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Generalized Edgeworth–Pareto Principle

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Abstract—The general multicriteria choice problem with m individual preference relations and an asymmetric collective preference relation is considered. The concept of a k-effective alternative is introduced, which coincides with an effective alternative for $k = 1$ and represents a weakly effective alternative for $k = m$. For the other integer values of k, it lies somewhere in between. In terms of the general multicriteria choice problem, the Pareto axiom and the exclusion axiom for dominated alternatives are stated. Assuming that these axioms hold, a generalized Edgeworth–Pareto principle is established, which was earlier introduced by the author in the special case $k = 1$. The results are extended to a fuzzy collective preference relation and to a fuzzy set of initial alternatives.

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1. INTRODUCTION

At present, the Edgeworth–Pareto principle is a fundamental tool used to solve multicriteria choice problems of various natures. To many researchers, it frequently seems so obvious and natural that, in their views, it does not require any substantiation or special argumentation. In fact, they state that the best choice always has to be sought within the Pareto set. This approach can be called a "naive" Edgeworth–Pareto principle. It has been used and continues to be applied by thousands of researchers.

It was found over time that this principle is not universal: there are problems in which it "does not work." In this context, there appeared an urgent necessity to differentiate choice problems in which the naive principle is valid from problems in which the Pareto optimal solution is not necessarily the best one. In other words, there was an urgent need to pass from the naive Edgeworth–Pareto principle to an axiomatic one.

The Edgeworth–Pareto principle was axiomatized by the author in [1, 2]. Later, the class of problems for which this principle is valid was substantially expanded (see [3–8]).

Below, we consider the general multicriteria choice problem with m individual strict preference relations that can be interpreted as relations of a set of m players or economy participants and with a strict collective preference relation. The last is assumed to be unknown. The concept of a k-effective alternative is introduced in Section 1. This concept coincides with an effective (Pareto optimal) alternative for $k = 1$ and represents a weakly effective (Slater optimal) alternative for $k = m$. For the other integer values of k, it is intermediate between the above two. In Section 2, the Pareto axiom and the exclusion axiom for dominated alternatives are stated for the general multicriteria choice problem. Assuming that these axioms hold, a generalized Edgeworth–Pareto principle is established, which was earlier introduced by the author in the special case $k = 1$. In Section 3, the results are extended to a fuzzy collective preference relation and a fuzzy set of initial alternatives.

2. k-EFFECTIVE ALTERNATIVES IN THE GENERAL MULTICRITERIA CHOICE PROBLEM

Consider the general multicriteria choice model $\langle Y_1, \dots, Y_m, \succ_1, \dots, \succ_m, Y, \succ \rangle$, where

Y_i is the scale of the ith criterion (abstract set), $i = 1, 2, \dots, m$;

$\succ_i$ is an irreflexive, transitive, and weakly connected binary relation defined on the set Y_i, $i = 1, 2, \dots, m$;

Y is the set of feasible alternatives (tuples) to choose from, $Y \subset \hat{Y} = Y_1 \times \dots \times Y_m$;

$\succ$ is an asymmetric binary strict preference relation defined on $\hat{Y}$; as a rule, it is not known in applied choice problems.

Recall that a binary relation $\mathfrak{R}$ defined on a set A is called

irreflexive if $x\mathfrak{R}x$ is false for any $x \in A$;

transitive if, for any $x, y, z \in A$, $x\mathfrak{R}y$ and $y\mathfrak{R}z$ imply $x\mathfrak{R}z$;

weakly connected if for any $x, y \in A$ such that $x \neq y$, either $x\mathfrak{R}y$ or $y\mathfrak{R}x$.

The solution of the multicriteria choice problem is a set called a *set of chosen alternatives*, which is denoted by $C(Y) \subset Y$. Any attempt to define it rigorously is ineffective. In such conditions, it seems reasonable to derive an upper bound for the unknown set $C(X)$ with the help of certain incomplete information on the basic objects participating in the multicriteria choice problem. It is this task that is performed on the basis of the Edgeworth–Pareto principle.

In terms of the above-mentioned set, the binary relation $\succ$ can be characterized as $y' \succ y'' \Leftrightarrow C(\{y', y''\}) = \{y'\}$.

