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An Effective Method for Numerical Solution and Numerical Derivatives for Sixth Order Two-Point Boundary Value Problems[1]

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Abstract—In this paper, we study an effective quintic polynomial spline method for numerical solution, and first order to fifth order numerical derivatives of the analytic solution at the knots for a class of sixth order two-point boundary value problems. Our new method is based on a quintic spline interpolation problem. It is easy to implement and is able to provide sixth order accurate numerical results at the knots. Numerical tests show that our method is very practical and effective.

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1. INTRODUCTION

The following sixth order two-point boundary value problem

$$\begin{aligned} y^{(6)}(x) &= f(x, y(x)), \quad x \in [a, b], \\ y(a) = a_0, \quad y''(a) &= a_2, \quad y^{(4)}(a) = a_4, \\ y(b) = b_0, \quad y''(b) &= b_2, \quad y^{(4)}(b) = b_4, \end{aligned} \tag{1.1}$$

often arises in many fields in engineering and astrophysics, such as hydromagnetics, hydrodynamics, stellar convection dynamics, and so on, see [1–5]. The conditions for the existence and uniqueness of solutions of such problems have been discussed in [6].

Generally, it is difficult to obtain the analytic solution of (1.1) for arbitrary $f(x, y(x))$. Hence, numerical methods are desired. Currently, there have been some numerical methods for (1.1). For example, the modified decomposition method [7], the homotopy perturbation method [8], the variational iteration method [9], the spline methods [10–13] and the fourth order finite difference method [14] have been presented by some scholars respectively. However, the error orders of some of these methods are not higher. Moreover, there are few available effective numerical methods for derivatives approximation for (1.1). Actually, except [14], the other methods do not provide the numerical approximation to the derivatives for (1.1). [14] is able to provide fourth order accurate numerical approximation to $y(x)$, $y''(x)$ and $y^{(4)}(x)$ at the knots for (1.1). But, [14] does not give the numerical approximation to $y'(x)$, $y^{(3)}(x)$ and $y^{(5)}(x)$.

In order to increase the error orders of the existing methods and also to provide more accurate numerical derivatives of (1.1), we develop a new effective quintic spline method for (1.1) in this paper. The method is based on a quintic spline interpolation problem and is easy to implement. It can provide sixth order accurate numerical approximation to $y(x)$, $y'(x)$, $y''(x)$, $y^{(3)}(x)$, $y^{(4)}(x)$ and $y^{(5)}(x)$ at the knots for (1.1). Moreover, using the obtained numerical data and differentiating $y^{(6)}(x) = f(x,y(x))$, we also can get sixth order accurate numerical approximation to the other high order derivatives of $y(x)$.

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries for a quintic spline interpolation problem; in Section 3, we study the new effective method; in Section 4, we test our method with three numerical examples, numerical tests show that the method is very effective in obtaining numerical solution and numerical derivatives for (1.1); finally, we conclude this paper in Section 5.

[1] The article is published in the original.

Table 1. The values of $B_i^{(k)}(x)$, $i = -2, -1, \ldots, n+2$; $k = 0, 1, 2, 3, 4$, at the knots

$B_i(x)$ \ x	x_{i-3}	x_{i-2}	x_{i-1}	x_i	x_{i+1}	x_{i+2}	x_{i+3}	else
$B_i(x)$	0	$\frac{1}{120}$	$\frac{26}{120}$	$\frac{66}{120}$	$\frac{26}{120}$	$\frac{1}{120}$	0	0
$B_i'(x)$	0	$\frac{1}{24h}$	$\frac{10}{24h}$	0	$-\frac{10}{24h}$	$-\frac{1}{24h}$	0	0
$B_i''(x)$	0	$\frac{1}{6h^2}$	$\frac{2}{6h^2}$	$-\frac{6}{6h^2}$	$\frac{2}{6h^2}$	$\frac{1}{6h^2}$	0	0
$B_i^{(3)}(x)$	0	$\frac{1}{2h^3}$	$-\frac{2}{2h^3}$	0	$\frac{2}{2h^3}$	$-\frac{1}{2h^3}$	0	0
$B_i^{(4)}(x)$	0	$\frac{1}{h^4}$	$-\frac{4}{h^4}$	$\frac{6}{h^4}$	$-\frac{4}{h^4}$	$\frac{1}{h^4}$	0	0

2. QUINTIC SPLINE INTERPOLATION

For an interval $I = [a, b]$, divide it into n subintervals $I_i = [x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$, by the equidistant knots $x_i = a + ih$, $i = 0, 1, \ldots, n$, where $h = \frac{b-a}{n}$. The quintic spline space is defined as follows

$$S_5(I) = \{s(x) \in C^4(I) \,|\, s(x)_{|I_i} \in P_5,\ i = 0, 1, \ldots, n-1\},$$

where $s(x)_{|I_i}$ denotes the restriction of $s(x)$ over I_i, and P_5 denotes the set of univariate quintic polynomials. $S_5(I)$ is a linear space and its dimension is $n + 5$.

