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# Optimal Control of Linear Systems with Interval Constraints

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**Abstract**—For linear systems with interval constraints, a method for computing a time-optimal control is proposed. The method is based on transforming a quasi-optimal control. The properties and features of the quasi-optimal control are examined. A technique is described for dividing the domain of initial conditions into reachable sets over different times and for approximating each set by a family of hyperplanes. An iterative method for computing an optimal control with interval constraints is developed. The convergence of the method is proved, and a sufficient condition for the convergence of the computational process is obtained. The radius of local quadratic convergence is found. Numerical results are presented.

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**Keywords:** optimal control, quasi-optimal control, interval constraints, time-optimal control, reachable set, adjoint system, convergence, iterative process.

## 1. INTRODUCTION

The development of optimal control theory is great theoretical and practical interest (see, for example, [1]). Various aspects of this theory [2–6], including the computation of optimal control with interval constraints [7–11], have been considered by researchers in our country and abroad. Below, a method based on transforming a quasi-optimal control [12, 13] is proposed for computing a time-optimal control with interval constraints imposed on the components of the control vector. The quasi-optimal control represents an alternating sequence of control functions with different polarity whose values are proportional to the initial conditions and the switching times are close to those of the time-optimal control. The quasi-optimal control has a number of important properties. For example, it is easy to implement and is capable of driving a system to the origin from any initial state belonging to the controllability domain. The complexity of implementing the quasi-optimal control increases only slightly as the order of the controlled system and the number of control parameters grow. The control is formed in real time for high-order systems, which makes it possible to control fast-acting objects and rapidly proceeding processes.

## 2. FORMULATION OF THE PROBLEM

Let a controlled system be governed by the linear differential equation

$$\dot{x} = A(t)x + B(t)u, \quad x(t_0) = x_0, \quad x_0 \in D, \quad D \subset V, \tag{2.1}$$

where $x$ is the $n$-dimensional state vector; $A(t)$ and $B(t)$ are continuous $n \times n$ and $n \times m$ matrices, respectively; and $u$ is an $m$-dimensional control vector whose components are piecewise continuous functions satisfying the interval constraints

$$M_j^1 \le |u_j| \le M_j^2, \quad M_j^1, M_j^2 > 0, \quad M_j^2 > M_j^1, \quad j = \overline{1, m}. \tag{2.2}$$

It is assumed that system (2.1) is completely controllable, i.e.,

$$\operatorname{rank}\left[\int_{t_0}^{t_k} \Phi(t_k, \tau) B(\tau) B^*(\tau) \Phi^*(t_k, \tau) d\tau\right] = n, \tag{2.3}$$

and can be driven to the origin from a bounded domain $D$ of initial conditions; $V$ is the controllability domain, $\Phi(t_k, t_0)$ is the fundamental solution matrix of the homogeneous differential equation, and * denotes the transpose.

**Problem.** Find an admissible control $u^0(t)$ that satisfies interval constraints (2.2) and drives system (2.1) from the initial state $x(t_0) = x_0$ to the origin $x(t_k) = 0$ over the minimum time $T = t_k - t_0$.

## 3. NUMERICAL METHOD FOR SOLVING THE PROBLEM

### *3.1. Formation of Quasi-Time-Optimal Control*

Consider variable constraints depending on the initial conditions. Assume that the components of the control vector satisfy the constraint

$$|u_j| \le \sum_{i=1}^{n} N_{ij}|x_i(t_0)|, \quad N_{ij} > 0, \quad j = \overline{1, m}, \tag{3.1}$$

where $N_{ij}$ are weighting coefficients. Let the initial point $x(t_0) = x_0$ be on the $i$th axis of the state space, i.e., the vector of initial conditions $x^{(i)}(t_0)$ contains only one nonzero component $x_i(t_0)$ for some fixed $i$. Consider the formation of a control from the initial condition for the $i$th state coordinate. For the considered initial condition, constraint (3.1) becomes

$$\left|u_j^{(i)}\right| \le N_{ij}|x_i(t_0)|, \quad i = \overline{1, n}, \quad j = \overline{1, m}. \tag{3.2}$$

Consider the linear time-optimal control problem. To find the minimum time $T^{(i)} = t_k^{(i)} - t_0$ required for driving system (2.1) from the initial state $x^{(i)}(t_0) = (0, \ldots, 0, x_i(t_0), 0, \ldots, 0)$ to the zero terminal state $x(t_k^{(i)}) = 0$, we use the maximum principle [1]. The Pontryagin function is written as $H(x(t), \psi(t), u(t)) = \psi^*A(t)x + \psi^*B(t)u$, where $\psi^*$ is the transposed solution vector of the adjoint system $\dot{\psi} = -A^*(t)\psi$, $\psi(t_0) = \psi_0$. The Pontryagin function is maximal if the components of the control vector under constraint (3.2) satisfy the relation

$$u_j^{(i)}(t) = N_{ij}|x_i(t_0)|\operatorname{sgn}[B_j(t)]^*\psi(t), \quad i = \overline{1, n}, \quad j = \overline{1, m}. \tag{3.3}$$

Let $x_i(t_0) = x_i^+(t_0) > 0$. Then (3.3) can be written as

$$u_j^{(i)}(t) = N_{ij}x_i^+(t_0)\operatorname{sgn}[B_j(t)]^*\psi^{(i)}(t), \quad i = \overline{1, n}, \quad j = \overline{1, m}, \tag{3.4}$$

where $\psi^{(i)}(t)$ is the solution of the adjoint system corresponding to the positive value $x_i(t_0)$. If $x_i(t_0) = -x_i^+(t_0) < 0$, then

$$u_j^{(i)}(t) = N_{ij}x_i^+(t_0)\operatorname{sgn}[B_j(t)]^*\hat{\psi}^{(i)}(t), \quad i = \overline{1, n}, \quad j = \overline{1, m}, \tag{3.5}$$

where $\hat{\psi}^{(i)}(t)$ is the solution of the adjoint system corresponding to the symmetric point $[-x_i^+(t_0)]$. Since the manifolds of switching times of the optimal control are symmetric about the origin, we have $\hat{\psi}^{(i)}(t) = -\psi^{(i)}(t)$. Now (3.5) is written as

$$u_j^{(i)}(t) = -N_{ij}x_i^+(t_0)\operatorname{sgn}[B_j(t)]^*\psi^{(i)}(t), \quad i = \overline{1, n}, \quad j = \overline{1, m}. \tag{3.6}$$

Combining (3.6) and (3.4) for an arbitrary value $x_i(t_0)$, we obtain

$$u_j^{(i)}(t) = N_{ij}x_i(t_0)\operatorname{sgn}[B_j(t)]^*\psi^{(i)}(t), \quad i = \overline{1, n}, \quad j = \overline{1, m}. \tag{3.7}$$

The solution of differential equation (2.1) at the terminal time $t = t_k^{(i)}$ is given by

$$x(t_k^{(i)}) = \Phi(t_k^{(i)}, t_0)x^{(i)}(t_0) + \sum_{j=1}^{m}\int_{t_0}^{t_k^{(i)}}\Phi(t_k^{(i)}, \tau)B_j(\tau)u_j^{(i)}d\tau, \quad i = \overline{1, n}. \tag{3.8}$$

Let $\Gamma^{(i)}(t, t_0)$ denote the $i$th column vector of the fundamental solution matrix $\Phi(t, t_0)$. Then the first term on the right-hand side of Eq. (3.8) can be written as

$$\Phi(t_k^{(i)}, t_0)x^{(i)}(t_0) = \Gamma^{(i)}(t_k^{(i)}, t_0)x_i(t_0), \quad i = \overline{1, n}. \tag{3.9}$$

The solution of the adjoint system can be expressed in terms of the fundamental solution matrix of the direct system and the initial condition as $\psi^{(i)}(t) = [\Phi^{-1}(t, t_0)]^*\psi^{(i)}(t_0)$, $i = \overline{1, n}$. Substituting (3.9), (3.7), and the terminal condition $x(t_k^{(i)}) = 0$ into (3.8), we obtain an equation that relates the switching times of control (3.7), the driving time $T^{(i)} = t_k^{(i)} - t_0$, and the initial value $x_i(t_0)$ to the parameters of the controlled system. We obtain a system of $n$ linear equations with $n$ unknowns, which are the initial conditions $\psi_\xi^{(i)}(t_0)$, $\xi = 1, n$:

$$\sum_{j=1}^{m}\left[\int_{t_0}^{t_k^{(i)}}\Phi(t_k^{(i)}, \tau)B_j(\tau)N_{ij}\operatorname{sgn}[B_j(\tau)]^*[\Phi^{-1}(\tau, t_0)]^*\psi^{(i)}(t_0)d\tau + \Gamma^{(i)}(t_k^{(i)}, t_0)\right]x_i(t_0) = 0, \quad i = \overline{1, n}.$$