Let $I = \{1, 2, \dots, m\}$. The binary relation $\rhd_k$ on the set $\hat{Y}$ is defined according to the rule

$$y' \rhd_k y'' \quad \Leftrightarrow \quad y_i' \succ_i y_i'' \text{ at least for some } k \in I \text{ indices } i \text{ from } I$$
$$\text{and } y_i' = y_i'' \text{ for the other } i.$$

This relation can be interpreted as a collective (group) preference relation if $\succ_1, \dots, \succ_m$ are treated as individual preference relation of m participants (for example, in an economy or game).

For Y, the set of k-effective alternatives is defined as

$$E^k(Y) = \{y^* \in Y \mid \text{ there is no } y \in Y \text{ such that } y \rhd_k y^*\},$$

and the set of nondominated alternatives is defined as

$$\operatorname{Ndom} Y = \{y^* \in Y \mid \text{ there is no } y \in Y \text{ such that } y \succ y^*\}.$$

$E^1(Y)$ is the set of effective (i.e., Pareto optimal) alternatives, while $E^m(Y)$ is the set of weakly effective (Slater optimal) alternatives (see [9]). For $k \in \{2, \dots, m-1\}$ we obtain sets that are intermediate between these two concepts. As was noted above, in applications, the relation $\succ$ is usually not known and, hence, the set $\operatorname{Ndom} Y$ is not defined.

It should be noted that, for k greater than a half of m, the binary relation $\rhd_k$, though somewhat similar to the well-known majority relation (see [10]), differs fundamentally from it. Indeed, the majority relation between a pair of tuples y', y'' holds if $y_i' \succ_i y_i''$ for more than a half of the indices i irrespective of whether or not analogous relations hold for the other indices i. Meanwhile, for $y' \rhd_k y''$ to hold, it is required that $y_i'' \succ_i y_i'$ be true for none of the indicated other indices. It can be seen that $\rhd_k$ is contained (as a set of pairs) in the majority relation, but not vice versa. Moreover, in contrast to the majority relation, $\rhd_k$ is transitive. We can say that, for the indicated k, the relation $\rhd_k$ is the maximal (by inclusion) transitive part of the majority relation.

On the set $\hat{Y}$, the relation $\rhd_k'$ is defined by the equivalence

$$y' \rhd_k' y'' \quad \Leftrightarrow \quad y_i' \succ_i y_i'' \text{ for some } k \in I \text{ indices } i \text{ from } I \text{ and } y_i' = y_i'' \text{ for the other } i.$$

Clearly, this relation is a subset of $\rhd_k$, but not vice versa.

The relation $\succ_i$ is said to be consistent with the relation $\succ$ if for any two tuples

$$y' = (y_1', \dots, y_{i-1}', y_i', y_{i+1}', \dots, y_m'), \quad y' = (y_1', \dots, y_{i-1}', y_i'', y_{i+1}', \dots, y_m') \in \hat{Y}, \quad y_i' \succ_i y_i''$$

it is true that $y' \succ y''$.

Lemma. *Let each of the relations* $\succ_1, \dots, \succ_m$ be *consistent with the relation* $\succ$, *which is transitive on* $\hat{Y}$. *Then* $y' \rhd_k' y'' \Rightarrow y' \succ y''$ *for any* $y', y'' \in \hat{Y}$ *implies* $y' \rhd_k y'' \Rightarrow y' \succ y''$ *for all* $y', y'' \in \hat{Y}$.

Proof. Assume without loss of generality that, by virtue of $y' \rhd_k y''$, we have $y_i' \succ_i y_i''$, $i = 1, \dots, l \geqq k$; $y_i' = y_i''$, $i = l+1, \dots, m$.

By assumption, $y' = (y_1', \dots, y_k', y_{k+1}', \dots, y_m') \succ (y_1'', \dots, y_k'', y_{k+1}', \dots, y_m')$. Using the consistency property, we obtain the chain of relations

$$y' \succ (y_1'', \dots, y_k'', y_{k+1}', y_{k+2}', \dots, y_m') \succ (y_1', \dots, y_{k+1}'', y_{k+2}'', y_{k+3}', \dots, y_m') \succ \dots \succ (y_1'', \dots, y_l'', y_{l+1}', \dots, y_m') = y''.$$

Since the relation $\succ$ is transitive, it follows that $y' \succ y''$, as required. The lemma is proved.