Extend $I = [a, b]$ to $\tilde{I} = [a - 5h, b + 5h]$ with the equidistant knots $x_i = a + ih$, $i = -5, -4, \ldots, n + 5$. By the results in [15–17], we give the explicit representations of the typical quintic B-spline $B_i(x)$, $i = -2, -1, \ldots, n + 2$, as follows

$$B_i(x) = \frac{1}{120h^5}\begin{cases} (x - x_i + 3h)^5, & \text{if } x \in [x_{i-3}, x_{x-2}], \\ (x - x_i + 3h)^5 - 6(x - x_i + 2h)^5, & \text{if } x \in [x_{i-2}, x_{x-1}], \\ (x - x_i + 3h)^5 - 6(x - x_i + 2h)^5 + 15(x - x_i + h)^5, & \text{if } x \in [x_{i-1}, x_x], \\ (-x + x_i + 3h)^5 - 6(-x + x_i + 2h)^5 + 15(-x + x_i + h)^5, & \text{if } x \in [x_i, x_{x+1}], \\ (-x + x_i + 3h)^5 - 6(-x + x_i + 2h)^5, & \text{if } x \in [x_{i+1}, x_{x+2}], \\ (-x + x_i + 3h)^5, & \text{if } x \in [x_{i+2}, x_{x+3}], \\ 0, & \text{else.} \end{cases}$$

They are the basis splines of $S_5(I)$. See Table 1 for the values of $B_i^{(k)}(x)$, $i = -2, -1, \ldots, n + 2$; $k = 0, 1, 2, 3, 4$, at the knots.

Given a sufficiently smooth function $y(x)$, there exists a unique quintic spline $s(x) = \sum_{i=-2}^{n+2} c_i B_i(x) \in S_5(I)$ satisfying the following interpolation conditions

$$s(x_i) = y(x_i), \quad i = 0, 1, \ldots, n, \tag{2.1}$$

$$s''(a) = y''(a) + \frac{1}{720}h^4 y^{(6)}(a), \tag{2.2}$$

$$s''(b) = y''(b) + \frac{1}{720}h^4 y^{(6)}(b), \tag{2.3}$$

$$s^{(4)}(a) = y^{(4)}(a) - \frac{1}{12}h^2 y^{(6)}(a) + \frac{1}{240}h^4 y^{(8)}(a), \tag{2.4}$$

$$s^{(4)}(b) = y^{(4)}(b) - \frac{1}{12}h^2 y^{(6)}(b) + \frac{1}{240}h^4 y^{(8)}(b). \tag{2.5}$$

In the following, we derive the interpolation errors at the knots. For $j = 0, 1, \ldots, n$, let $y_j = s(x_j) = y(x_j)$, $m_j = s'(x_j)$, $M_j = s''(x_j)$, $T_j = s^{(3)}(x_j)$ and $F_j = s^{(4)}(x_j)$ for short. Since $s(x) = \sum_{i=-2}^{n+2} c_i B_i(x_i)$, by Table 1, we have

$$y_j = s(x_j) = \sum_{i=-2}^{n+2} c_i B_i(x_j) = \frac{1}{120}(c_{j-2} + 26c_{j-1} + 66c_j + 26c_{j+1} + c_{j+2}),$$

$$m_j = s'(x_j) = \sum_{i=-2}^{n+2} c_i B_i'(x_j) = \frac{1}{24h}(-c_{j-2} - 10c_{j-1} + 10c_{j+1} + c_{j+2}),$$

$$M_j = s''(x_j) = \sum_{i=-2}^{n+2} c_i B_i''(x_j) = \frac{1}{6h^2}(c_{j-2} + 2c_{j-1} - 6c_j + 2c_{j+1} + c_{j+2}),$$

$$T_j = s^{(3)}(x_j) = \sum_{i=-2}^{n+2} c_i B_i^{(3)}(x_j) = \frac{1}{2h^3}(-c_{j-2} + 2c_{j-1} - 2c_{j+1} + c_{j+2}),$$

$$F_j = s^{(4)}(x_j) = \sum_{i=-2}^{n+2} c_i B_i^{(4)}(x_j) = \frac{1}{h^4}(c_{j-2} - 4c_{j-1} + 6c_j - 4c_{j+1} + c_{j+2}).$$

By comparing the linear combination of c_i, $i = -2, -1, \ldots, n + 2$, we have

$$\frac{h}{5}(m_{j-2} + 26m_{j-1} + 66m_j + 26m_{j+1} + m_{j+2}) = -y_{j-2} - 10_{j-1} + 10y_{j+1} + y_{j+2},$$

$$\frac{h^2}{20}(M_{j-2} + 26M_{j-1} + 66M_j + 26M_{j+1} + M_{j+2}) = y_{j-2} + 2y_{j-1} - 6y_j + 2y_{j+1} + y_{j+2},$$

$$\frac{h^3}{60}(T_{j-2} + 26T_{j-1} + 66T_j + 26T_{j+1} + T_{j+2}) = -y_{j-2} + 2y_{j-1} - 2y_{j+1} + y_{j+2},$$

$$\frac{h^4}{120}(F_{j-2} + 26F_{j-1} + 66F_j + 26F_{j+1} + F_{j+2}) = y_{j-2} - 4y_{j-1} + 6y_j - 4y_{j+1} + y_{j+2}.$$

