt$$

and the norm $\|\overline{\vartheta}\|_H = \sqrt{<\overline{\vartheta},\overline{\vartheta}>_H} = \sqrt{\|p\|_{L_2}^2 + \|\vartheta\|_{L_2}^2}$, where $\overline{\vartheta}^k = (p^k, \vartheta^k)$, $k = 1,2$.

The set of admissible controls is defined as

$$V = \{(p,\vartheta) \in H : 0 \le p_i \le A_i, |\vartheta_j| \le B_j, i = \overline{1,n}, j = \overline{1,r}\}, \tag{1.5}$$

where $A_i > 0$ $(i = \overline{1,n})$ and $B_j > 0$ $(j = \overline{1,r})$ are given numbers. Consider the functional

$$J(\tilde{\vartheta}) = \int_0^l \int_0^T [u(x,t) - \tilde{u}(x,t)]^2 dx dt + \alpha_1 \sum_{k=1}^{n} \int_0^T [p_k(t) - \tilde{p}_k(t)]^2 dt + \alpha_2 \sum_{m=1}^{r} \int_0^T [\vartheta_m(t) - \tilde{\vartheta}_m(t)]^2 dt, \tag{1.6}$$

where $\alpha_1, \alpha_2 \geq 0, \alpha_1 + \alpha_2 > 0$ are given parameters and $\omega = (\tilde{p}(t), \tilde{\vartheta}(t)) \in H$, $\tilde{u}(x,t)) \in L_2(\Omega)$, $\tilde{p}(t) \in L_2(0,T;R^n)$, and $\tilde{\vartheta}(t) \in L_2(0,T;R^r)$ are given functions.

Let us formulate the following problem: find an admissible control $\overline{\vartheta} = (p(t), \vartheta(t))$ from the set V and a corresponding solution $(u(x,t), s(t))$ of problem (1.1)–(1.4) that minimize functional (1.6).

2. WELL-POSEDNESS OF THE PROBLEM

Before analyzing the well-posedness of the optimal control problem (1.1)–(1.6), we state an auxiliary theorem from [8].

Theorem 1 (see [8]). *Let H be a uniformly convex Banach space, V be a closed bounded subset of H, $I(\vartheta)$ be a lower semicontinuous functional that is bounded below on V, and $\alpha > 0$ and $\beta \geq 1$ be given numbers. Then there exists a dense subset K of H such that, for any $\omega \in K$, the functional $J_\alpha(\vartheta) = I(\vartheta) + \alpha\|\vartheta - \omega\|_H^\beta$ reaches its least value on V. If $\beta > 1$, then the minimum value of $J_\alpha(\vartheta)$ on V is reached at a single element.*

Definition 1. The weak solution of problem (1.1)–(1.4) with control $\overline{\vartheta} = (p(t), \vartheta(t)) \in V$ is a pair of functions $(u(x,t), s(t))$ from $(V_2^{1,0}(\Omega), C[0,T])$, where $u = u(x,t)$ satisfies the integral identity

$$\int_0^l \int_0^T \left[-u\frac{\partial \eta}{\partial t} + a^2 \frac{\partial u}{\partial x}\frac{\partial \eta}{\partial x}\right] dxdt = \int_0^l \varphi(x)\eta(x,0)dx + \sum_{k=1}^n \int_0^T p_k(t)\eta(s_k(t),t)dt \tag{2.1}$$

for any $\eta = \eta(x,t) \in W_2^{1,1}(\Omega)$ with $\eta(x,T) = 0$, while $s_k(t) = s_k(t;\vartheta)$ satisfies the integral equation

$$s_k(t) = \int_0^t f_k(s(\tau), \vartheta(\tau), \tau)d\tau + s_{k0}, \quad 0 \leq t \leq T, \quad k = \overline{1,n}. \tag{2.2}$$

It follows from the results of [9] that, for each fixed control $\overline{\vartheta} \in V$, the boundary value problem (1.1)–(1.4) has a unique solution from $(V_2^{1,0}(\Omega), C[0,T])$. Let the conditions stated in the formulation of problem (1.1)–(1.6) be satisfied. Then problem (1.1)–(1.6) has at least one solution. Note that problem (1.1)–(1.6) with $\alpha_j = 0$, $j = \overline{1,2}$, is ill-posed in the classical sense [10]. Nevertheless, the following result holds.

Theorem 2. *There is a dense subset K of H such that, for any $\omega \in K$, problem (1.1)–(1.6) with $\alpha_i > 0, i = \overline{1,2}$, has a unique solution.*

Proof. Let us prove the continuity of the functional

$$J_0(\overline{\vartheta}) = \|u(x,t) - \tilde{u}(x,t)\|^2_{L_2(\Omega)}.$$

Let $\Delta\overline{\vartheta} = (\Delta p, \Delta\vartheta) \in V$ be the increment of the control at an element $\overline{\vartheta} = (p, \vartheta) \in V$ such that $\overline{\vartheta} + \Delta\overline{\vartheta} \in V$. Define

$$\Delta u \equiv \Delta u(x,t) = u(x,t;\overline{\vartheta} + \Delta\overline{\vartheta}) - u(x,t;\overline{\vartheta}),$$
$$\Delta s_k \equiv \Delta s_k(t) = s_k(t;\overline{\vartheta} + \Delta\overline{\vartheta}) - s_k(t;\overline{\vartheta}).$$