The implication $y' \rhd_k y'' \Rightarrow y' \succ y''$ means that the relation $\rhd_k$ is contained in $\succ$ and, hence, can be used to construct an upper bound for the unknown relation $\succ$. According to the lemma, under the lemma assumptions, an upper bound for the set of nondominated alternatives $\mathrm{Ndom}\, Y$ can be constructed using the set $E^k(Y)$ with $\rhd_k$ replaced by the simpler relation $\rhd'_k$ (see Corollary 1 below).

3. GENERALIZED EDGEWORTH–PARETO PRINCIPLE

Let the following axiom hold.

Axiom 1 (exclusion axiom). *For any pair of alternatives $y', y'' \in Y$ satisfying the relation $y' \succ y''$, it is true that $y'' \notin C(Y)$.*

According to Axiom 1, the alternative not chosen from a pair is not chosen from the entire set Y. Though natural, this axiom is not universal and can be violated in some applications (see [2]).

Assuming the opposite, it is easy to see that, for any set $C(Y)$ satisfying Axiom 1, we have the inclusion

$$C(Y) \subset \mathrm{Ndom}\, Y. \tag{1}$$

In other words, the final choice is made within the set of nondominated alternatives.

Axiom 2 (generalized Pareto axiom). *Let $k \in I$. For any two vectors $y', y'' \in Y$ satisfying the relation $y' \rhd_k y''$, it is true that $y' \succ y''$.*

For $k = 1$, this axiom coincides with a well-known version of the Pareto axiom (see [4]). A straightforward consequence of Axiom 2 is that, for any set $\mathrm{Ndom}\, Y$,

$$\mathrm{Ndom}\, Y \subset E^k(Y). \tag{2}$$

Combining (1) and (2) yields the following result.

Generalized Pareto principle. *Let $k \in I$ and Axioms 1 and 2 hold. Then, for any set $C(Y)$,*

$$C(Y) \subset E^k(Y). \tag{3}$$

Note that this principle holds irrespective of whether or not the relation $\succ$ is transitive. Note also that Axioms 1 and 2 involve the index $k \in I$. Therefore, the generalized Pareto principle contains m generally different assertions depending on the particular numerical value of k. For $k = 1$, we have the well-known Edgeworth–Pareto principle (see [4]). For $k = m$, the last assertion can be referred to as the *Slater principle*: given two tuples, if one of them can be chosen only in the case of domination over all components simultaneously (i.e., for $y'_i \succ_i y''_i, i = 1, \dots, m$), then the final choice has to be made within the set of weakly effective alternatives (Slater set). For the other k, we obtain a collection of intermediate principles.

Applying the lemma yields the following result.

Corollary 1. Let $k \in I$, each of the relations $\succ_1, \dots, \succ_m$ be consistent with the transitive relation $\succ$, and Axioms 1 and 2 hold with $\rhd_k$ in the latter replaced by the relation $\rhd'_k$. Then, for any set $C(Y)$,

$$\mathrm{Ndom}\, Y \subset E^k(Y).$$

It can be seen that Corollary 1 is similar to the Pareto generalized principle, although it is based on the relation $\rhd'_k$, which is a subset of $\rhd_k$. This was achieved due to the additional assumption that the relation $\succ$ is consistent and transitive.

Consider the standard multicriteria choice problem (model) $\langle X, f, \succ_X \rangle$, where

X is the set of feasible alternatives (solutions) to choose from;

$f = (f_1, \dots, f_m)$ $(m \geqq 2)$ is a set of numerical criteria (vector criterion) defined on X and taking values in the arithmetic vector space R^m;

$\succ_X$ is an asymmetric binary strict preference relation defined on X. This relation is usually assigned to a decision maker (DM), although it can also be interpreted as a collective preference relation if the criteria $f_1, \dots, f_m$ are associated with m different participants (players).

Recall that a binary relation $\Re$ defined on a set A is called asymmetric if, for any elements $x, y \in A$, $x \Re y$ implies that the relation $y \Re x$ is false.

The feasible alternatives from X to be found by solving the multicriteria choice problem are called chosen alternatives. They form a set of chosen alternatives (chosen decisions), which is denoted by $C(X) \subset X$.