Let $Ey(x) = y(x + h)$ be the shift operator, $Dy(x) = y'(x)$ be the differential operator, and $Iy(x) = y(x)$ be the identity operator. These operators are very useful in numerical analysis, see [18–20]. For a positive integer m, we have

$$E^m y(x) = y(x + mh), \quad E^{-m} y(x) = y(x - mh), \quad D^m y(x) = y^{(m)}(x), \quad I^m y(x) = y(x).$$

Moreover, we have

$$Ey(x) = y(x+h) = \sum_{n=0}^{\infty} \frac{h^n y^{(n)}(x)}{n!} = \left[\sum_{m=0}^{\infty} \frac{(hD)^n}{n!}\right] y(x) = e^{hD} y(x),$$

which implies that $E = e^{hD}$. Similarly, we have

$$E^{-1} = e^{-hD}, \quad E^{m} = e^{mhD} \quad \text{and} \quad E^{-m} = e^{-mhD}.$$

By using these operators and expanding them in powers of hD, we obtain

$$m_j = \frac{5}{h}\left(\frac{-E^{-2} - 10E^{-1} + 10E + E^{2}}{E^{-2} + 26E^{-1} + 66I + 26E + E^{2}}\right)y_i = \frac{5}{h}\left(\frac{-e^{-2hD} - 10e^{-hD} + 10e^{hD} + e^{2hD}}{e^{-2hD} + 26e^{-hD} + 66I + 26e^{hD} + e^{2hD}}\right)y_j$$

$$= \frac{5}{h}\left(\frac{24(hD) + 6(hD^{3}) + \frac{7}{10}(hD)^{5} + \frac{23}{420}(hD)^{7} + \dots}{120I + 30(hD)^{2} + \frac{7}{2}(hD)^{4} + \frac{1}{4}(hD)^{6} + \dots}\right)y_i = \frac{1}{h}\left(hD + \frac{1}{5040}(hD)^{7} + \dots\right)y_j = y'(x_j) + O(h^{6}).$$

Similarly, we get

$$M_j = \frac{20}{h^{2}}\left(\frac{E^{-2} + 2E^{-1} - 6I + 2E + E^{2}}{E^{-2} + 26E^{-1} + 66I + 26E + E^{2}}\right)y_j = y''(x_j) + \frac{1}{720}h^{4}y^{(6)}(x_j) + O(h^{6}),$$

$$T_j = \frac{60}{h^{3}}\left(\frac{-E^{-2} + 2E^{-1} - 2E + E^{2}}{E^{-2} + 26E^{-1} + 66I + 26E + E^{2}}\right)y_j = y^{(3)}(x_j) - \frac{1}{240}h^{4}y^{(7)}(x_j) + O(h^{6}),$$

$$F_j = \frac{120}{h^{4}}\left(\frac{E^{-2} - 4E^{-1} + 6I - 4E + E^{2}}{E^{-2} + 26E^{-1} + 66I + 26E + E^{2}}\right)y_j = y^{(4)}(x_j) - \frac{1}{12}h^{2}y^{(6)}(x_j) + \frac{1}{240}h^{4}y^{(8)}(x_j) + O(h^{6}).$$

Furthermore, we can use F_j, $j = 0, 1, \dots, n$, to construct numerical formulae for $y^{(5)}(x_j)$ and $y^{(6)}(x_j)$, $j = 1, 2, \dots, n - 1$, as follows

$$\frac{F_{j+1} - F_{j-1}}{2h} = \frac{1}{2h^{5}}(-c_{j-3} + 4c_{j-2} - 5c_{j-1} + 5c_{j+1} - 4c_{j+2} + c_{j+3})$$

$$= \frac{60}{h^{5}}\left(\frac{-E^{-3} + 4E^{-2} - 5E^{-1} + 5E - 4E^{2} + E^{3}}{E^{-2} + 26E^{-1} + 66I + 26E + E^{2}}\right)y_j = y^{(5)}(x_j) + \frac{1}{12}h^{2}y^{(7)}(x_j) - \frac{1}{720}h^{4}y^{(9)}(x_j) + O(h^{6}),$$

$$\frac{F_{j+1} - 2F_j + F_{j-1}}{2h} = \frac{1}{h^{6}}(c_{j-3} - 6c_{j-2} + 15c_{j-1} - 20c_j + 15c_{j+1} - 6c_{j+2} + c_{j+3})$$

$$= \frac{120}{h^{6}}\left(\frac{E^{-3} - 6E^{-2} + 15E^{-1} - 20I + 15E - 6E^{2} + E^{3}}{E^{-2} + 26E^{-1} + 66I + 26E + E^{2}}\right)y_j = y^{(6)}(x_j) + O(h^{6}).$$