It follows from (1.1)–(1.4) that the function Δu is a weak solution of the boundary value problem

$$\frac{\partial \Delta u}{\partial t} = a^2 \frac{\partial^2 \Delta u}{\partial x^2} + \sum_{k=1}^n [(p_k + \Delta p_k)\delta(x - (s_k + \Delta s_k)) - p_k\delta(x - s_k)], \quad (x,t) \in \Omega, \tag{2.3}$$

$$\frac{\partial \Delta u(0,t)}{\partial x} = 0, \quad \frac{\partial \Delta u(l,t)}{\partial x} = 0, \quad 0 < t \leq T, \tag{2.4}$$

$$\Delta u\big|_{t=0} = 0, \quad x \in [0,l], \tag{2.5}$$

while the functions $\Delta s_k, k = \overline{1,n}$, are the solution of the Cauchy problem

$$\frac{d\Delta s_k(t)}{dt} = \Delta f_k(s(t), \vartheta(t), t), \quad \Delta s_k(0) = 0, \quad k = \overline{1,n}, \tag{2.6}$$

where $\Delta f_k(s(t), \vartheta(t), t) = f_k(s + \Delta s, \vartheta + \Delta\vartheta, t) - f_k(s, \vartheta, t)$.

Let us prove that $\Delta u(x,t)$ satisfies the estimate

$$\|\Delta u\|_{V_2^{1,0}(\Omega)} \leq c_1 \|\Delta\overline{\vartheta}\|_H, \tag{2.7}$$

where $c_1 > 0$ is a constant. Multiplying both sides of Eq. (2.3) by $\eta = \eta(x,t)$ and integrating the result by parts, we obtain the relation

$$\int_0^l\int_0^T\left[-\Delta u\frac{\partial\eta}{\partial t} + a^2\frac{\partial\Delta u}{\partial x}\frac{\partial\eta}{\partial x}\right]dxdt = \sum_{k=1}^n\int_0^T[(p_k + \Delta p_k)\eta(s_k + \Delta s_k, t) - p_k\eta(s_k,t)]dt. \tag{2.8}$$

Let $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$. Setting

$$\eta(x,t) = \begin{cases}\Delta u(x,t), & t\in(t_1,t_2],\\ 0, & t\in[0,t_1]\cup(t_2,T]\end{cases}$$

in (2.8) yields the integral identity

$$\frac{1}{2}\int_0^l|\Delta u(x,t)|^2dx\Bigg|_{t=t_1}^{t=t_2} + a^2\int_0^l\int_0^t\left|\frac{\partial\Delta u}{\partial x}\right|^2dxdt\Bigg|_{t=t_1}^{t=t_2} = \sum_{k=1}^n\int_0^T[(p_k + \Delta p_k)\eta(s_k + \Delta s_k, t) - p_k\eta(s_k,t)]dt. \tag{2.9}$$

By applying the mean value theorem to $\Delta u(s_k + \Delta s_k, t)$ in the form

$$\Delta u(s_k + \Delta s_k, t) = \Delta u(s_k,t) + \frac{\partial\Delta u(\overline{s}_k,t)}{\partial x}\Delta s_k, \quad \overline{s}_k = s_k + \theta\Delta s_k, \quad \theta\in[0,1],$$

the right-hand side of (2.9) can be represented as

$$\sum_{k=1}^n\int_{t_1}^{t_2}[(p_k + \Delta p_k)\Delta u(s_k + \Delta s_k, t) - p_k\Delta u(s_k,t)]dt$$

$$= \sum_{k=1}^n\int_{t_1}^{t_2}\left[(p_k + \Delta p_k)\left(\Delta u(s_k,t) + \frac{\partial\Delta u(\overline{s}_k,t)}{\partial x}\Delta s_k\right) - p_k\Delta u(s_k,t)\right]dt$$

$$= \sum_{k=1}^n\int_{t_1}^{t_2}\left[\frac{\partial\Delta u(\overline{s}_k,t)}{\partial x}(p_k + \Delta p_k)\Delta s_k + \Delta u(s_k,t)\Delta p_k\right]dt.$$

Taking into account (2.9), we obtain the following energy balance equation for problem (2.3)–(2.6):

$$\frac{1}{2}\|\Delta u(x,t)\|_{L_2(0,l)}^2\Bigg|_{t=t_1}^{t=t_2} + a^2\left\|\frac{\partial\Delta u(x,t)}{\partial x}\right\|_{L_2(\Omega_t)}^2\Bigg|_{t=t_1}^{t=t_2} = \sum_{k=1}^n\int_{t_1}^{t_2}\left[(p_k + \Delta p_k)\Delta s_k\frac{\partial\Delta u(\overline{s}_k,t)}{\partial x} + \Delta p_k\Delta u(s_k,t)]dt\right],$$

where $\overline{s}_k = s_k + \theta\Delta s_k$, $\theta\in[0,1]$.