In what follows, we also use the set of feasible vectors $Y = f(X) \subset R^m$ and the set of chosen vectors $C(Y) = f(C(X)) \subset Y$. It is natural to assume that the strict preference relation $\succ$ is defined on the set of vectors $\hat{Y} = f_1(X) \times \ldots \times f_m(X)$, and this relation is adjoint to (consistent with) $\succ_X$ as follows:

$$x_1 \succ_X x_2 \Leftrightarrow f(x_1) \succ f(x_2) \quad \text{for all } x_1 \in \tilde{x}_1, \quad x_2 \in \tilde{x}_2; \quad \tilde{x}_1, \tilde{x}_2 \in \tilde{X},$$

where $\tilde{X}$ is the collection of equivalence classes generated by the equivalence relation $x_1 \sim x_2 \Leftrightarrow f(x_1) = f(x_2)$ on X. It should be noted that the set $\hat{Y} \backslash f(X)$ is not necessarily empty, but the particular form of $\succ$ on this difference is of no matter in the context of this paper.

In terms of vectors, the standard multicriteria choice problem $\langle Y, \succ \rangle$ contains the set of feasible vectors Y and the strict preference relation $\succ$ defined on Y, while its solution is the set of vectors $C(Y) \subset Y$.

Pareto axiom 1. *For two any vectors $y', y'' \in Y$ such that $y' \geq y''$, it is true that $y' \succ y''$.*

Here, the inequality $y' \geq y''$ means that each component of the first vector is greater than or equal to the corresponding component of the second vector; moreover, $y' \neq y''$. This Pareto axiom seems quite natural if the DM tries to obtain as large as possible values in each of the criteria.

Relying on the above results, for the standard multicriteria choice problem under consideration, the generalized Pareto principle for $k = 1$ can be stated as follows.

Edgeworth–Pareto principle. Let Axiom 1 and the Pareto axiom hold. Then, for any set $C(X)$, we have the inclusion

$$C(X) \subset P_f(X), \tag{4}$$

where the right-hand side is the Pareto set defined as

$$P_f(X) = \{x^* \in Y \mid \text{ there is no } x \in X \text{ such that } f(x) \geq f(x^*)\}.$$

For the first time, this principle was presented in the form of (4) in [1]. It should be noted that the "best" alternatives were chosen from the Pareto set even before the overwhelming majority of studies concerning multicriteria problems had appeared. In fact, researchers followed the naive Edgeworth–Pareto principle, i.e., they treated it as an axiom, although it is not an intuitively evident statement, as required by the axiom concept. Meanwhile, from the DM's point of view, Axiom 1 and the Pareto axiom are fairly clear statements that can be accepted or rejected with reason. Moreover, it was found that these two axioms make up a minimum set in the sense that the elimination of at least one of them may lead to the violation of the Edgeworth–Pareto principle. It follows that at least one of them will necessarily be broken by choosing the best alternative outside the Pareto set. Accordingly, making such a choice, the DM must realize this situation. Similar conclusions can be drawn for the generalized Edgeworth–Pareto principle.

4. GENERALIZED FUZZY PARETO PRINCIPLE

Recall (see, e.g., [11]) that a fuzzy set X on a set A is defined by a membership function $\lambda(\cdot)$ given on A and taking values in the interval [0, 1], while a fuzzy relation on A is defined by a membership function $\mu(\cdot,\cdot)$ given on $A \times A$ and taking values from [0, 1]. The support of a fuzzy set X is defined by the equality $\operatorname{supp} X = \{x \mid \lambda(x) > 0\}$. A fuzzy relation μ is asymmetric if the inequality $\mu(x,y) > 0$ always implies that $\mu(y,x) = 0$ for all $x, y \in A$. A fuzzy set A with a membership function $\lambda(\cdot)$ is a subset of a fuzzy set B with a membership function $\nu(\cdot)$ (i.e., $A \subset B$) if $\lambda(x) \leqq \nu(x)$ for all $x \in A$.

Let us show how the Pareto principle can be formulated for multicriteria choice problems with a fuzzy preference relation and fuzzy set of alternatives.

Consider a fuzzy multicriteria choice problem (model) $\langle Y_1, \ldots, Y_m, \succ_1, \ldots, \succ_m, Y, \lambda(\cdot), \mu(\cdot,\cdot) \rangle$, where

Y_i is the scale of the ith criterion (crisp set), $i = 1, 2, \ldots, m$;

$\succ_i$ is an irreflexive, transitive, and weakly connected binary relation defined on the set Y_i, $i = 1, 2, \ldots, m$;

Y is the fuzzy set of feasible alternatives with membership function $\lambda(\cdot)$ defined on the Cartesian product $\hat{Y} = Y_1 \times \ldots \times Y_m$;

$\mu(\cdot,\cdot)$ is the membership function of an asymmetric binary fuzzy relation of strict preference defined on $\hat{Y}$.