In a word, for $j = 0, 1, \dots, n$, we have

$$y(x_j) = \frac{1}{120}(c_{j-2} + 26c_{j-1} + 66c_j + 26c_{j+1} + c_{j+2}). \tag{2.6}$$

$$y'(x_j) = \frac{1}{24h}(-c_{j-2} - 10c_{j-1} + 10c_{j+1} + c_{j+2}) + O(h^{6}), \tag{2.7}$$

$$y''(x_j) = \frac{1}{6h^{2}}(c_{j-2} + 2c_{j-1} - 6c_j + 2c_{j+1} + c_{j+2}) - \frac{1}{720}h^{4}y^{(6)}(x_j) + O(h^{6}), \tag{2.8}$$

$$y^{(3)}(x_j) = \frac{1}{2h^{3}}(-c_{j-2} + 2c_{j-1} - 2c_{j+1} + c_{j+2}) + \frac{1}{240}h^{4}y^{(7)}(x_j) + O(h^{6}), \tag{2.9}$$

$$y^{(4)}(x_j) = \frac{1}{h^{4}}(c_{j-2} - 4c_{j-1} + 6c_j - 4c_{j+1} + c_{j+2}) + \frac{1}{12}h^{2}y^{(6)}(x_j) - \frac{1}{240}h^{4}y^{(8)}(x_j) + O(h^{6}), \tag{2.10}$$

and for $j = 1, 2, \ldots, n-1$, we have

$$y^{(5)}(x_j) = \frac{1}{2h^5}(-c_{j-3} + 4c_{j-2} - 5c_{j-1} + 5c_{j+1} - 4c_{j+2} + c_{j+3}) - \frac{1}{12}h^2 y^{(7)}(x_j) + \frac{1}{720}h^4 y^{(9)}(x_j) + O(h^6), \quad (2.11)$$

$$y^{(6)}(x_j) = \frac{1}{h^6}(c_{j-3} - 6c_{j-2} + 15c_{j-1} - 20c_j + 15c_{j+1} - 6c_{j+2} + c_{j+3}) + O(h^6). \quad (2.12)$$

These equations will be used in approximating the analytic solution and its derivatives of (1.1) at the knots.

3. NUMERICAL METHOD

Let $s(x) = \sum_{i=-2}^{n+2} c_i B_i(x)$ be the quintic spline solution of (1.1). Discretize (1.1) at the inner knots, we get

$$y^{(6)}(x_i) = f(x_i, y(x_i)), \quad i = 1, 2, \ldots, n-1.$$

By (2.6) and (2.12), we have

$$\frac{c_{i-3} - 6c_{i-2} + 15c_{i-1} - 20c_i + 15c_{i+1} - 6c_{i+2} + c_{i+3}}{h^6} = f\left(x_i, \frac{c_{i-2} + 26c_{i-1} + 66c_i + 26c_{i+1} + c_{i+2}}{120}\right). \quad (3.1)$$

Obviously, the truncated error of (3.1) is $O(h^6)$. (3.1) is equivalent to

$$c_{i-3} - 6c_{i-2} + 15c_{i-1} - 20c_i + 15c_{i+1} - 6c_{i+2} + c_{i+3} = h^6 f\left(x_i, \frac{c_{i-2} + 26c_{i-1} + 66c_i + 26c_{i+1} + c_{i+2}}{120}\right). \quad (3.2)$$

We still need six equations, which can be obtained from the boundary conditions. By $y(a) = a_0$ and $y(b) = b_0$, from (2.6), we have

$$c_{-2} + 26c_{-1} + 66c_0 + 26c_1 + c_2 = 120a_0, \quad (3.3)$$

$$c_{n-2} + 26c_{n-1} + 66c_n + 26c_{n+1} + c_{n+2} = 120b_0. \quad (3.4)$$

By $y''(a) = a_2$, $y''(b) = b_2$ and $y^{(6)}(a) = f(a, a_0)$, $y^{(6)}(b) = f(b, b_0)$, from (2.2), (2.3) and (2.8), we have

$$c_{-2} + 2c_{-1} - 6c_0 + 2c_1 + c_2 = 6a_2h^2 + \frac{h^6}{120}y^{(6)}(a), \quad (3.5)$$

$$c_{n-2} + 2c_{n-1} - 6c_n + 2c_{n+1} + c_{n+2} = 6b_2h^2 + \frac{h^6}{120}y^{(6)}(b). \quad (3.6)$$

By differentiating $y^{(6)}(x) = f(x, y(x))$, we get

$$y^{(7)}(x) = f_1'(x, y(x)) + f_2'(x, y(x))y'(x), \quad (3.7)$$

$$\begin{aligned} y^{(8)}(x) &= f_{11}''(x, y(x)) + 2f_{12}''(x, y(x))y'(x) + f_{22}''(x, y(x))(y'(x))^2 \\ &\quad + f_2'(x, y(x))y''(x) := G(x, y(x), y'(x), y''(x)), \end{aligned} \quad (3.8)$$

$$\begin{aligned} y^{(9)}(x) &= G_1'(x, y(x), y'(x), y''(x)) + G_2'(x, y(x), y'(x), y''(x))y'(x) + G_3'(x, y(x), y'(x), y''(x))y''(x) \\ &\quad + G_4'(x, y(x), y'(x), y''(x))y'''(x) := H(x, y(x), y'(x), y''(x), y'''(x)). \end{aligned} \quad (3.9)$$

Using $y^{(4)}(a) = a_4$, $y^{(4)}(b) = b_4$, $y^{(6)}(a) = f(a, a_0)$, $y^{(6)}(b) = f(b, b_0)$, from (2.4), (2.5) and (2.10), we have

$$c_{-2} - 4c_{-1} + 6c_0 - 4c_1 + c_2 = a_4h^4 - \frac{h^6}{12}y^{(6)}(a) + \frac{h^8}{240}y^{(8)}(a), \quad (3.10)$$

$$c_{n-2} - 4c_{n-1} + 6c_n - 4c_{n+1} + c_{n+2} = b_4h^4 - \frac{h^6}{12}y^{(6)}(b) + \frac{h^8}{240}y^{(8)}(b), \quad (3.11)$$

where

$$y^{(8)}(a) = G\left(a, a_0, \frac{-c_{-2} - 10c_{-1} + 10c_1 + c_2}{24h}, a_2\right),$$

$$y^{(8)}(b) = G\left(b, b_0, \frac{-c_{n-2} - 10c_{n-1} + 10c_{n+1} + c_{n+2}}{24h}, b_2\right),$$

are obtained from (3.8).