Applying the Cauchy–Schwarz inequality to the right-hand side of the equation produces

$$\frac{1}{2}\|\Delta u(x,t)\|_{L_2(0,l)}^2\Big|_{t=t_1}^{t=t_2} + a^2\left\|\frac{\partial\Delta u(x,t)}{\partial x}\right\|_{L_2(\Omega_t)}^2\Big|_{t=t_1}^{t=t_2}$$

$$\leq \sum_{k=1}^n\left[\left(\|p_k\|_{L_2(t_1,t_2)} + \|\Delta p_k\|_{L_2(t_1,t_2)}\right)\max_{t_1\leq t\leq t_2}|\Delta s_k(t)|\left\|\frac{\partial\Delta u(\overline{s}_k,t)}{\partial x}\right\|_{L_2(t_1,t_2)} + \|\Delta p_k\|_{L_2(t_1,t_2)}\|\Delta u(s_k,t)\|_{L_2(t_1,t_2)}\right] \tag{2.10}$$

$$= \sum_{k=1}^n\left[\left(\|p_k\|_{L_2(t_1,t_2)} + \|\Delta p_k\|_{L_2(t_1,t_2)}\right)\|\Delta s_k(t)\|_{C[t_1,t_2]}\left\|\frac{\partial\Delta u(\overline{s}_k,t)}{\partial x}\right\|_{L_2(t_1,t_2)} + \|\Delta p_k\|_{L_2(t_1,t_2)}\|\Delta u(s_k,t)\|_{L_2(t_1,t_2)}\right].$$

Since $\Delta s(t)$ is the solution of Cauchy problem (2.6), it follows from the properties of $f(s,\vartheta,t)$ that, for a sufficiently small $\varepsilon = t_2 - t_1$ [11, Section 6.3]

$$\|\Delta s_k(t)\|_{C[t_1,t_2]} \leq c_2\|\Delta\vartheta\|_{L_2(t_1,t_2)}, \quad 1\leq k\leq n.$$

Moreover, it is easy to show that

$$\|\Delta u(s_k,t)\|_{L_2(t_1,t_2)} \le c_3 \|\Delta u\|_{V_2^{1,0}(\Omega)},$$

$$\|\Delta u_x(\overline{s}_k,t)\|_{L_2(t_1,t_2)} \le c_4 \|\Delta u\|_{V_2^{1,0}(\Omega)},$$

where $c_2 > 0$, $c_3 > 0$, and $c_4 > 0$ are constants. Then the right-hand side of (2.10) can be bounded from above:

$$\frac{1}{2}\|\Delta u(x,t)\|^2_{L_2(0,l)}\Big|_{t=t_1}^{t=t_2} + a^2 \left\|\frac{\partial \Delta u(x,t)}{\partial x}\right\|^2_{L_2(\Omega_t)}\Big|_{t=t_1}^{t=t_2} \le c_5 \|\Delta\overline{\vartheta}\|_{L_2(t_1,t_2)} \|\Delta u\|_{V_2^{1,0}(\Omega)}, \tag{2.11}$$

where $c_5 > 0$ is a constant. Similarly [12, pp. 166–168 of the Russian edition], for an arbitrary $t \in [0,T]$, we divide the interval $[0,t]$ into a finite number of subintervals with inequality (2.11) satisfied on each of them. Then, summing up the resulting inequalities for all subintervals, we obtain

$$\frac{1}{2}\|\Delta u(x,t)\|^2_{L_2(0,l)} + a^2 \left\|\frac{\partial \Delta u(x,t)}{\partial x}\right\|^2_{L_2(\Omega)} \le c_4 \|\Delta\overline{\vartheta}\|_H \|\Delta u\|_{V_2^{1,0}(\Omega)},$$

which implies inequality (2.7). Then $\|\Delta u\|_{V_2^{1,0}(\Omega)} \to 0$ as $\|\Delta\overline{\vartheta}\|_H \to 0$. Therefore, $\|\Delta u(x,t)\|_{L_2(\Omega)} \to 0$ as $\|\Delta\overline{\vartheta}\|_H \to 0$.

The increment of the functional $J_0(\overline{\vartheta})$ can be represented as

$$J_0(\overline{\vartheta} + \Delta\overline{\vartheta}) - J_0(\overline{\vartheta}) = 2\int_0^l\int_0^T [u(x,t) - \tilde{u}(x,t)]\Delta u(x,t)dxdt + \|\Delta u(x,t)\|^2_{L_2(\Omega)}.$$

Combining this relation with the fact that $\|\Delta u(x,t)\|_{L_2(\Omega)} \to 0$ as $\|\Delta\overline{\vartheta}\|_H \to 0$, we conclude that $J_0(\overline{\vartheta})$ is continuous.

The functional $J_0(\overline{\vartheta})$ is bounded below and, by what was proved above, is continuous in V. Moreover, H is a Hilbert space that is a uniformly convex and reflexive Banach space [13]. Then Theorem 1 implies the existence of a dense subset K of H such that, for any $\omega = (\tilde{p}(t), \tilde{\vartheta}(t)) \in H$, problem (1.1)–(1.6) with $\alpha_i > 0, i = \overline{1,2}$, has a unique solution. The theorem is proved.