The solution of this problem is a fuzzy set of chosen alternatives with a membership function $\lambda^C(\cdot)$ defined on $\hat{Y}$ such that $\lambda^C(y) \leqq \lambda(y)$ for all $y \in \hat{Y}$. The last inequality means that the set of chosen alternatives is a subset of Y.

Pareto axiom 2. *Let $k \in I$. For any two alternatives $y', y'' \in Y$ satisfying $y' \rhd_k y''$, it is always true that* $\mu(y', y'') = 1$.

Fuzzy exclusion axiom. *For any pair of alternatives $y', y'' \in Y$ such that $\mu(y', y'') = \mu^* \in [0,1]$, it is true that* $\lambda^C(y'') \leqq 1 - \mu^*$.

The membership function of a set of k-effective vectors is defined as

$$\lambda_k^E(y) = \begin{cases} \lambda(y) & \text{if } y \in E^k(Y), \\ 0 & \text{otherwise} \end{cases}$$

and the membership function of a fuzzy set of nondominated alternatives is defined as

$$\lambda^N(y) = \min\{1 - \sup_{z \in Y} \mu(z, y), \lambda(y)\} \quad \forall y \in Y.$$

Then the generalized fuzzy Pareto principle can be formulated as follows.

Theorem. *Let $k \in I$. Under the Pareto and fuzzy exclusion axioms, for any fuzzy set of chosen alternatives with a membership function* $\lambda^C(\cdot)$,

$$\lambda^C(y) \leqq \min\{\lambda_k^E(y), \lambda(y)\} \quad \forall y \in Y. \tag{5}$$

Proof. Consider an arbitrary fixed fuzzy set of chosen alternatives with a membership function $\lambda^C(\cdot)$. First, we prove the inequality

$$\lambda^C(y) \leqq \lambda^N(y) \quad \forall y \in Y. \tag{6}$$

By the fuzzy exclusion axiom, $\lambda^C(y) \leqq 1 - \mu(z, y)$ for all $y, z \in Y$. Passing to the infimum over $z \in Y$ on the right-hand side of this inequality, we obtain

$$\lambda^C(y) \leqq \inf_{z \in Y}(1 - \mu(z, y)) = 1 - \sup_{z \in Y} \mu(z, y) \quad \forall y \in Y.$$

Combining this result with $\lambda^C(y) \leqq \lambda(y)$ for all $y \in Y$ yields (5).

Now we prove the inequality

$$\lambda^N(y) \leqq \min\{\lambda_k^E(y), \lambda(y)\} \quad \forall y \in Y. \tag{7}$$

Assume the opposite, i.e., for some $y \in Y$, it holds that $\lambda^N(y) > \min\{\lambda_k^E(y), \lambda(y)\}$. Then, obviously, $\lambda^N(y) > 0$ and either $\lambda^N(y) > \lambda_k^E(y)$ or $\lambda^N(y) > \lambda(y)$. The alternative $\lambda^N(y) > \lambda(y)$ is not possible by the definition of $\lambda^N(\cdot)$. Therefore, $\lambda^N(y) > \lambda_k^E(y)$. By the definition of $\lambda_k^E(\cdot)$, its value at y is equal to $\lambda(y)$ or 0. Since the inequality $\lambda^N(y) > \lambda(y)$ is not possible, as was proved above, we conclude that $\lambda_k^E(y) = 0$. Therefore, there is an alternative $y' \in Y$ such that the relation $y' \rhd_k y$ holds. Combining this result with the Pareto axiom yields $\mu(y', y) = 1$ and, by the definition of a fuzzy set of nondominated alternatives, we obtain the contradiction $\lambda^N(y) \leqq 1 - \mu(y', y) = 0$, which proves (7).

Inequalities (6) and (7) imply (5), as required.

Corollary 2. Under the conditions of the theorem, let the set Y be crisp. Then, for any fuzzy set of chosen alternatives $C(Y)$ with a membership function $\lambda^C(\cdot)$, it is true that $\lambda^C(y) \leqq \lambda_k^E(y)$ for all $y \in Y$, which means that inclusion (3) holds for fuzzy sets.

It is easy to show that, for a crisp collective preference relation and a crisp set Y, the assertion of the theorem completely coincides with the generalized Edgeworth–Pareto principle obtained in Section 3.

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