Take (3.3), (3.5), (3.10), (3.2), and (3.11), (3.6), (3.4) together, we obtain a nonlinear system composed of $n + 5$ nonlinear equations with c_i, $i = -2, -1, \ldots, n + 2$, as unknowns. Let

$$C = [c_{-2}, c_{-1}, \ldots, c_{n+2}]^{\mathrm{T}},$$

$$F = \left[120a_0, 6a_2h^2 + \frac{h^6}{120}y^{(6)}(a), a_4h^4 - \frac{h^6}{12}y^6(a) + \frac{h^8}{240}y^8(a), h^6\Phi_1, \ldots\right.$$
$$\left. h^6\Phi_{n-1}, b_4h^4 - \frac{h^6}{12}y^{(6)}(b) + \frac{h^8}{240}y^{(8)}(b), 6b_2h^2 + \frac{h^6}{120}y^{(6)}(b), 120b_0\right]^{\mathrm{T}},$$

where $\Phi_i = f\left(x_i, \frac{c_{i-2} + 26c_{i-1} + 66c_i + 26c_{c+1} + c_{i+2}}{120}\right)$, $i = 1, 2, \ldots, n - 1$, and

$$A = \begin{pmatrix} 1 & 26 & 66 & 26 & 1 & & & & & \\ 1 & 2 & -6 & 2 & 1 & & & & & \\ 1 & -4 & 6 & -4 & 1 & & & & & \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & -6 & 15 & -20 & 15 & -6 & 1 \\ & & & & & 1 & -4 & 6 & -4 & 1 \\ & & & & & 1 & 2 & -6 & 2 & 1 \\ & & & & & 1 & 26 & 66 & 26 & 1 \end{pmatrix}_{(n+5)\times(n+5)}. \quad (3.12)$$

Then, the nonlinear system can be written in matrix notations as

$$AC = F. \quad (3.13)$$

If (1.1) is a linear boundary value problem as

$$\begin{aligned} y^{(6)}(x) &= p(x)y(x) + q(x), \quad x \in [a, b], \\ y(a) &= a_0, \quad y''(a) = a_2, \quad y^{(4)}(a) = a_4, \\ y(b) &= b_0, \quad y''(b) = b_2, \quad y^{(4)}(b) = b_4, \end{aligned} \quad (3.14)$$

then (3.2) becomes

$$\begin{aligned} &c_{i-3} - 6c_{i-2} + 15c_{i-1} - 20c_i + 15c_{i+1} - 6c_{i+2} + c_{i+3} \\ &- p(x_i)h^6\frac{c_{i-2} + 26c_{i-1} + 66c_i + 26c_{c+1} + c_{i+2}}{120} = q(x_i)h^6. \end{aligned} \quad (3.15)$$

(3.8) becomes

$$y^{(8)}(x) = p''(x)y(x) + 2p'(x)y'(x) + p(x)y''(x) + q''(x), \quad (3.16)$$

hence, (3.10) becomes

$$\begin{aligned} &c_{-2} - 4c_{-1} + 6c_0 - 4c_1 + c_2 - \frac{1}{120 \times 24}h^7 p'(a)(-c_{-2} - 10c_{-1} + 10c_1 + c_2) \\ &= a_4h^4 - \frac{1}{12}y^{(6)}(a)h^6 + \frac{1}{240}h^8(p''(a)a_0 + p(a)a_2 + q''(a)), \end{aligned} \tag{3.17}$$

(3.11) becomes

$$\begin{aligned} &c_{n-2} - 4c_{n-1} + 6c_n - 4c_{n+1} + c_{n+2} = \frac{1}{120 \times 24}h^7 p'(b)(-c_{n-2} - 10c_{n-1} + 10c_{n+1} + c_{n+2}) \\ &= b_4h^4 - \frac{1}{12}y^{(6)}(b)h^6 + \frac{1}{240}h^8(p''(b)b_0 + p(b)b_2 + q''(b)), \end{aligned} \tag{3.18}$$