Since $x_i(t_0) \neq 0\ \forall i = \overline{1, n}$, we obtain the basic equation relating the switching times of the control

$$u_j^{(i)}(t) = N_{ij}x_i(t_0)\operatorname{sgn}[B_j(t)]^*[\Phi^{-1}(t, t_0)]^*\psi^{(i)}(t_0), \quad i = \overline{1, n}, \quad j = \overline{1, m} \tag{3.10}$$

to the parameters of the controlled system

$$\sum_{j=1}^{m}\int_{t_0}^{t_k^{(i)}}\Phi(t_k^{(i)}, \tau)B_j(\tau)N_{ij}\operatorname{sgn}[B_j(\tau)]^*[\Phi^{-1}(\tau, t_0)]^*\psi^{(i)}(t_0)d\tau + \Gamma^{(i)}(t_k^{(i)}, t_0) = 0, \quad i = \overline{1, n}. \tag{3.11}$$

Solving system (3.11), we find the initial conditions $\psi_\xi^{(i)}(t_0)$, $\xi = \overline{1, n}$, for the adjoint system. They determine all the switching times of control (3.7), which are specified by the switching function $[B_j(t)]^*\psi^{(i)}(t)$. It follows directly from (3.11) that the switching times of control (3.10) are independent of the values $x_i(t_0)$ $\forall i = \overline{1, n}$.

In the case of constant matrices $A$ and $B$, from (3.11), we derive the following basic equation, which relates the switching times of control (3.7) to the matrices $A$ and $B$ and the coefficient $N_{ij}$:

$$\sum_{j=1}^{m}\int_{t_0}^{T^{(i)}}e^{A(T^{(i)}-\tau)}B_jN_{ij}\operatorname{sgn}[B_j]^*\psi^{(i)}(\tau)d\tau + \Gamma^{(i)}(T^{(i)}) = 0, \quad i = \overline{1, n}. \tag{3.12}$$

It follows directly from (3.12) that the switching times are independent of the initial time $t_0$ or the initial condition $x_i(t_0)$ ($\forall i = \overline{1, n}$) and are constants. Thus, we have proved the following result.

**Theorem 1.** *The switching times of control* (3.7) *generated from the initial condition for the ith state coordinate are independent of* $x_i(t_0)$ $\forall i = \overline{1, n}$. *In the case of constant matrices A and B, the switching times are independent of the initial time* $t_0$ *or the initial condition* $x_i(t_0)$ ($\forall i = \overline{1, n}$) *and are constants.*

Denote the points where the boundary of the reachable set $D_T$ over the time $T = t_k - t_0$ intersects the axes of the state space by $\pm(x_i(t_0))_{\max}$, $i = \overline{1, n}$. The weighting coefficients $N_{ij}$ are determined by the condition $N_{ij}|x_i(t_0)|_{\max} = M_j$, $i = \overline{1, n}$, $j = \overline{1, m}$. As a result, the driving time $T^{(i)}$ is equal to $T$; i.e., $T^{(i)} = T$ and $t_k^i = t_k$.

Each of the components of control (3.7) constrained by (3.2) represents an alternating sequence of pulses of different polarity whose values are directly proportional to the initial condition $x_i(t_0)$, while the switching times are equal to those of the time-optimal control for the maximum deviation $|x_i(t_0)|_{\max}$. Thus, the quasi-optimal control coincides with the optimal one under maximally admissible "axial" initial conditions and preserves these switching times and the driving time. Moreover, for each component, the quasi-optimal control does not exceed the maximally admissible value $M_j$, $j = \overline{1, m}$, and is proportional to the initial condition. Control (3.7) drives system (2.1) from any initial point lying on the $i$th state axis to the origin over the fixed time $T = t_k - t_0$. Since the switching function $\operatorname{sgn}[B_j]^*\psi^{(i)}(t) = \pm 1$ specifies the switching times, the quasi-optimal control can be written in the simple form $u_j^{(i)}(t)_{kv} = \pm N_{ij}x_i(t_0), j = \overline{1, m}$, $i = \overline{1, n}$.

In the general case, when the vector of initial conditions $x(t_0)$ contains all (or some of) the nonzero components, the control vector components are formed by summing the components generated from each state coordinate

$$u_j^{\mathrm{kv}}(t) = \sum_{i=1}^{n} N_{ij}x_i(t_0)\,\mathrm{sgn}[B_j(t)]^*\psi^{(i)}(t), \quad j = \overline{1, m}. \tag{3.13}$$

Since the superposition principle holds for linear systems, control (3.13) drives system (2.1) from any initial state $x(t_0)$ to the origin $x(t_k) = 0$ over the fixed time $T = t_k - t_0$.

Suppose that the switching times remain independent of the initial conditions $x_i(t_0)$, $i = \overline{1, n}$, but another method is used to generate a quasi-optimal control under which the complexity of computing the optimal control is substantially reduced. Such a quasi-optimal control is reasonable to use as a good initial approximation in the iterative procedure for finding a time-optimal control; as a result, convergence is improved and the number of iterations is reduced (see [14, 15]).

The interval $[t_0, t_k]$ is divided by arbitrarily switchings points $\nu_j^p$ ($j = \overline{1, m}$, $p = \overline{1, r_j - 1}$). For the uniformity of the notation, we set $\nu_j^0 = t_0$ and $\nu_j^{r_j} = t_k$. The switching times of the quasi-optimal control are set identical for all state coordinates. Then the weighting coefficients $N_{ij}$ take different values $N_{ij}^p$ on each $p$th interval where the control components have a definite sign; here $r_j$ is the number of such intervals for the $j$th component.

For switching times identical for all state coordinates, the quasi-optimal control is generated according to the algorithm

$$u_j^{(i)}(t) = N_{ij}^p x_i(t_0)\,\mathrm{sgn}[B_j(t)]^*\psi^{(i)}(t_0), \quad j = \overline{1, m}, \quad p = \overline{1, r_j}, \quad t \in [\nu_j^{p-1}, \nu_j^p]. \tag{3.14}$$

In the general case, when the vector of initial conditions $x(t_0)$ contains all (or some of) the nonzero components, the control components are formed by summing components (3.14) generated from the initial values of each state coordinate

$$u_j^{\mathrm{kv}}(t) = \sum_{i=1}^{n} N_{ij}^p x_i(t_0)\,\mathrm{sgn}[B_j(t)]^*\psi^{(i)}(t), \quad j = \overline{1, m}, \quad p = \overline{1, r_j}, \quad t \in [\nu_j^{p-1}, \nu_j^p]. \tag{3.15}$$

Since the superposition principle holds for linear systems, total control (3.15) drives system (2.1) from any initial state $x(t_0)$ to the origin $x(t_k) = 0$ over the fixed time $T = t_k - t_0$. Define $\hat{N}_{ij}^p = N_{ij}^p\,\mathrm{sgn}[B_j(t)]^*\psi^{(i)}(t)$. For the quasi-optimal control (3.15), we obtain the simple expression

$$u_j^{\mathrm{kv}}(t) = \sum_{i=1}^{n} \hat{N}_{ij}^p x_i(t_0), \quad j = \overline{1, m}, \quad p = \overline{1, r_j}, \quad t \in [\nu_j^{p-1}, \nu_j^p]. \tag{3.16}$$

Control (3.16) is a sequence of pulses of different polarity generated from the initial values of each state coordinate. The values of the pulses are proportional to the initial conditions taken with some weight $\hat{N}_{ij}^p$, which is different on each definite-sign interval. The weighting coefficients $N_{ij}^p$ ($i = \overline{1, n}$, $j = \overline{1, m}$, $p = \overline{1, r_j}$) are related to the parameters of system (2.1) by the following equation (similar to (3.11)):

$$\sum_{j=1}^{m} \int_{t_0}^{t_k} \Phi(t_k, \tau) B_j(\tau) N_{ij}^p\,\mathrm{sgn}[B_j(t)]^*\psi^{(i)}(t) + \Gamma(t_k, t_0) = 0, \quad i = \overline{1, n}. \tag{3.17}$$

Since $\Gamma(t_k, t_0) = \Phi(t_k, t_0)I_i$, where $I_i$ is the $i$th column vector of the identity matrix, and the fundamental matrix $\Phi(t_k, t_0)$ is nonsingular, system (3.17) can be represented as

$$\sum_{j=1}^{m}\sum_{p=1}^{r_j}\hat{N}_{ij}^{p}\int_{\nu_j^{p-1}}^{\nu_j^{p}}\Phi^{-1}(\tau, t_0)B_j(\tau)d\tau + I_i = 0, \quad i = \overline{1, n}. \tag{3.18}$$