3. NECESSARY OPTIMALITY CONDITIONS

Let $\psi = \psi(x,t)$ be a solution from $V_2^{1,0}(\Omega)$ of the adjoint of problem (1.1)–(1.3), namely,

$$\frac{\partial\psi}{\partial t} + a^2\frac{\partial^2\psi}{\partial x^2} = -2[u(x,t) - \tilde{u}(x,t)], \quad (x,t) \in \Omega, \tag{3.1}$$

$$\frac{\partial\psi(0,t)}{\partial x} = 0, \quad \frac{\partial\psi(l,t)}{\partial x} = 0, \quad t \in [0,T), \tag{3.2}$$

$$\psi(x,T) = 0,\ 0 \le x \le l, \tag{3.3}$$

and let $q_k(t)$ be a solution from $C[0,T]$ of the adjoint of problem (1.4), namely,

$$\frac{dq_k(t)}{dt} = -\sum_{i=1}^n \frac{\partial f_i}{\partial s_k} q_i(t) + \frac{\partial\psi(s_k(t),t)}{\partial x} p_k(t), \quad 0 \le t < T, \quad q_k(T) = 0, \quad k = \overline{1,n}. \tag{3.4}$$

Definition 2. The weak solution of problem (3.1)–(3.4) with a control $\overline{\vartheta} = (p(t), \vartheta(t)) \in H$ is a pair of functions $(\psi(x,t), q(t))$ from $(V_2^{1,0}(\Omega), C[0,T])$, where $\psi = \psi(x,t)$ satisfies the integral identity

$$\int_0^l\int_0^T \left[\psi\frac{\partial\eta_1}{\partial t} + a^2\frac{\partial\psi}{\partial x}\frac{\partial\eta_1}{\partial x}\right]dxdt = 2\int_0^l\int_0^T [u(x,t) - \tilde{u}(x,t)]\eta_1(x,t)dxdt \tag{3.5}$$

for any $\eta_1 = \eta_1(x,t) \in W_2^{1,1}(\Omega)$ such that $\eta_1(x,0) = 0$, while the function $q_k(t)$ satisfies the integral equation

$$q_k(t) = \int_t^T \left[\sum_{i=1}^n \frac{\partial f_i}{\partial s_k} q_i(\tau) - p_k(\tau)\psi_x(s_k(\tau),\tau) \right] d\tau, \quad 0 \le t \le T, \quad k = \overline{1,n}. \tag{3.6}$$

The function

$$H(t,s,\psi,q,\overline{\vartheta}) = \left\{ \sum_{k=1}^n [f_k(s(t),\vartheta(t),t)q_k(t) - \psi(s_k(t),t)p_k(t) - \alpha_1(p_k(t) - \tilde{p}_k(t))^2] - \alpha_2 \sum_{m=1}^r \vartheta_m(t) - \tilde{\vartheta}_m(t))^2 \right\} \tag{3.7}$$

is referred to as the Hamilton–Pontryagin function of problem (1.1)–(1.6).

Theorem 3. *Let the following assumptions hold*:

(1) *The functions* $f_k(s,\vartheta,t), k = \overline{1,n}$, *are continuous with respect to their arguments and have continuous bounded partial derivatives with respect to* s *and* ϑ *for* $(s,\vartheta,t) \in R^n \times R^r \times [0,T]$.

(2) *The functions* $f_k(s,\vartheta,t)$, $f_{ks} = \dfrac{\partial f_k(s,\vartheta,t)}{\partial s}$, *and* $f_{k\vartheta} = \dfrac{\partial f_k(s,\vartheta,t)}{\partial \vartheta}$, $k = \overline{1,n}$, *satisfy the Lipschitz condition with respect to* s and ϑ, *i.e.*,

$$|f_k(s+\Delta s,\vartheta+\Delta\vartheta,t) - f_k(s,\vartheta,t)| \le L(|\Delta s| + |\Delta\vartheta|),$$
$$|f_{ks}(s+\Delta s,\vartheta+\Delta\vartheta,t) - f_{ks}(s,\vartheta,t)| \le L(|\Delta s| + |\Delta\vartheta|),$$
$$|f_{k\vartheta}(s+\Delta s,\vartheta+\Delta\vartheta,t) - f_{k\vartheta}(s,\vartheta,t)| \le L(|\Delta s| + |\Delta\vartheta|)$$

for all $(s+\Delta s,\vartheta+\Delta\vartheta,t),(s,\vartheta,t) \in E^n \times E^r \times [0,T]$, *where* $L = \text{const} \ge 0$.

If $(\psi(x,t),q(t))$ *is the solution of adjoint problem* (3.1)–(3.4), *then functional* (1.6) *is Fréchet differentiable and its gradient is given by*

$$J'(\overline{\vartheta}) = \left(\frac{\partial J(\overline{\vartheta})}{\partial p}, \frac{\partial J(\overline{\vartheta})}{\partial \vartheta} \right) = \left(-\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial \vartheta} \right). \tag{3.8}$$

Proof. Consider the increment of J:

$$\begin{aligned} \Delta J \equiv J(\overline{\vartheta} + \Delta\overline{\vartheta}) - J(\overline{\vartheta}) &= 2\int_0^l \int_0^T [u(x,t) - \tilde{u}(x,t)]\Delta u(x,t)dxdt + \int_0^l \int_0^T |\Delta u(x,t)|^2 dxdt \\ &+ \sum_{k=1}^n \left\{ 2\alpha_1 \int_0^T [p_k(t) - \tilde{p}_k(t)]\Delta p_k(t)dt + \alpha_1 \int_0^T |\Delta p_k(t)|^2 dt \right\} \\ &+ \sum_{m=1}^r \left\{ 2\alpha_2 \int_0^T [\vartheta_m(t) - \tilde{\vartheta}_m(t)]\Delta\vartheta_m(t)dt + \alpha_2 \int_0^T |\Delta\vartheta_m(t)|^2 dt \right\}, \end{aligned} \tag{3.9}$$

where $\overline{\vartheta} = (p,\vartheta) \in V$, $\overline{\vartheta} + \Delta\overline{\vartheta} \in V$, $\Delta u(x,T) \equiv u(x,T;\overline{\vartheta} + \Delta\overline{\vartheta}) - u(x,T;\overline{\vartheta})$, and $u \equiv u(x,T;\overline{\vartheta})$.