Take (3.3), (3.5), (3.17), (3.15), and (3.18), (3.6), (3.4) together, we get a linear system as follows

$$\left(A - \frac{1}{120}h^6 PB\right)C = Q, \tag{3.19}$$

where A is given in (3.12), and

$$P = diag\left(0, 0, \frac{h}{24}p'(a), p(x_1), \ldots, p(x_{n-1}), \frac{h}{24}p'(b), 0, 0\right),$$

$$B = \begin{pmatrix} 0 & & & & & & & & & \\ & 0 & & & & & & & & \\ -1 & -10 & 0 & 10 & 1 & 0 & & & & \\ 0 & 1 & 26 & 66 & 26 & 1 & 0 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & 0 & 1 & 26 & 66 & 26 & 1 & 0 \\ & & & & 0 & -1 & -10 & 0 & 10 & 1 \\ & & & & & & & & 0 & \\ & & & & & & & & & 0 \end{pmatrix}_{(n+5)\times(n+5)},$$

$$\begin{aligned} Q = \Big[&120a_0, 6a_2h^2 + \frac{h^6}{120}y^{(6)}(a), a_4h^4 - \frac{1}{12}y^6(a)h^6 + \frac{1}{240}h^8(p''(a)a_0 + p(a)a_2 + q''(a)), q(x_1)h^6, \ldots \\ &q(x_{n-1})h^6, b_4h^4 - \frac{1}{12}y^{(6)}(b)h^6 + \frac{1}{240}h^8(p''(b)b_0 + p(b)b_2 + q''(b)), 6b_2h^2 + \frac{h^6}{120}y^{(6)}(b), 120b_0\Big]^{\mathrm{T}}. \end{aligned}$$

After solving the nonlinear system (3.13) or the linear system (3.19), we obtain c_i $(i = -2, -1, \ldots, n+2)$, and $s(x) = \sum_{i=-2}^{n+2} c_iB_i(x)$ is the approximation solution. By the quintic spline interpolation theory, we know $s^{(k)}(x)$ can approximate $y^{(k)}(x)$ over $[a, b]$ with $Q(h^{6-k})$ error, where $k = 0, 1, \ldots, 4$. To get better approximation at the knots, especially, we can use (2.6), (2.7), (2.8), (2.9), (2.10) and (2.11) to get the sixth order accurate numerical approximation to $y(x)$, $y'(x)$, $y''(x)$, $y^{(3)}(x)$, $y^{(4)}(x)$ and $y^{(5)}(x)$ at the knots for (1.1) respectively, where $y^{(6)}(x_j)$, $y^{(7)}(x_j)$, $y^{(8)}(x_j)$ and $y^{(9)}(x_j)$ are computed by (2.12), (3.7), (3.8) and (3.9) respectively. We give the following computational procedure.

Step 1. Solve (3.13) or (3.19) by using "fsolve" command of Matlab (refer to http://www.mathworks.com/help/optim/ug/fsolve.html; or enter "help fsolve" in the command window to find the description and usage).

Step 2. By (2.6), use $y_j = \dfrac{c_{j-2} + 26c_{j-1} + 66c_j + 26c_{j+1} + c_{j+2}}{120}$ approximate $y(x_j)$.

Step 3. By (2.7), use $m_j = \dfrac{-c_{j-2} - 10c_{j-1} + 10c_{j+1} + c_{j+2}}{24h}$ to approximate $y'(x_j)$.

Step 4. By (2.8) and (2.12), use

$$\tilde{M}_j = \frac{-c_{j-3} + 126c_{j-2} + 225c_{j-1} - 700c_j + 225c_{j+1} + 126c_{j+2} - c_{j+3}}{720h^2}$$

to approximate $y''(x_j)$.

Step 5. By (2.9) and (3.7), use

$$\tilde{T}_j = \frac{1}{2h^3}(-c_{j-2} + 2c_{j-1} - 2c_{j+1} + c_{j+2}) + \frac{1}{240}h^4(f_1'(x_j, y_j) + f_2'(x_j, y_j)m_j)$$

to approximate $y^{(3)}(x_j)$.

Step 6. By (2.10), (2.12) and (3.8), use

$$\tilde{F}_j = \frac{c_{j-3} + 6c_{j-2} - 33c_{j-1} + 52c_j - 33c_{j+1} + 6c_{j+2} + c_{j+3}}{12h^4} - \frac{1}{240}h^4 G(x_j, y_j, m_j, \tilde{M}_j)$$

to approximate $y^{(4)}(x_j)$.

Step 7. By (2.11), (3.7) and (3.9), use

$$\tilde{W}_j = \frac{-c_{j-3} + 4c_{j-2} - 5c_{j-1} + 5c_{j+1} - 4c_{j+2} + c_{j+3}}{2h^5}$$
$$- \frac{1}{12}h^2(f_1'(x_j, y_j) + f_2'(x_j, y_j)m_j) + \frac{1}{720}h^4 H(x_j, y_j, m_j, \tilde{M}_j, \tilde{T}_j)$$

to approximate $y^{(5)}(x_j)$.

Step 8. If $\bar{x} \in [a, b]$ is not a knot and $y^{(k)}(\bar{x})$, $k = 0, 1, ..., 4$, is needed, we can use $s^{(k)}(\bar{x}) = \sum_{i=-2}^{n+2} c_i B_i^{(k)}(\bar{x})$ to approximate $y^{(k)}(\bar{x})$.