Assume that we have determined all the switching times $\nu_j^p$ ($j = \overline{1, m}$, $p = \overline{1, r_j - 1}$), which are as many as $\sum_{j=1}^{m} r_j - m \ge n$. The number of parameters $\hat{N}_{ij}^{p}$ in (3.18) can be larger than the number of equations. Therefore, for the $i$th state coordinate on $n$ control definite-sign intervals, the coefficients $\hat{N}_{ij}^{p}$, $j = \overline{1, m}$, $p = \overline{1, q_j}$, where $\sum_{j=1}^{m} q_j = n$, are regarded as unknowns, while, on the other intervals, we set $\hat{N}_{ij}^{p+1} = -\hat{N}_{ij}^{p}$, $p = \overline{q_j, r_j}$. The change in the sign of the coefficients is explained by the change in the sign of the optimal control on each interval. As a result, we obtain $n$ systems, each consisting of $n$ linear algebraic equations with $n$ unknowns $\hat{N}_{ij}^{p}$, $p = \overline{1, q_j}$, $\sum_{j=1}^{m} q_j = n$:

$$\sum_{j=1}^{m}\left\{\sum_{p=1}^{q_j}\hat{N}_{ij}^{p}\int_{\nu_j^{p-1}}^{\nu_j^{p}}\Phi^{-1}(\tau, t_0)B_j(\tau)d\tau + \sum_{p=q_j+1}^{r_j}(-1)^{p-q_j}\hat{N}_{ij}^{q_j}\int_{\nu_j^{p-1}}^{\nu_j^{p}}\Phi^{-1}(\tau, t_0)B_j(\tau)d\tau\right\} + I_i = 0, \quad i = \overline{1, n}. \tag{3.19}$$

Figure 1 shows the qualitative pattern of quasi-optimal control formation for a third-order system with two control parameters $u_1$ and $u_2$. Each component $u_j^{\mathrm{kv}}(t)$, $j = \overline{1, 2}$, is formed by summing three components $u_j^{(i)}(t)$, $i = \overline{1, 3}$. The value $M_j^0 = \frac{1}{2}(M_j^1 + M_j^2)$, $j = \overline{1, 2}$, is the mean value.

The quasi-optimal control (3.16) has a number of important properties. First, it drives the system from any initial state $x(t_0) = x_0$ to the origin $x(t_k) = 0$ over the given fixed time $T = t_k - t_0$. Second, it follows directly from (3.19) that the weighting coefficients $\hat{N}_{ij}^{p}$ are independent of the initial conditions; consequently, they can be found preliminarily, i.e., prior to the control process. This substantially simplifies the implementation of quasi-optimal control (3.16), which is formed nearly instantaneously (several dozens of multiplication and addition operations). Third, the quasi-optimal control is kind of a piecewise constant approximation to the desired optimal control and contains information on its structure. It is this property that underlies the determination of the optimal control.

In the general case, the quasi-optimal and optimal controls have different switching times and, hence, different control amplitudes: the optimal control functions $\pm M_j^0$, $j = \overline{1, m}$, are all equal (in absolute value), while the quasi-optimal control functions are different on each definite-sign interval. The method for computing the time-optimal control is based on gradually smoothing the quasi-optimal control values so that they become equal to the corresponding limiting optimal-control functions $\pm M_j^0$, $j = \overline{1, m}$. As the control function values vary, the switching times also vary with the help of the adjoint system and the sequence of quasi-optimal controls tends to the optimal control.

Summarizing, specifying arbitrary switching times, we solve the systems of linear algebraic equations (3.19) to find the weighting coefficients $\hat{N}_{ij}^{p}$. Quasi-optimal controls are obtained using formula (3.16).

### *3.2. Approximation of the Quasi-Optimal Control to the Time-Optimal Control*

The approximation is based on a special choice of switching times. For this purpose, in each octant of the state space [sgn($\pm x_1$); sgn($\pm x_2$); ...; sgn($\pm x_n$)], we find a maximally remote boundary point $x_*^{\alpha}(T)$, i.e., the point on the boundary of the reachable set with a maximal normalized distance:

$$\sum_{i=1}^{n}\left|\frac{x_i}{x_{(i)}^{*}}\right| - 1 = \max_{x_i \in G}.$$

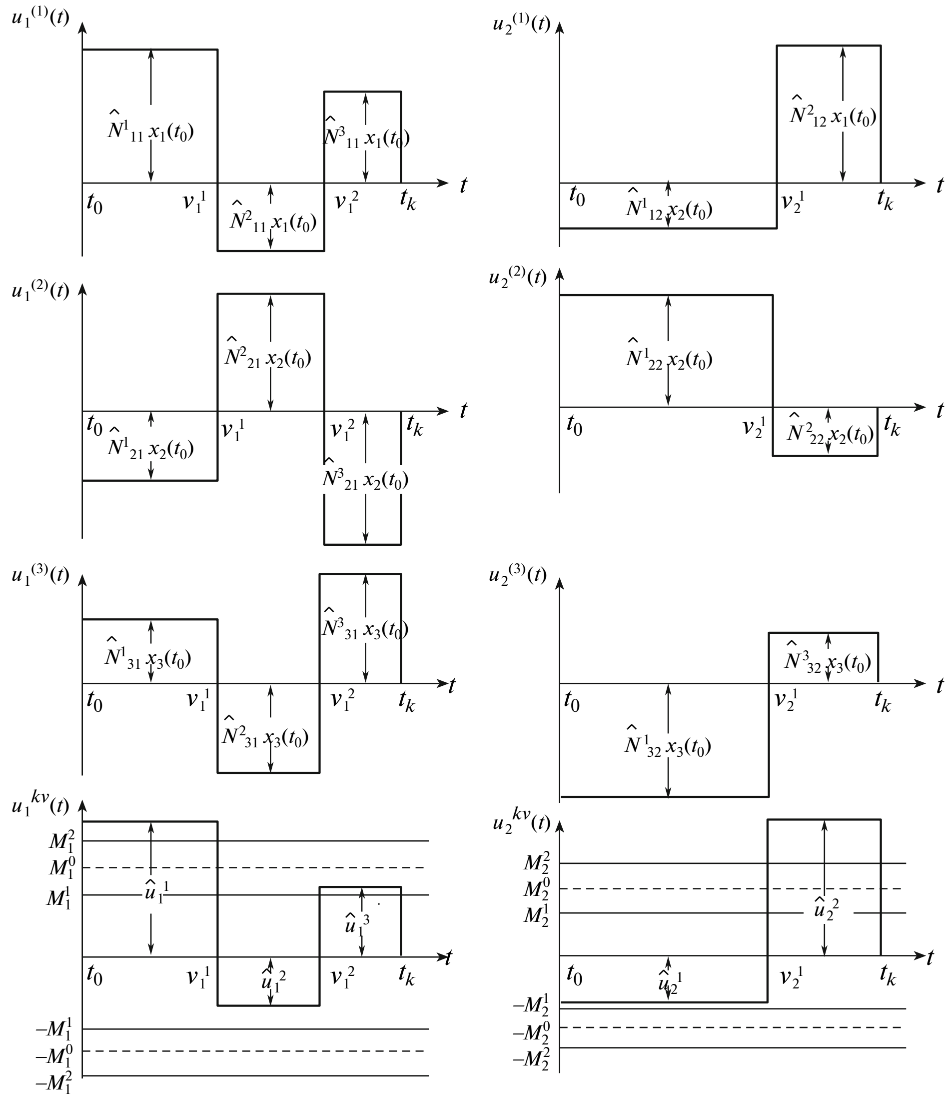

**Fig. 1.**

As a result, for a system of order $n$ in each octant $\alpha$ out of $2^n$ octants of the state space, we have $(n + 1)$ boundary points: $n$ points on the state axes and one maximally remote point $x_*^\alpha(T)$. Then we make $n$ combinations of $n$ boundary points on the state axes taken $(n - 1)$ at a time. Each combination is supplemented with the boundary point $x_*^\alpha(T)$. As a result, we obtain $n$ different combinations, each consisting of $n$ boundary points. Drawing a hyperplane through each combination of $n$ points, we obtain a collection of $n$ hyperplanes. Each hyperplane passes through $x_*^\alpha(T)$ and $(n - 1)$ different boundary points on the state axes and is described by an equation of the form

$$\sum_{i=1}^{n} c_{ik}^{\alpha} x_i - 1 = 0, \quad k = \overline{1, n}, \tag{3.20}$$

where $k$ is the combination index and $\alpha$ is the octant index. The numerical values of $n$ coefficients $c_{ik}^\alpha$, $i = \overline{1, n}$, for each $k \in [1, n]$ are found as follows. The coordinates of each boundary point with a given hyperplane passing through it are substituted into Eq. (3.20). There are $n$ such boundary points in each combination. As a result, we obtain a system of $n$ linear inhomogeneous algebraic equations with $n$ unknown