Setting $\eta_1 = \Delta u(x,t)$ in (3.5) and $\eta = \psi(x,t)$ in (2.8) and subtracting the resulting relations from one another, we obtain

$$2\int_0^l \int_0^T [u(x,t) - \tilde{u}(x,t)]\Delta u(x,t)dxdt = \sum_{k=1}^n \int_0^T [(p_k + \Delta p_k)\psi(s_k + \Delta s_k,t) - p_k\psi(s_k,t)]dt. \tag{3.10}$$

It follows from (2.6) that the function $\Delta s_k(t)$ satisfies the integral identity

$$\int_0^T \left[\frac{d\theta_k(t)}{dt} \Delta s_k(t) + \Delta f_k(s(t),\vartheta(t),t)\theta_k(t) \right] dt = 0 \tag{3.11}$$

for any $\theta_k(t) \in C^1[0,T]$ with $\theta_k(T) = 0, k = \overline{1,n}$.

It follows from (3.4) that the function $q_k(t)$ satisfies the integral identity

$$\int_0^T \left[\frac{d\theta_{1k}(t)}{dt} q_k(t) - \left(\sum_{i=1}^n \frac{\partial f_i}{\partial s_k} q_i(t) - \frac{\partial \psi(s_k(t),t)}{\partial x} p_k(t)\right)\theta_{1k}(t)\right] dt = 0 \tag{3.12}$$

for any $\theta_{1k}(t) \in C^1[0,T]$ with $\theta_{1k}(0) = 0, k = \overline{1,n}$.

Setting $\theta_{1k}(t) = \Delta s_k(t)$ in (3.12) and $\theta_k(t) = q_k(t)$ in (3.11) and summing up the resulting relations, we obtain

$$[\Delta s_k(t) q_k(t)]\Big|_{t=0}^{t=T} = \int_0^T \left[\left(\sum_{i=1}^n \frac{\partial f_i}{\partial s_k} q_i(t) - \frac{\partial \psi(s_k(t),t)}{\partial x} p_k(t)\right)\Delta s_k(t) - \Delta f_k q_k(t)\right] dt.$$

By the assumption of the theorem, the function $\Delta f_k = \Delta f_k(s(t), \vartheta(t), t)$ can be represented in the form

$$\Delta f_k = \sum_{i=1}^n \frac{\partial f_k}{\partial s_i}\Delta s_i + \sum_{m=1}^r \frac{\partial f_k}{\partial \vartheta_m}\Delta \vartheta_m + R_1,$$

where $R_1 = o(\sqrt{\|\Delta s\|^2_{L_2(0,T)} + \|\Delta\vartheta\|^2_{L_2(0,T)}})$. From the last equality, it follows that

$$[\Delta s_k(t) q_k(t)]\Big|_{t=0}^{t=T} = \int_0^T \left[\left(\sum_{i=1}^n \frac{\partial f_i}{\partial s_k} q_i(t) - \frac{\partial \psi(s_k(t),t)}{\partial x} p_k(t)\right)\Delta s_k(t)\right.$$
$$\left. - \sum_{m=1}^r \frac{\partial f_k}{\partial \vartheta_m}\Delta\vartheta_m(t) q_k(t) - \sum_{i=1}^n \frac{\partial f_k}{\partial s_i}\Delta s_i(t) q_k(t)\right] dt + R_1,$$

which, in view of (2.6) and (3.4), is equivalent to the equation

$$\int_0^T \frac{\partial \psi(s_k(t),t)}{\partial x} p_k(t)\Delta s_k(t) dt = -\sum_{m=1}^r \int_0^T \frac{\partial f_k}{\partial \vartheta_m}\Delta\vartheta_m(t) q_k(t) dt$$
$$- \sum_{i=1}^n \int_0^t \left[\frac{\partial f_k}{\partial s_i} q_k(t)\Delta s_i(t) - \frac{\partial f_i}{\partial s_k} q_i(t)\Delta s_k(t)\right] dt + R_1. \tag{3.13}$$

Clearly, under the assumptions made above, the Taylor formula yields

$$\psi(s_k + \Delta s_k, t) = \psi(s_k, t) + \frac{\partial \psi(s_k,t)}{\partial x}\Delta s_k + o(\|\Delta s_k\|_{C[0,T]}).$$

In view of this expansion, it follows from (3.10) that

$$2\int_0^l \int_0^T [u(x,t) - \tilde{u}(x,t)]\Delta u(x,t) dx dt = \sum_{k=1}^n \int_0^T \left[\frac{\partial \psi(s_k(t),t)}{\partial x} p_k(t)\Delta s_k(t)\right.$$
$$\left. + \psi(s_k(t),t)\Delta p_k(t) + \frac{\partial \psi(s_k(t),t)}{\partial x}\Delta p_k(t)\Delta s_k(t) + o(\|\Delta s_k\|_{C[0,T]})\right] dt.$$