4. NUMERICAL TESTS

In this section, we examine our method by Matlab with three same numerical examples that have been studied by other authors for the sake of comparison. For every example, we first give the maximum absolute errors $E(n, \mu)$, $\mu = 0, 1, ..., 5$, at the knots of our method for different steps, where

$$E(n, 0) = \max_{1 \le j \le n-1} |y(x_j) - y_j|, \quad E(n, 1) = \max_{1 \le j \le n-1} |y'(x_j) - m_j|,$$
$$E(n, 2) = \max_{1 \le j \le n-1} |y''(x_j) - \tilde{M}_j|, \quad E(n, 3) = \max_{1 \le j \le n-1} |y^{(3)}(x_j) - \tilde{T}_j|,$$
$$E(n, 4) = \max_{1 \le j \le n-1} |y^{(4)}(x_j) - \tilde{F}_j|, \quad E(n, 5) = \max_{1 \le j \le n-1} |y^{(5)}(x_j) - W_j|.$$

For the sake of integrity, we also give the global maximum absolute errors E_μ, $\mu = 0, 1, ..., 4$, over $[a, b]$ of our method for different steps, where

$$E_\mu = \max_{a \le x \le b} |y^{(\mu)}(x) - s^{(\mu)}(x)| = \max_{a \le x \le b} \left| y^{(\mu)}(x) - \sum_{i=-2}^{n+2} c_i B_i^{(\mu)}(x) \right|.$$

Example 4.1. Consider the following linear six order boundary value problem

$$y^{(6)}(x) = -y(x) + 12x\cos x + 30\sin x, \quad x \in [0, 1],$$
$$y(0) = 0, \quad y''(0) = 0, \quad y^{(4)}(0) = 0,$$
$$y(1) = 0, \quad y''(1) = 2\sin 1 + 4\cos 1, \quad y^{(4)}(1) = -12\sin 1 - 8\cos 1,$$

Table 2. Our maximum absolute errors at the knots of Example 4.1

n \ E	$E(n, 0)$	$E(n, 1)$	$E(n, 2)$	$E(n, 3)$	$E(n, 4)$	$E(n, 5)$
4	2.717e–7	2.865e–6	2.943e–7	1.067e–5	3.367e–7	4.436e–6
8	4.285e–9	4.413e–8	4.559e–9	1.704e–7	5.293e–9	6.973e–8
16	6.945e–11	6.927e–10	5.495e–11	2.731e–9	1.030e–10	1.715e–9

Table 3. The maximum absolute errors at the knots in [12–14] of Example 4.1

n \ E	$E(n, 0)$ ([12])	$E(n, 0)$ ([13])	$E(n, 0)$ ([14])	$E(n, 2)$ ([14])	$E(n, 4)$ ([14])
4	–	–	1.121e–6	1.102e–5	1.036e–4
8	8.151e–5	1.652e–8	6.855e–8	6.745e–7	6.493e–6
16	2.105e–5	2.497e–10	4.262e–9	4.194e–8	4.117e–7

Table 4. Our maximum absolute errors at the knots of Example 4.2

n \ E	$E(n, 0)$	$E(n, 1)$	$E(n, 2)$	$E(n, 3)$	$E(n, 4)$	$E(n, 5)$
4	1.222e–7	1.488e–6	6.868e–8	2.849e–6	1.105e–7	8.840e–7
8	1.914e–9	2.347e–8	1.085e–9	5.006e–8	1.702e–9	1.418e–8
16	2.933e–11	3.661e–10	1.553e–11	8.099e–10	3.382e–10	2.498e–9

where the analytic solution is $y(x) = (x^2 - 1)\sin x$, see Table 2 for our maximum absolute errors at the knots.

Example 4.2. Consider the following linear six order boundary value problem

$$y^{(6)}(x) = y(x) - 6e^x, \quad x \in [0, 1],$$
$$y(0) = 1, \quad y''(0) = -1, \quad y^{(4)}(0) = -3,$$
$$y(1) = 0, \quad y''(1) = -2e, \quad y^{(4)}(1) = -4e,$$

where the analytic solution is $y(x) = (1 - x)e^x$, see Table 4 for our maximum absolute errors at the knots.

Example 4.3. Consider the following nonlinear six order boundary value problem

$$y^{(6)}(x) = e^{-x}y^2(x), \quad x \in [0, 1],$$
$$y(0) = 1, \quad y''(0) = 1, \quad y^{(4)}(0) = 1,$$
$$y(1) = e, \quad y''(1) = e, \quad y^{(4)}(1) = e,$$

where the analytic solution is $y(x) = e^x$, see Table 6 for our maximum absolute errors at the knots.

The results in Table 2, Table 4 and Table 6 are consistent with our expectation. It is easy to find that $E(n, \mu)$, $\mu = 0, 1, ..., 5$, decreases about by 1/64 when the original interval is refined by 1/2 step by step, without considering the round off errors. It indicates that the numerical results are of sixth order accurate.

Example 4.1 was also studied in [12–14], see Table 3 for the respective maximum absolute errors at the knots. Obviously, our results are better than that of Table 3. Furthermore, we remark that our new method has lower computational complexity. In fact, the method in [12] was based on sextic spline, the method in [13] was based on septic non-polynomial spline. They are higher degree spline based methods, hence, they have higher computational complexity. At the same time, the method in [14] requires solving a system with $3n$ equations and $3n$ unknowns, while our method only requires solving a system with $n + 5$ equations and $n + 5$ unknowns.