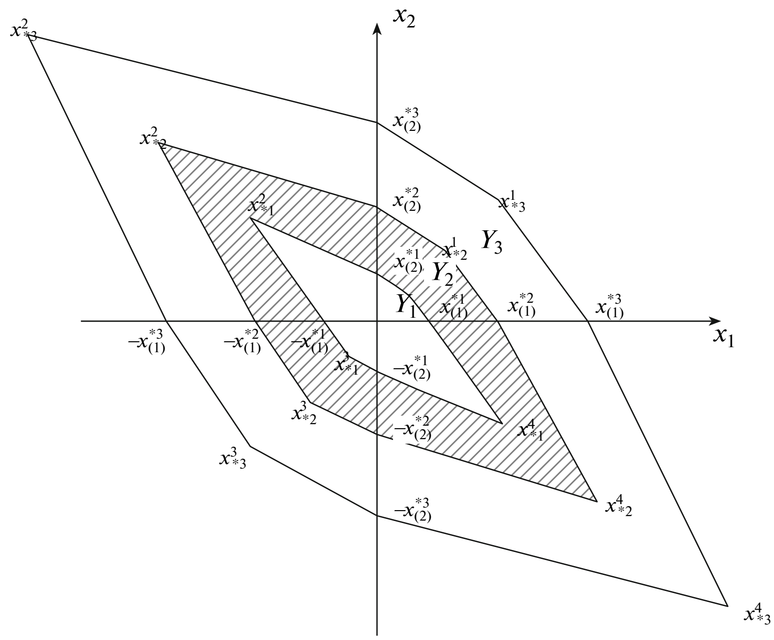


**Fig. 2.**

coefficients $c_{ik}^{\alpha}$, $i = \overline{1, n}$. Since the boundary points on the axes have all zero values, except for one, each of $n$ coefficients is found directly from one of $n$ equations. Thus, the computation of the coefficients is substantially simplified by choosing boundary points on the state axes.

Importantly, to determine whether the initial condition $x(t_0) = x_0$ belongs to the reachable set, it is necessary and sufficient to verify the membership conditions only in the octant containing this initial condition, i.e., in the octant $[\mathrm{sgn}x_1(t_0); \mathrm{sgn}x_2(t_0); \ldots; \mathrm{sgn}x_n(t_0)]$. If the initial condition is in the octant $\alpha$ of the state space, then the initial condition $x(t_0) = x_0$ belongs to the reachable set over the time $T$ if all $n$ inequalities

$$\sum_{i=1}^{n} c_{ik}^{\alpha} x_i(t_0) - 1 \le 0, \quad k = \overline{1, n} \tag{3.21}$$

are satisfied.

### *3.3. Division of the Domain of Initial Conditions into Reachable Sets*

Let us divide the entire bounded set of initial conditions $x_0 \in D$ into $q$ reachable sets over different times $T_s$, $s = \overline{1, q}$, where $T_{s-1} < T_s$. Each reachable subset over the time $T_s$ is approximated by a collection of hyperplanes. Each hyperplane is described in the octant $\alpha$ of the state space by one of the following equations with fixed $k$ and $s$:

$$\sum_{i=1}^{n} c_{ik}^{\alpha s} x_i - 1 = 0, \quad k = \overline{1, n}, \quad s = \overline{1, q}.$$

Figure 2 shows how the domain of initial conditions is divided into reachable sets over different times.

The subset $Y_s$ is determined as follows. We find the minimum value $s$ at which inequality (3.21) holds for each $k = \overline{1, n}$ and, for $(s - 1)$, there exists at least one $k \in [1, n]$ for which (3.21) does not hold:

$$Y_s = \left[ x(t_0) : \sum_{i=1}^{n} c_{ik}^{\alpha s} x_i(t_0) - 1 \le 0, k = \overline{1, n}; \exists k \in [1, n] \sum_{i=1}^{n} c_{ik}^{\alpha(s-1)} x_i(t_0) - 1 > 0 \right]. \tag{3.22}$$

If, for an arbitrary $s$, the relations

$$\sum_{i=1}^{n} c_{ik}^{\alpha s} x_i(t_0) - 1 < 0, \quad \sum_{i=1}^{n} c_{ik}^{\alpha(s-1)} x_i(t_0) - 1 \le 0, \quad k = \overline{1, n},$$

hold for each $k = \overline{1, n}$, then $s$ has to be decreased. If there is at least one $k \in [1, n]$ such that

$$\sum_{i=1}^{n} c_{ik}^{\alpha s} x_i(t_0) - 1 > 0, \quad \sum_{i=1}^{n} c_{ik}^{\alpha(s-1)} x_i(t_0) - 1 > 0, \quad \exists k \in [1, n],$$

then $s$ has to be increased. The choice of $s$ is completed when conditions (3.22) are satisfied.

It was proved in [14] that, as a supporting hyperplane, we have to use one for which the normalized distance $d_k = \sum_{i=1}^{n} c_{ik} x_i(t_0) - 1$, $k = \overline{1, n}$, from the initial condition $x(t_0) = x_0$ to the supporting hyperplane $\Gamma_r$, $r \in [1, n]$, of the reachable set over the time $T_{s-1}$ is nonnegative ($d \ge 0$) and maximal: $d_r = \max_k d_k$.

### 3.4. Choice of Switching Times and Quasi-Optimal Control Time

Assume that the initial condition $x(t_0) = x_0$ of system (2.1) is on the boundary of the reachable set over the time $T$. Through the point $x(t_0) = x_0$, we draw a supporting hyperplane $\Gamma_1$ of the reachable set. The normal vector to the hyperplane at the point $x(t_0) = x_0$ directed inward into the reachable set is the vector $\psi(t_0)$ (see [1]). It should be emphasized that, since the Pontryagin function is homogeneous with respect to the adjoint system, only the direction of the vector $\psi(t_0)$ is important for the time-optimal control problem, while its magnitude is of no matter. Since the reachable set is approximated by hyperplanes, the normal vector to the corresponding supporting hyperplane is the vector $\psi(t_0)$. The components of the unit normal vector $\hat{\psi}(t_0)$ are determined in terms of direction cosines. However, the time-optimal control is critical (sensitive) to the direction of $\hat{\psi}(t_0)$. Therefore, to improve the accuracy of the initial approximation, it is reasonable to compute approximate switching times and then to use them to find $\hat{\psi}(t_0)$. For the given initial condition $x(t_0) = x_0$, we compute a supporting hyperplane. It passes through $n$ boundary points, of which $(n - 1)$ are on the state axes and one is maximally remote. For each boundary point, we know an optimal control; i.e., all the switching times are known. The averaged $p$th switching time of the $j$th optimal control component for any initial condition belonging to the supporting hyperplane $(\Gamma_1)$ in the octant $\alpha$ is computed as

$$\nu_{j(\text{оп})}^{p} = \frac{1}{n}\left[\sum_{\xi=1}^{n-1} \nu_j^p(x_\xi^*(T_{s-1})) + \nu_j^p(x_*^\alpha(T_{s-1}))\right], \quad p = \overline{1, r_j}. \tag{3.23}$$

Here, $\nu_j^p(x_\xi^*(T_{s-1}))$ is the $p$th switching time of the $j$th optimal control component for the $\xi$th boundary point belonging to the supporting hyperplane and lying on a state axis, while $\nu_j^p(x_*^\alpha(T_{s-1})$ is the $p$th switching time of the $j$th optimal control component for the maximally remote boundary point belonging to the same supporting hyperplane in the same octant $\alpha$. The averaged $p$th switching time for initial conditions belonging to a “parallel” hyperplane $\Gamma_2$ is computed as

$$\nu_{j(\text{pr})}^{p} = \frac{1}{n}\left[\sum_{\xi=1}^{n-1} \nu_j^p(x_\xi^*(T_s)) + \nu_j^p(x_*^\alpha(T_s))\right], \quad p = \overline{1, r_j}. \tag{3.24}$$