Since

$$\sum_{k=1}^n \sum_{i=1}^n \left[\frac{\partial f_k}{\partial s_i} q_k(t)\Delta s_i(t) - \frac{\partial f_i}{\partial s_k} q_i(t)\Delta s_k(t)\right] = 0,$$

combining the last equality and relation (3.13), we find

$$2\int_0^l \int_0^T [u(x,t) - \tilde{u}(x,t)]\Delta u(x,t) dx dt = \sum_{k=1}^n \int_0^T \left[-\sum_{m=1}^r \frac{\partial f_k}{\partial \vartheta_m} q_k(t)\Delta\vartheta_m(t) + \psi(s_k,t)\Delta p_k\right] dt + R_2, \tag{3.14}$$

where $R_2 = \sum_{k=1}^{n} \int_0^T \left[\frac{\partial \psi(s_k(t), t)}{\partial x} \Delta p_k(t) \Delta s_k(t) + o(\|\Delta s_k\|_{C[0,T]}) \right] dt + R_1$.

Following the standard scheme [11, p. 94], we can prove the estimate

$$\|\Delta s\|_{C[0,T]} \le c_6 \|\Delta \vartheta\|_{L_2(0,T)}, \tag{3.15}$$

where $c_6 > 0$ is a constant.

From this, we have $R_2 = o(\|\Delta\bar{\vartheta}\|_H)$. On the other hand, estimate (2.7) implies that $\|\Delta u(x,t)\|_{L_2(\Omega)} = O(\|\Delta\bar{\vartheta}\|_H)$. Substituting the resulting relations into (3.9) yields

$$\Delta J(\bar{\vartheta}) = \sum_{k=1}^{n} \left(J_1(k) + \sum_{m=1}^{r} J_2(k,m) \right) + o(\|\Delta\bar{\vartheta}\|_H) \quad \text{as} \quad \|\Delta\bar{\vartheta}\|_H \to 0,$$

where

$$J_1(k) = \int_0^T [\psi(s_k(t), t) + 2\alpha_1 (p_k(t) - \tilde{p}_k(t))] \Delta p_k(t) dt,$$

$$J_2(k,m) = \int_0^T \left[-\frac{\partial f_k(s(t), \vartheta(t), t)}{\partial \vartheta_m} q_k(t) + 2\alpha_2 \left(\vartheta_m(t) - \tilde{\vartheta}_m(t) \right) \right] \Delta\vartheta_m(t) dt.$$

Combining this result with the expressions for the Hamilton–Pontryagin function, we obtain

$$\Delta J(\bar{\vartheta}) = \left(-\frac{\partial H}{\partial \bar{\vartheta}}, \Delta\bar{\vartheta} \right)_H + o(\|\Delta\bar{\vartheta}\|_H) \quad \text{as} \quad \|\Delta\bar{\vartheta}\|_H \to 0,$$

which implies the Fréchet differentiability of functional (1.6) and the validity of formula (3.8). The theorem is proved.

Theorem 4. *Let all the conditions of Theorem* 2 *hold. Then a necessary condition for the optimality of a control* $\bar{\vartheta}^* = (p^*(t), \vartheta^*(t))$ *is that*

$$\begin{aligned} \langle J'(\bar{\vartheta}^*), \bar{\vartheta} - \bar{\vartheta}^* \rangle_H = \int_0^T \sum_{k=1}^{n} \Big[&(\psi^*(s_k^*(t), t) + 2\alpha_1 (p_k^*(t) - \tilde{p}_k(t)), p_k(t) - p_k^*(t)) \\ &+ \sum_{m=1}^{r} \left(-\frac{\partial f_k(s^*(t), \vartheta^*(t), t)}{\partial \vartheta_m} q_k^*(t) + 2\alpha_2 (\vartheta_m^*(t) - \tilde{\vartheta}_m(t)), \vartheta_m(t) - \vartheta_m^*(t) \right) \Big] dt \ge 0 \end{aligned} \tag{3.16}$$

for any $\bar{\vartheta} = (p(t), \vartheta(t)) \in V$. *Here,* $\psi^*(s_k^*(t), t)$ *and* $q_k^*(t)$ *are the respective solutions of problems* (3.1)–(3.3) *and* (3.4) *with* $\bar{\vartheta} = \bar{\vartheta}^*(p^*(t), \vartheta^*(t))$.

Proof. By the well-known theorem [11, p. 28], a necessary condition for the optimality of $\bar{\vartheta}^* = (p^*, \vartheta^*) \in V$ is that

$$\langle J'(\bar{\vartheta}^*), \bar{\vartheta} - \bar{\vartheta}^* \rangle_H \ge 0 \quad \text{for all} \quad \bar{\vartheta} \in V.$$

Using relation (3.8) and the expressions for the Hamilton–Pontryagin function, we calculate the gradient of functional (1.6) and, substituting it into the above inequality, conclude the validity of inequality (3.16). The theorem is proved.