Example 4.2 was also studied in [11] by septic spline method, see Table 5 for the maximum absolute errors at the knots. The numerical results in Table 5 are only of second order accurate, which are lower

Table 5. The maximum absolute errors at the knots in [11] of Example 4.2

n \ E	$E(n, 0)$	$E(n, 1)$	$E(n, 2)$	$E(n, 3)$	$E(n, 4)$	$E(n, 5)$
8	1.37e–6	4.67e–6	5.11e–5	2.36e–4	1.30e–3	9.80e–3
16	1.08e–7	3.70e–7	2.46e–6	3.01e–5	4.48e–4	2.60e–3
32	2.25e–8	7.79e–8	5.37e–7	9.14e–6	1.16e–4	6.08e–4
64	7.04e–9	2.43e–8	1.69e–7	2.81e–6	1.74e–4	7.11e–2

Table 6. Our maximum absolute errors at the knots of Example 4.3

n \ E	$E(n, 0)$	$E(n, 1)$	$E(n, 2)$	$E(n, 3)$	$E(n, 4)$	$E(n, 5)$
4	1.600e–8	2.048e–7	7.196e–9	3.271e–7	1.181e–8	8.628e–8
8	2.507e–10	3.224e–9	1.135e–10	5.674e–9	1.801e–10	1.364e–9
16	5.844e–11	5.515e–11	2.528e–11	1.462e–10	5.224e–10	3.210e–9

Table 7. Comparison results of Example 4.3

Errors \ x_i	0.1	0.2	0.3	0.4	0.5
our errors ($n = 10$)	2.091e–11	3.842e–11	5.211e–11	6.152e–11	6.618e–11
errors in [7–9]	1.233e–4	2.354e–4	3.257e–4	3.855e–4	4.086e–4

Table 8. Comparison results of Example 4.3

Errors \ x_i	0.6	0.7	0.8	0.9	1.0
our errors ($n = 10$)	6.559e–11	5.924e–11	4.656e–11	2.702e–11	0
errors in [7–9]	3.919e–4	3.361e–4	2.459e–4	1.299e–4	2.000e–9

Table 9. The running time (in seconds)

n \ Example	4.1	4.2	4.3
4	0.41 s	0.41 s	0.42 s
8	0.41 s	0.42 s	0.43 s
16	0.41 s	0.42 s	0.50 s

than ours. To get a similar error, [11] must use a bigger n (the number of the subintervals). It shows that the computational cost of [11] is higher than our method.

Example 4.3 was also studied in [7–9] by modified decomposition method, homotopy perturbation method and variational iteration method, respectively. These methods, which involve many numerical and symbolic computations, developed a same polynomial of degree 12 over [0, 1] as the approximation solution. The absolute errors at $x_i = \frac{i}{10}$, $i = 1, 2, \ldots, 10$, are given in Table 7 and Table 8. Our errors, which are obtained with $n = 10$, are dramatically better than that of [7–9]. It shows that the efficiency of our method is higher.

We also point out that the results in Table 2, Table 4 and Table 6 can be obtained instantaneously by Matlab on a personal computer (1.97 Ghz CPU, 1 G Memory). See Table 9 for the running time.

Table 10. E_μ, $\mu = 0, 1, ..., 4$, of the examples

Example \ E	E_0	E_1	E_2	E_3	E_4
4.1 ($n = 4$)	7.438e–5	9.658e–4	1.485e–4	6.737e–4	1.431e–1
4.1 ($n = 8$)	1.156e–6	2.920e–5	1.015e–5	4.333e–5	3.903e–2
4.1 ($n = 16$)	1.808e–8	9.125e–7	6.557e–7	2.728e–6	1.007e–2
4.2 ($n = 4$)	2.787e–5	3.185e–4	6.610e–5	2.297e–4	6.313e–2
4.2 ($n = 8$)	4.386e–7	1.095e–5	4.779e–6	1.673e–5	1.833e–2
4.2 ($n = 16$)	6.857e–9	3.453e–7	3.213e–7	1.126e–6	4.934e–3
4.3 ($n = 4$)	4.646e–6	5.309e–5	1.149e–5	3.412e–5	1.099e–2
4.3 ($n = 8$)	7.310e–8	1.825e–6	8.135e–7	2.435e–6	3.121e–3
4.3 ($n = 16$)	1.143e–9	5.755e–8	5.412e–8	1.623e–7	8.311e–4

Before we end this section, we also give E_μ for the examples, see Table 10. By the quintic spline interpolation theory, we know $E_\mu = O(h^{6-\mu})$. It is easy to observe that the decrease rates of E_0, E_1, E_2, E_3 and E_4 are about $\frac{1}{64}, \frac{1}{32}, \frac{1}{16}, \frac{1}{8}$ and 1/4 respectively, when the original interval is halved step by step. These results are also consistent with expectation.

In a word, our quintic spline method is very effective for approximating the solution and the derivatives of the solution for (1.1). The super numerical products are the sixth order accurate numerical results at the knots.

5. CONCLUSIONS

In this paper, quintic spline is properly used to develop an effective numerical method for solving sixth order two-point boundary value problems (1.1). The main advantage of our new method is that it can provide sixth order accurate numerical results to approximate $y^{(\mu)}(x)$, $\mu = 0, 1, ..., 5$, at the knots. It shows that quintic spline is very powerful and effective for solving boundary value problems (1.1). Moreover, we believe quintic spline can also be used to solve some other kinds of differential equations. Some works are under consideration.

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