By a parallel hyperplane, we mean one passing through $(n - 1)$ boundary points lying on the same state axes as those for the supporting hyperplane. The point $x(t_0) = x_0$ and the origin are on the same side of the parallel hyperplane $\Gamma_2$ and on different sides of the supporting hyperplane $\Gamma_1$. The normalized distance from the point $x(t_0) = x_0$ to $\Gamma_1$ is given by

$$d_1 = \sum_{i=1}^{n} c_{ik}^{\alpha(s-1)} x_i(t_0) - 1, \quad d_1 \geq 0.$$

The normalized distance from the point $x(t_0) = x_0$ to $\Gamma_2$ is given by

$$d_2 = \sum_{i=1}^{n} c_{ik}^{\alpha s} x_i(t_0) - 1, \quad d_2 < 0.$$

The point $x(t_0) = x_0$ lies between $\Gamma_2$ and $\Gamma_1$, which are separated by the distance $\Delta T = T_s - T_{s-1}$. The time required for driving system (2.1) from the given initial state $x(t_0) = x_0$ to the origin is proportional to the distance and is approximately given by

$$T \approx T_{s-1} + \frac{d_1}{d_1 + |d_2|}(T_s - T_{s-1}). \tag{3.25}$$

Similarly, the switching times of the quasi-optimal control are approximately calculated as

$$\nu_j^p \approx \nu_{j(\text{оп})}^p + \frac{d_1}{d_1 + |d_2|}(\nu_{j(\text{pr})}^p - \nu_{j(\text{оп})}^p), \quad p = \overline{1, r_j}. \tag{3.26}$$

Substituting these switching times into (3.19), we compute the weighting coefficients $\hat{N}_{ij}^{p,s}$, which are used to form the quasi-optimal control (3.16). As a result, the proximity of the quasi-optimal control to the time-optimal one is estimated as

$$0 \leq |T_{\text{kv}} - T_{\text{opt}}| \leq \Delta T, \quad \Delta T = T_s - T_{s-1}, \quad s = \overline{1, q}.$$

It should be emphasized that $d_1$ and $d_2$ are calculated by verifying simple relations simultaneously with determining $Y_s$. Thus, the determination of switching times and the quasi-optimal control time does not require complicated computations.

### *3.5. Computation of Normalized Initial Condition of the Adjoint System*

For linear system (2.1), in the case of control components satisfying the severe constraints $|u_j| \leq M_j^0$, $j = \overline{1, m}$, the time-optimal control is given by the expression

$$u_j^0(t) = M_j^0 \operatorname{sgn}[B_j(t)]^* \psi(t), \quad j = \overline{1, m}.$$

The switching times $\nu_j^p$ ($j = \overline{1, m}$, $p = \overline{1, r_j}$) of the optimal control components and their number $r_j$ on the interval $[t_0, t_k]$ are uniquely determined by the switching functions $[B_j(t)]^*\psi(t)$, $j = \overline{1, m}$, if we know the solution $\psi(t)$; i.e., the initial conditions $\psi_i(t_0)$, $i = \overline{1, n}$, of the adjoint system are known. At the switching times, the switching function is equal to zero; i.e.,

$$[B_j(\nu_j^p)]^* \hat{\Phi}(\nu_j^p, t_0) \psi(t_0) = 0, \quad j = \overline{1, m}, \quad p = \overline{1, r_j}.$$

Define $\hat{\psi}(t_0) = \psi(t_0)/\psi_\beta(t_0)$, where $\beta \in [1, n]$. Here, $\psi_\beta(t_0)$ is a nonzero initial value of the $\beta$-coordinate at the time $t_0$. Moreover, $\beta$ is any value from the set $[1, n]$ for which $\psi_\beta(t_0) \neq 0$. As a result, we obtain the system

$$[B_j(\nu_j^p)]^* \hat{\Phi}(\nu_j^p, t_0) \hat{\psi}(t_0) = 0, \quad j = \overline{1, m}, \quad p = \overline{1, r_j}, \tag{3.27}$$

which relates the switching times $\nu_j^p$ ($j = \overline{1, m}$, $p = \overline{1, r_j}$) to the initial condition $\hat{\psi}(t_0)$ of the normalized adjoint system.

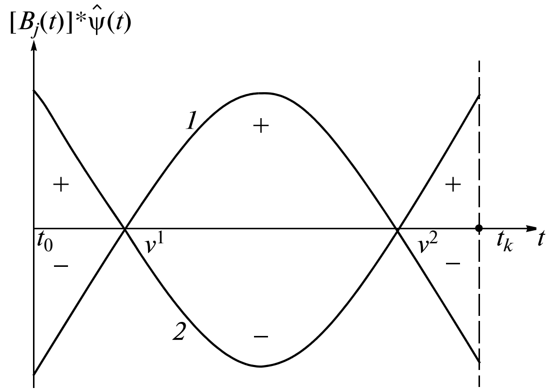


**Fig. 3.**

Next, $(n-1)$ approximate switching times $\nu_j^i$, $i = \overline{1, (n-1)}$ calculated by formula (3.23) are substituted into (3.27) to obtain a system of $(n-1)$ linear inhomogeneous algebraic equations with $(n-1)$ unknowns, which are the components of the normalized initial condition $\hat{\psi}(t_0)$ of the adjoint system (taking into account that $\hat{\psi}_\beta(t_0) = 1$).

Importantly, this method for normalizing the initial condition vector of the adjoint system eliminates a fundamental difficulty arising in the approximation of the reachable set by hyperplanes, namely, the task of constructing a supporting hyperplane through remote boundary points on different state axes for which the optimal control is opposite in sign. In the method proposed, two symmetric normalized switching functions satisfying two different symmetric sequences of control signs can be drawn through the same $(n-1)$ switching times. Figure 3 shows two symmetric normalized switching functions that pass through the switching times $\nu^1$ and $\nu^2$ and satisfy different (symmetric) sequences of optimal-control signs on the interval $t \in [t_0, t_k]$.

Corresponding to the given initial condition $x(t_0) = x_0$, the sequence of signs of the desired optimal control is specified with the help of the quasi-optimal control. In view of the simplicity of implementing the quasi-optimal control (3.16), the signs of the components of the desired optimal control corresponding to the given initial condition $x(t_0) = x_0$ can be determined if, instead of the arbitrary values of $\nu_j^p$ and $T = t_k - t_0$, we use their approximate values given by formulas (3.25) and (3.26).

According to the method proposed for finding the optimal control, the quasi-optimal controls are smoothed to become equal to the limiting values $\pm M_j^0$, $j = \overline{1, m}$, which are used to form the time-optimal control for linear system (2.1). The smoothing of the control functions leads to changes in the switching times of the quasi-optimal control, which tend to the switching times of the optimal control.

### *3.6. Iterative Method for Computing the Optimal Control with Interval Constraints*

The deviation of the quasi-optimal control from the corresponding limiting value $M_j^0 S_j(p) = M_j^0 \operatorname{sgn}[B_j(t)]^*\psi(t)$, $t \in [\nu_j^{p-1}, \nu_j^p]$ in optimal control with severe constraints on the $j$th component in the $p$th interval is given by

$$\Delta u_j^p = M_j^0 S_j(p) - \sum_{i=1}^{n} \hat{N}_{ij}^p x_i(t_0), \quad j = \overline{1, m}, \quad p = \overline{1, r_j}, \quad t \in [\nu_j^{p-1}, \nu_j^p].$$

Consider only the $\rho$th part of this deviation. The deviation $\rho\Delta u_j^p$, $0 < \rho \le 1$, generates the following deviations of the state coordinates at the terminal time $t = t_k$:

$$\Delta\hat{x}(t_k) = \sum_{j=1}^{m}\sum_{p=1}^{r_j} \int_{\nu_j^{p-1}}^{\nu_j^p} \Phi(t_k, \tau) B_j(\tau)\rho\left[M_j^0 S_j(p) - \sum_{i=1}^{n} \hat{N}_{ij}^p x_i(t_0)\right] d\tau. \tag{3.28}$$

Variations in the switching times on $\Delta\nu_j^p$ and in the terminal time on $\Delta t_k$ for a piecewise constant control $u(t)$ whose components are switched at the times $\nu_j^p$ and take the values $u_j(t) = \hat{u}_j^p$, $t \in [\nu_j^{p-1}, \nu_j^p]$, give rise to the following deviations of the state coordinates at the terminal time $t = t_k$:

$$\Delta\tilde{x}(t_k) = \sum_{j=1}^{m}\sum_{p=1}^{r_j-1}\int_{\nu_j^{p-1}}^{\nu_j^p+\Delta\nu_j^p}\Phi(t_k,\tau)B_j(\tau)[\hat{u}_j^p - \hat{u}_j^{p+1}]d\tau + \sum_{j=1}^{m}\int_{t_k}^{t_k+\Delta t_k}\Phi(t_k,\tau)B_j(\tau)\hat{u}_j^{r_j}d\tau.$$

If $\Delta\nu_j^p$ and $\Delta t_k$ are small (which can be achieved by choosing a suitable parameter $\rho$), we can write the approximate relation

$$\Delta\tilde{x}(t_k) \approx \sum_{j=1}^{m}\sum_{p=1}^{r_j-1}\Phi(t_k,\nu_j^p)B_j(\nu_j^p)[u_j^p - u_j^{p+1}]\Delta\nu_j^p + \sum_{j=1}^{m}B_j(t_k)u_j^{r_j}\Delta t_k. \tag{3.29}$$