Theorem 5. *Let all the conditions of Theorem* 2 *hold and the pairs of functions* $(u^*(x,t), s^*(t))$ *and* $(\psi^*(x,t), q^*(t))$ *be the solutions of problems* (1.1)–(1.4) *and* (3.1)–(3.4) *with* $\bar{\vartheta} = \bar{\vartheta}^* \in V$, *respectively. Then a necessary condition for the optimality of the control* $\bar{\vartheta}^*$ is that

$$H(t, s^*, \psi^*, q^*, \bar{\vartheta}^*) = \max_{\bar{\vartheta} \in V} H(t, s^*, \psi^*, q^*, \bar{\vartheta}) \quad \forall (x,t) \in \Omega.$$

Proof. Let (σ, θ) be a Lebesgue point inside Ω_T for all the functions included in the conditions of problems (1.1)–(1.4) and (3.1)–(3.4). Let

$$\Pi^\varepsilon \equiv \left\{(x,t) : \sigma - \frac{\varepsilon}{2} < x < \sigma + \frac{\varepsilon}{2},\ \theta - \frac{\varepsilon}{2} < t < \theta + \frac{\varepsilon}{2}\right\} \subset \Omega_T,$$

where $\varepsilon > 0$ is a sufficiently small number.

The impulse variation of the control is defined as

$$\bar{\vartheta}^\varepsilon \equiv \left(p^\varepsilon, \vartheta^\varepsilon\right) = \begin{cases} \bar{\vartheta}, & (x,t) \in \Pi^\varepsilon, \\ \bar{\vartheta}^*, & (x,t) \notin \Pi^\varepsilon, \end{cases}$$

where $\bar{\vartheta}$ is a constant vector. Let $\Delta u^\varepsilon \equiv u^\varepsilon(x,t) - u^*(x,t)$, where $u^\varepsilon(x,t) = u(x,t;\bar{\vartheta}^\varepsilon)$ and $u^*(x,t) = u(x,t;\bar{\vartheta}^*)$. Then the function Δu^ε satisfies the identity

$$\int_0^l \int_0^T [-\Delta u^\varepsilon \eta_t + a^2 \Delta u_x^\varepsilon \eta_x] dx dt = \sum_{k=1}^n \int_0^T [p_k^\varepsilon + \Delta p_k^\varepsilon) \eta(s_k^\varepsilon + \Delta s_k^\varepsilon, t) - p_k^\varepsilon \eta(s_k^\varepsilon, t)] dt \tag{3.17}$$

for any $\eta = \eta(x,t) \in W_2^{1,1}(\Omega)$ with $\eta(x,T) = 0$.

Proceeding as in the proof of estimate (2.7), we show that $\Delta u^\varepsilon(x,t)$ satisfies the estimate

$$\left\|\Delta u^\varepsilon\right\|_{V_2^{1,0}(\Omega_T)} \le c_7 \left\|\Delta \bar{\vartheta}^\varepsilon\right\|_H,$$

where $c_7 > 0$ is a constant and $\Delta\bar{\vartheta}^\varepsilon = (\Delta p^\varepsilon, \Delta\vartheta^\varepsilon)$. From this and the fact that $(\sigma, \theta) \in \Omega_T$ is a Lebesgue point, we conclude that $\Delta u^\varepsilon \to 0$ in $V_2^{1,0}(\Omega)$ as $\varepsilon \to 0$. Moreover, it follows from (3.15) that the function $\Delta s^\varepsilon = \Delta s^\varepsilon(t)$ satisfies the estimate

$$\left\|\Delta s^\varepsilon\right\|_{C[0,T]} \le c_8 \left\|\Delta \vartheta^\varepsilon\right\|_{L_2(0,T)},$$

where $c_8 > 0$ is a constant. From this, we conclude that $\Delta s^\varepsilon \to 0$ in $C[0,T]$ as $\varepsilon \to 0$.

Let $\psi^\varepsilon = \psi^\varepsilon(x,t) \in V_2^{1,0}(\Omega_T)$ be the solution of the integral identity

$$\int_0^l \int_0^T \left(\psi^\varepsilon \frac{\partial \eta_1}{\partial t} + a^2 \frac{\partial \psi^\varepsilon}{\partial x} \frac{\partial \eta_1}{\partial x}\right) dx dt = 2 \int_0^l \int_0^T \left[u^\varepsilon(x,t) - \tilde{u}(x,t) + \frac{1}{2} \Delta u^\varepsilon(x,t)\right] \eta_1(x,t) dx dt \tag{3.18}$$

for any $\eta_1 = \eta_1(x,t) \in W_2^{1,1}(\Omega)$ with $\eta_1(x,T) = 0$. The difference $\psi^\varepsilon - \psi^*$ satisfies an integral identity similar to (3.18). Combining this result with the fact that $\Delta u^\varepsilon \to 0$ in $V_2^{1,0}(\Omega)$ as $\varepsilon \to 0$, we conclude that $\psi^\varepsilon \to \psi^*$ in $V_2^{1,0}(\Omega)$ as $\varepsilon \to 0$.

Let $q^\varepsilon = q^\varepsilon(t) \in C[0,T]$ be the solution of the identity

$$q_k^\varepsilon(t) = \int_t^T \left[\sum_{i=1}^n \frac{\partial f_i^\varepsilon}{\partial s_k} q_i^\varepsilon(\tau) - p_k^\varepsilon(\tau) \psi_x^\varepsilon(s_k(\tau), \tau)\right] d\tau, \quad 0 \le t \le T, \quad k = \overline{1, n}. \tag{3.19}$$

Since the difference $q^\varepsilon - q^*$ satisfies an identity similar to (3.19) and since $\Delta s^\varepsilon \to 0$ in $C[0,T]$ as $\varepsilon \to 0$, we obtain $q^\varepsilon \to q^*$ in $C[0,T]$ as $\varepsilon \to 0$.