The deviation $\Delta\hat{x}(t_k)$ caused by variations in the control functions has to be balanced by the deviation $\Delta\tilde{x}(t_k)$ caused by variations in the switching times and the terminal time:

$$\Delta\hat{x}(t_k) + \Delta\tilde{x}(t_k) = 0. \tag{3.30}$$

Substituting (3.28) and (3.29) into (3.30) yields a system of $n$ linear algebraic equations

$$\begin{gathered}\sum_{j=1}^{m}\sum_{p=1}^{r_j-1}\Phi(t_k,\nu_j^p)B_j(\nu_j^p)[\hat{u}_j^p - \hat{u}_j^{p+1}]\Delta\nu_j^p + \sum_{j=1}^{m}B_j(t_k)\hat{u}_j^{r_j}\Delta t_k \\ + \sum_{j=1}^{m}\sum_{p=1}^{r_j}\int_{\nu_j^{p-1}}^{\nu_j^p}\Phi(t_k,\tau)B_j(\tau)\rho\left[M_j^0S_j(p) - \sum_{i=1}^{n}\hat{N}_{ij}^p x_i(t_0)\right]d\tau = 0.\end{gathered} \tag{3.31}$$

To find an optimal (rather than merely admissible) control, the adjoint system is used to determine the number and locations of switching times. For this purpose, we need to find the relation between the deviations $\Delta\nu_j^p$ and $\Delta\hat{\psi}(t_0)$. According to [16], this relation is given

$$\Delta\nu_j^p \approx \{\{[B_j(\nu_j^p)]^*A^*(\nu_j^p) - [\dot{B}_j(\nu_j^p)]^*\}[\Phi^{-1}(\nu_j^p,t_0)]^*\hat{\psi}(t_0)\}^{-1}[B_j(\nu_j^p)]^*[\Phi^{-1}(\nu_j^p,t_0)]^*\Delta\hat{\psi}(t_0),$$
$$j = \overline{1,m}, \quad p = \overline{1,r_j-1}.$$

More compactly, this expression is written as

$$\Delta\nu_j^p \approx \mathcal{L}(\nu_j^p)\Delta\hat{\psi}(t_0), \quad j = \overline{1,m}, \quad p = \overline{1,r_j-1}. \tag{3.32}$$

Substituting (3.32) into (3.31) gives a system of $n$ linear algebraic equations with $n$ unknowns, which are $(n-1)$ deviations $\Delta\hat{\psi}(t_0)$ and the deviation $\Delta t_k$:

$$\begin{gathered}\sum_{j=1}^{m}\sum_{p=1}^{r_j-1}\Phi(t_k,\nu_j^p)B_j(\nu_j^p)[\hat{u}_j^p - \hat{u}_j^{p+1}]\mathcal{L}(\nu_j^p)\Delta\hat{\psi}(t_0) + \sum_{j=1}^{m}B_j(t_k)\hat{u}_j^{r_j}\Delta t_k \\ + \sum_{j=1}^{m}\sum_{p=1}^{r_j}\int_{\nu_j^{p-1}}^{\nu_j^p}\Phi(t_k,\tau)B_j(\tau)\rho\left[M_j^0S_j(p) - \sum_{i=1}^{n}\hat{N}_{ij}^p x_i(t_0)\right]d\tau = 0.\end{gathered} \tag{3.33}$$

Solving system (3.33), we find $\Delta\hat{\psi}(t_0)$ and $\Delta t_k$, i.e., the refined values of the switching times, terminal time, and the normalized initial condition of the adjoint system for the next iteration $(s+1)$: $\nu_j^{p,s+1} = \nu_j^{p,s} + \Delta\nu_j^{p,s}$; $t_k^{s+1} = t_k^s + \Delta t_k^s$; and $\hat{\psi}^{s+1}(t_0) = \hat{\psi}^s(t_0) + \Delta\hat{\psi}^s(t_0)$. The computations terminate if interval constraints (2.2) on the control components are satisfied, i.e., $M_j^1 \le |\hat{u}_j^i| \le M_j^2$, $i = \overline{1,r_j}$, $j = \overline{1,m}$.

The convergence of this computational process is proved in the Appendix.

*3.7. Nonuniqueness of Optimal Control*

Assume that $M_j^1 = M_j^2 = M_j^0$. Then interval constraints (2.2) degenerate into "severe" constraints $|u_j| = M_j^0$, $j = \overline{1, m}$. In this case, the time-optimal control of system (2.1) is given by $u_j^0(t) = M_j^0 \operatorname{sgn}[Bj(t)]^*\psi(t), j = \overline{1, m}$; i.e., the control functions take the limiting values $\pm M_j^0$, $j = \overline{1, m}$, and the switching times are determined with the help of the adjoint system $\psi(t)$ by means of the switching functions $[B_j(t)]^*\psi(t), j = \overline{1, m}$. The switching times of the quasi-optimal control are also determined by the switching functions, and the control functions at the end of the iterative process tend to the limiting values $\pm M_j^0$, $j = \overline{1, m}$. For a linear system, a time-optimal control exists if $x(t_0) \in V$, i.e., the initial condition belongs to the controllability domain. This condition holds (see (2.1)).

For linear nonstationary systems, an optimal control is unique if (i) the origin of the control domain is its interior point and (ii) the control is "regular," i.e., a linear combination of rows of the matrix $\Phi(t_k, t)B(t)$ is nontrivial at all points of the time interval $t \in [t_0, t_k]$, except for a finite set of points.

For the considered constraints on the components of control vector (2.2), the origin of the control domain is its interior point; i.e., condition (i) holds.

Regularity condition (ii) for linear stationary systems can be replaced by the general position condition (see [1]). In the case of rectangular box constraints on the components of control vector (2.2), as considered in this paper, the general position condition is adequate for componentwise complete controllability; i.e., the complete controllability condition (2.3) must be satisfied for each component of the control vector:

$$\operatorname{rank}\left[\int_{t_0}^{t_k} \Phi(t_k, \tau)B_j(\tau)B_j^*(\tau)\Phi^*(t_k, \tau)d\tau\right] = n, \quad j = \overline{1, m}. \tag{3.34}$$

Thus, the general position condition is more restrictive than the complete controllability condition (2.3). For a scalar control, the general position condition coincides with the complete controllability condition. Therefore, for a scalar control, since the time-optimal control for linear stationary systems is unique, the quasi-optimal control with limiting control function values $\pm M_j^0$, $j = \overline{1, m}$, that drives the system to the origin is found with the help of the adjoint system, satisfies the Pontryagin maximum principle (which is a necessary optimality condition), and is a time-optimal control. For a vector control, the optimal control can be nonunique, since only the complete controllability condition (2.3) is assumed to hold in this work. In this case, the sequence of quasi-optimal controls converges to one of the optimal controls.

If $M_j^1 \neq M_j^2$, $j = \overline{1, m}$, then any quasi-optimal control satisfying interval constraints (2.2) is an optimal control.

## 4. SIMULATION AND NUMERICAL RESULTS

Consider the system of linear differential equations

$$\begin{aligned}
\dot{x}_1 &= x_2, & x_1(t_0) &= x_{10},\\
\dot{x}_2 &= x_3, & x_2(t_0) &= x_{20},\\
\dot{x}_3 &= x_4, & x_3(t_0) &= x_{30},\\
\dot{x}_4 &= a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 + bu, & x_4(t_0) &= x_{40}, \quad M^1 \le |u| \le M^2.
\end{aligned}$$

For $a_{41} = -2.9684$, $a_{42} = -5.84$, $a_{43} = -6.33$, and $a_{44} = -3.4$, the matrix $A$ of this system has the complex conjugate eigenvalues $\lambda_{1,2} = -0.784 \pm j0.986$ and $\lambda_{3,4} = -0.916 \pm j1.016$. Let $b = 4$, $M^0 = 5$, $M^1 = 4.8$, $M^2 = 5.2$, and $T_{s+1} - T_s = 0.5$, $s = \overline{1, 10}$. Let $x(t_0) = (1, 0.8, -1.2, 2)$. The initial condition belongs to the octant $[+ + - +]$ of the state space. Checking conditions (3.22), we conclude that the given initial condition belongs to the reachable set over the time $T = 2.5$, but does not belong to the reachable set over the time $T = 2$. The averaged switching times (3.23) for initial conditions belonging to the supporting hyperplane are $\nu^1_{(\mathrm{supp})} = 0.5128$, $\nu^2_{(\mathrm{supp})} = 1.2524$, and $\nu^3_{(\mathrm{supp})} = 1.8$. The averaged switching times (3.24) for initial conditions belonging to the parallel hyperplane $\Gamma_2$ are $\nu^1_{(\mathrm{pr})} = 0.645$, $\nu^2_{(\mathrm{pr})} = 1.599$, and $\nu^3_{(\mathrm{pr})} = 2.2615$.

**Table 1**

| $h$ | $\nu^1$ | $\nu^2$ | $\nu^3$ | $t_k$ | $\hat{u}^1$ | $\hat{u}^2$ | $\hat{u}^3$ | $\hat{u}^4$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.621 | 1.536 | 2.177 | 2.409 | –4.986 | 3.815 | –3.590 | 3.706 |
| 1 | 0.674289 | 1.505826 | 2.100508 | 2.322964 | –4.956 | 4.911 | –4.868 | 4.889 |

**Table 2**

| $h$ | $\nu^1$ | $\nu^2$ | $\nu^3$ | $t_k$ | $\hat{u}^1$ | $\hat{u}^2$ | $\hat{u}^3$ | $\hat{u}^4$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 1.4505 | 2.789 | 3.556 | 3.867 | –5.307 | 4.540 | –4.539 | 4.459 |
| 1 | 1.526362 | 2.789784 | 3.548774 | 3.850736 | –5.018 | 5.022 | –5.002 | 4.995 |

Formulas (3.25) and (3.26) are used to compute approximate values of the driving time and switching times for the optimal control driving the system from the initial state $x(t_0) = (1, 0.8, -1.2, 2)$ to the origin: $\nu^1 = 0.621$, $\nu^2 = 1.536$, $\nu^3 = 2.177$, and $T = 2.409$. Table 1 presents the switching times $\nu^i$ ($i = \overline{1, 3}$), the terminal time $t_k$, and the quasi-optimal controls $\hat{u}^i$ ($i = \overline{1, 4}$) calculated at every iteration step $h$ on each definite-sign interval. A single iteration step is sufficient to satisfy the given interval constraint imposed on the control. The system is driven to the origin with $10^{-7}$ accuracy.

Consider another initial condition: $x(t_0) = (2.5, 3, 4, 7)$. This point lies in the octant $[+ + + +]$ of the state space. Checking inequalities (3.22), we find that the initial condition belongs to the reachable set over the time $T = 4$, but does not belong to the reachable set over the time $T = 3.5$. The averaged values of switching times (3.23) for initial conditions belonging to the supporting hyperplane are $\nu^1_{(\mathrm{supp})} = 1.313925$, $\nu^2_{(\mathrm{supp})} = 2.488759$, and $\nu^3_{(\mathrm{supp})} = 3.215372$. The averaged values of switching times (3.24) for initial conditions belonging to the parallel hyperplane $\Gamma_2$ are $\nu^1_{(\mathrm{pr})} = 1.499737$, $\nu^2_{(\mathrm{pr})} = 2.897506$, and $\nu^3_{(\mathrm{pr})} = 3.679024$. Formulas (3.25) and (3.26) are used to compute approximate values of the driving time and switching times for the optimal control driving the system from the initial state $x(t_0) = (2.5, 3, 4, 7)$ to the zero terminal state: $\nu^1 = 1.4505$, $\nu^2 = 2.789$, $\nu^3 = 3.556$, and $T = 3.867$. Table 2 presents the switching times $\nu^i$ ($i = \overline{1, 3}$), the terminal time $t_k$, and the quasi-optimal controls $\hat{u}^i$ ($i = \overline{1, 4}$) calculated at every iteration step $h$ on each definite-sign interval. A single iteration step is sufficient to satisfy the given interval constraint imposed on the control. The system is driven to the origin with $10^{-7}$ accuracy.

## CONCLUSIONS

The quasi-optimal control found drives a linear system to the origin from any initial state and does not require complicated computations. As a result, this quasi-optimal control can be directly used to satisfy interval constraints in the real-time control of fast-acting objects and rapidly proceeding technological processes. By dividing the domain of initial conditions into reachable sets over different times, the quasi-optimal control is made much closer to the time-optimal control and can be used as an initial approximation in the iterative computation of an optimal control. In the case of interval constraints imposed on the control components, one or several iteration steps are sufficient to compute an optimal control.

## *APPENDIX*

Below, we prove the convergence of the iterative computational process.

**Theorem.** *The sequence of quasi-optimal controls* (3.16), *i.e.*, $u_j^{\mathrm{kv}, h}(t) = \sum_{i=1}^{n} \hat{N}_{ij}^{p, h} x_i(t_0)$, $j = \overline{1, m}$, $p = \overline{1, r_j}$, $t \in [\nu_j^{p-1, h}, \nu_j^{p, h}]$, $h = 1, 2, 3, \ldots$, *converges at the iteration step $N$* (*where $N$ is not fixed*) *to the quasi-*

*time-optimal control* $u_j^{\mathrm{kv},N}(t) = \sum_{i=1}^{n} \hat{N}_{ij}^{p,N} x_i(t_0), j = \overline{1,m}, p = \overline{1,r_j}, t \in [\nu_j^{p-1,N}, \nu_j^{p,N}]$, *that drives system* (2.1) *to the origin* $x(t_k) = 0$ *and satisfies constraints* (2.2) $M_j^1 \le |u_j^{\mathrm{kv},N}| \le M_j^2, j = \overline{1,m}$.

**Proof.** For the iterative computational process to converge, the deviations at every iteration step must be small. The smaller the deviations, the more accurate the used approximate relations (3.29) and (3.32), which tend in the limit to the exact relations. On the $p$th definite-sign interval, the deviation of the quasi-optimal control from the optimal one for the $j$th component is

$$\Delta u_j^p = M_j^0 S_j(p) - \sum_{i=1}^{n} \hat{N}_{ij}^p x_i(t_0), \quad j = \overline{1,m}, \quad p = \overline{1,r_j}. \tag{A.1}$$

Consider the $\rho$th part of deviation (A.1):

$$\rho(\Delta u_j^p) = \rho\left[M_j^0 S_j(p) - \sum_{i=1}^{n} \hat{N}_{ij}^p x_i(t_0)\right], \quad 0 < \rho \le 1. \tag{A.2}$$

In this case, in smoothing the control functions, the quasi-optimal control on each definite-sign interval tends not to the limiting value $M_j^0 S_j(p)$, but rather to some intermediate value

$$\hat{u}_j^p = \sum_{i=1}^{n} \hat{N}_{ij}^p x_i(t_0) + \rho\left[M_j^0 S_j(p) - \sum_{i=1}^{n} \hat{N}_{ij}^p x_i(t_0)\right].$$

The deviation of the state coordinates caused by deviation $\rho(\Delta u_j^p)$ (A.2) is given by the formula

$$\Delta\hat{x}(t_k) = \sum_{j=1}^{m}\sum_{p=1}^{r_j}\int_{\nu_j^{p-1}}^{\nu_j^p} \Phi(t_k,\tau)B_j(\tau)\rho\left[M_j^0 S_j(p) - \sum_{i=1}^{n} \hat{N}_{ij}^p x_i(t_0)\right]d\tau. \tag{A.3}$$

If the quasi-optimal control functions deviate substantially from $M_j^0 S_j(p)$, an arbitrarily small deviation $\Delta\hat{x}(t_k)$ at every iteration step can be ensured by choosing a suitable parameter $\rho$. The deviation balance equation (3.30) implies that the deviation $\Delta\tilde{x}(t_k)$ will be small and, hence, the deviations $\Delta\hat{\psi}(t_0)$ and $\Delta t_k$ generating $\Delta\tilde{x}(t_k)$ will be arbitrarily small. Let us prove that the smallness of $\Delta\hat{\psi}(t_0)$ and $\Delta t_k$ at every iteration step guarantees that the sequence of quasi-optimal controls converges to a quasi-optimal control with limiting values $\pm M_j^0, j = \overline{1,m}$.