The increment of functional (1.6) is computed as

$$\Delta J(\bar{\vartheta}^*) \equiv J(\bar{\vartheta}^\varepsilon) - J(\bar{\vartheta}^*) = 2\int_0^l\int_0^T \left[u^*(x,t) - \tilde{u}(x,t) + \frac{1}{2}\Delta u^\varepsilon(x,t)\right]\Delta u^\varepsilon(x,t)dxdt$$
$$+\sum_{k=1}^n\left\{2\alpha_1\int_0^T [p_k^*(t) - \tilde{p}_k(t)][p_k^\varepsilon(t) - p_k^*(t)]dt + \alpha_1\int_0^T [p_k^\varepsilon(t) - p_k^*(t)]^2 dt\right\} \tag{3.20}$$
$$+\sum_{m=1}^r\left\{2\alpha_2\int_0^T [\vartheta_m^*(t) - \tilde{\vartheta}_m(t)][\vartheta_m^\varepsilon(t) - \vartheta_m^*(t)]dt + \alpha_2\int_0^T [\vartheta_m^\varepsilon(t) - \vartheta_m^*(t)]^2 dt\right\}.$$

Proceeding as in the proof of (3.14) and using identity (3.18), we obtain

$$2\int_0^l\int_0^T \left[u^*(x,t) - \tilde{u}(x,t) + \frac{1}{2}\Delta u^\varepsilon(x,t)\right]\Delta u^\varepsilon(x,t)dxdt = \sum_{k=1}^n \int_{\Pi^\varepsilon}\left[\psi^\varepsilon(s_k^\varepsilon,t)\Delta p_k^\varepsilon - \sum_{m=1}^r \frac{\partial f_k^\varepsilon(s(t),\vartheta(t),t)}{\partial\vartheta_m^\varepsilon} q_k^\varepsilon(t)\Delta\vartheta_m^\varepsilon\right]dt.$$

Combining this relation with (3.20) yields

$$\Delta J(\bar{\vartheta}^*) \equiv J(\bar{\vartheta}^\varepsilon) - J(\bar{\vartheta}^*) = \sum_{k=1}^n \int_{\Pi^\varepsilon}\left[\psi^\varepsilon(s_k^\varepsilon,t)\Delta p_k^\varepsilon - \sum_{m=1}^r \frac{\partial f_k^\varepsilon(s(t),\vartheta(t),t)}{\partial\vartheta_m^\varepsilon} q_k^\varepsilon(t)\Delta\vartheta_m^\varepsilon\right]dt$$
$$+\sum_{k=1}^n\left\{2\alpha_1\int_{\Pi^\varepsilon} [p_k^*(t) - \tilde{p}_k(t)][p_k^\varepsilon(t) - p_k^*(t)]dt + \alpha_1\int_{\Pi^\varepsilon} [p_k^\varepsilon(t) - p_k^*(t)]^2 dt\right\}$$
$$+\sum_{m=1}^r\left\{2\alpha_2\int_{\Pi^\varepsilon} [\vartheta_m^*(t) - \tilde{\vartheta}_m(t)][\vartheta_m^\varepsilon(t) - \vartheta_m^*(t)]dt + \int_{\Pi^\varepsilon} [\vartheta_m^\varepsilon(t) - \vartheta_m^*(t)]^2 dt\right\}.$$

Then it follows from the form of the Hamilton–Pontryagin function (3.7) that

$$\Delta J(\bar{\vartheta}^*) = -\int_{\Pi^\varepsilon} [H(t, s^\varepsilon, \psi_\varepsilon, q^\varepsilon, \bar{\vartheta}^\varepsilon) - H(t, s^*, \psi_\varepsilon, q^\varepsilon, \bar{\vartheta}^*)]dt.$$

Since $\psi^\varepsilon \to \psi^*$ in $V_2^{1,0}(\Omega)$ and $q^\varepsilon \to q^*$ in $C[0,T]$ as $\varepsilon \to 0$, we derive the following formula for the variation of functional (1.6):

$$\delta J(\bar{\vartheta}^*) = \lim_{\varepsilon\to 0}\frac{\Delta J(\bar{\vartheta}^*)}{\varepsilon} = -[H(\theta, s^*, \psi^*, q^*, \bar{\vartheta}) - H(\theta, s^*, \psi^*, q^*, \bar{\vartheta}^*)].$$

The optimality of the control $\bar{\vartheta} \in V$ implies that $\delta J(\bar{\vartheta}^*) \geq 0$. From this and the fact that Lebesgue points are dense everywhere in Ω_T, we obtain the assertion of the theorem. The theorem is proved.

4. CONCLUSIONS

The optimal control of processes governed by a parabolic equation and a system of ordinary differential equations was studied. For this optimal control problem, we proved an existence and uniqueness theorem, established sufficient conditions for the Fréchet differentiability of the cost functional, derived an expression for its gradient, and obtained necessary optimality conditions in the form of pointwise and integral maximum principles.

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