The deviation $\Delta\tilde{x}(t_k)$ of the state coordinates are given by

$$\Delta\tilde{x}(t_k) = \sum_{j=1}^{m}\sum_{p=1}^{r_j-1}\int_{\nu_j^p}^{\nu_j^p+\Delta\nu_j^p} \Phi(t_k,\tau)B_j(\tau)[\hat{u}_j^p - \hat{u}_j^{p+1}]d\tau + \sum_{j=1}^{m}\int_{t_k}^{t_k+\Delta t_k} \Phi(t_k,\tau)B_j(\tau)\hat{u}_j^{r_j}d\tau.$$

Since the integrand is continuous on each interval $[\nu_j^p, \nu_j^p + \Delta\nu_j^p]$, the mean value theorem implies that

$$\Delta\tilde{x}(t_k) = \sum_{j=1}^{m}\sum_{p=1}^{r_j-1}\Phi(t_k,\nu_j^p+\mu_j^p\Delta\nu_j^p)B_j(\nu_j^p+\mu_j^p\Delta\nu_j^p)[\hat{u}_j^p - \hat{u}_j^{p+1}]\Delta\nu_j^p$$
$$+\sum_{j=1}^{m}\Phi(t_k,t_k+\mu_j\Delta t_k)B_j(t_k+\mu_j\Delta t_k)\hat{u}_j^{r_j}\Delta t_k, \quad 0 < \mu_j^p,\ \mu_j < 1.$$

Applying Lagrange's mean value theorem yields

$$\begin{aligned}\Delta\tilde{x}(t_k) &= \sum_{j=1}^{m}\sum_{p=1}^{r_j-1}\Phi(t_k,\nu_j^p)B_j(\nu_j^p)[\hat{u}_j^p-\hat{u}_j^{p+1}]\Delta\nu_j^p+\sum_{j=1}^{m}B_j(t_k)\hat{u}_j^{r_j}\Delta t_k\\ &+\sum_{j=1}^{m}\sum_{p=1}^{r_j-1}\mu_j^p[\hat{u}_j^p-\hat{u}_j^{p+1}]\frac{d\{\Phi(t_k,\nu_j^p+\theta_j^p\mu_j^p\Delta\nu_j^p)B_j(\nu_j^p+\theta_j^p\mu_j^p\Delta\nu_j^p)\}}{dt}(\Delta\nu_j^p)^2\\ &+\sum_{j=1}^{m}\mu_j\hat{u}_j^{r_j}\frac{d\{\Phi(t_k,t_k+\theta_j\mu_j\Delta t_k)B_j(t_k+\theta_j\mu_j\Delta t_k)\}}{dt}(\Delta t_k)^2,\quad 0<\theta_j^p,\ \theta_j<1,\quad 0<\mu_j^p,\ \mu_j<1.\end{aligned}\tag{A.4}$$

Define

$$\Delta\nu_j^p=\Delta\tilde{\nu}_j^p+\sigma\nu_j^p,\quad \Delta t_k=\Delta\tilde{t}_k+\sigma t_k,\tag{A.5}$$

where $\Delta\nu_j^p$ and $\Delta t_k$ are the exact (true) deviations, $\Delta\tilde{\nu}_j^p$ and $\Delta\tilde{t}_k$ are the computed deviations, and $\sigma\nu_j^p$ and $\sigma t_k$ are the errors of the computed deviations. The values $\Delta\tilde{\nu}_j^p$ and $\Delta\tilde{t}_k$ are determined by solving the system of linear algebraic equations (3.29), which is exact for the computed values:

$$\sum_{j=1}^{m}\sum_{p=1}^{r_j-1}\Phi(t_k,\nu_j^p)B_j(\nu_j^p)[\hat{u}_j^p-\hat{u}_j^{p+1}]\Delta\tilde{\nu}_j^p+\sum_{j=1}^{m}B_j(t_k)\hat{u}_j^{r_j}\Delta\tilde{t}_k = \Delta\tilde{x}(t_k).\tag{A.6}$$

Substituting (A.5) into (A.4) and taking into account (A.6), we obtain the equation

$$\begin{aligned}&\sum_{j=1}^{m}\sum_{p=1}^{r_j-1}\Phi(t_k,\nu_j^p)B_j(\nu_j^p)[\hat{u}_j^p-\hat{u}_j^{p+1}]\sigma\nu_j^p+\sum_{j=1}^{m}B_j(t_k)\hat{u}_j^{r_j}\sigma t_k\\ &+\sum_{j=1}^{m}\sum_{p=1}^{r_j-1}\mu_j^p[\hat{u}_j^p-\hat{u}_j^{p+1}]\frac{dF(t_k,\Delta\nu_j^p)}{dt}(\Delta\nu_j^p)^2+\sum_{j=1}^{m}\mu_j\hat{u}_j^{r_j}\frac{dF(t_k,\Delta t_k)}{dt}(\Delta t_k)^2 = 0,\end{aligned}\tag{A.7}$$

where, for notational brevity, we used the notation

$$F(t_k,\Delta\nu_j^p) = \Phi(t_k,\nu_j^p+\theta_j^p\mu_j^p\Delta\nu_j^p)B_j(\nu_j^p+\theta_j^p\mu_j^p\Delta\nu_j^p),$$

$$F(t_k,\Delta t_k) = \Phi(t_k,t_k+\theta_j\mu_j\Delta t_k)B_j(t_k+\theta_j\mu_j\Delta t_k),\quad 0<\theta_j^p,\theta_j<1,\quad 0<\mu_j^p,\ \mu_j<1.$$

The following assertion is important for the proof of the convergence of the computational process: the errors $\sigma\nu_j^p$ and $\sigma t_k$ found at the $h$th iteration step are the exact (true) deviations at the $(h+1)$th iteration step; i.e.,

$$\sigma\nu_j^{p,h}=\Delta\nu_j^{p,h+1},\quad \sigma t_k^h = \Delta t_k^{h+1}.\tag{A.8}$$

Substituting (A.8) into (A.7), we obtain an expression relating the deviations at the $h$th and $(h+1)$th iterations:

$$\begin{aligned}&\sum_{j=1}^{m}\sum_{p=1}^{r_j-1}\Phi(t_k^h,\nu_j^{p,h})B_j(\nu_j^{p,h})[\hat{u}_j^{p,h}-\hat{u}_j^{p+1,h}]\Delta\nu_j^{p,h+1}+\sum_{j=1}^{m}B_j(t_k^h)\hat{u}_j^{r_j,h}\Delta t_k^{h+1}\\ &= -\sum_{j=1}^{m}\sum_{p=1}^{r_j-1}\mu_j^{p,h}[\hat{u}_j^{p,h}-\hat{u}_j^{p+1,h}]\frac{dF(t_k^h,\Delta\nu_j^{p,h})}{dt}(\Delta\nu_j^{p,h})^2-\sum_{j=1}^{m}\mu_j^h\hat{u}_j^{r_j,h}\frac{dF(t_k^h,\Delta t_k^h)}{dt}(\Delta t_k^h)^2.\end{aligned}\tag{A.9}$$

Substituting (3.32) into (A.9) yields a system of $n$ equations with $n$ unknowns, which are $(n-1)$ deviations $\Delta\hat{\psi}_i^{h+1}(t_0)$ of the initial conditions in the normalized adjoint system and the deviation $\Delta t_k^{h+1}$ of the terminal time $t = t_k^h$:

$$\sum_{j=1}^{m}\sum_{p=1}^{r_j-1}\Phi(t_k^h, \nu_j^{p,h})B_j(\nu_j^{p,h})[\hat{u}_j^{p,h} - \hat{u}_j^{p+1,h}]\mathcal{L}(\nu_j^{p,h})\Delta\hat{\psi}^{h+1}(t_0) + \sum_{j=1}^{m}B_j(t_k^h)\hat{u}_j^{r_j,h}\Delta t_k^{h+1}$$
$$= -\sum_{j=1}^{m}\sum_{p=1}^{r_j-1}\mu_j^{p,h}[\hat{u}_j^{p,h} - \hat{u}_j^{p+1,h}]\frac{dF(t_k^h, \Delta\hat{\psi}^h)}{dt}(\mathcal{L}(\nu_j^{p,h})\Delta\hat{\psi}^h(t_0))^2 - \sum_{j=1}^{m}\mu_j^h\hat{u}_j^{r_j,h}\frac{dF(t_k^h, \Delta t_k^h)}{dt}(\Delta t_k^h)^2. \quad (A.10)$$

System (A.10) relates the deviations at the $h$th and $(h+1)$th iteration steps. To write it in a unified and compact form, we introduce the notation $\Delta z = (\Delta\hat{\psi}(t_0), \Delta t_k)$. Then system (A.10) can be compactly represented as

$$\Delta z^{h+1} = D(t_k^h, \nu_j^{p,h})\Delta\tilde{z}^h\Delta z^h, \quad (A.11)$$

where $\Delta z^h$ is an $n$-dimensional column vector with components $\Delta\hat{\psi}^h(t_0)$ and $\Delta t_k^h$, $\Delta\tilde{z}^h$ is a diagonal $n \times n$ matrix with above-indicated diagonal elements, and $D(t_k^h, \nu_j^{p,h})$ is an $n \times n$ matrix.

The controls are improved in the course of the computations [17] if the following sufficient condition is satisfied:

$$\|\Delta z^{h+1}\| < \|\Delta z^h\|, \quad h = 1, 2, 3\dots. \quad (A.12)$$

It follows directly from (A.11) that, by virtue of the quadratic dependence, there are values $\Delta z_*^h$ such that the sufficient condition for the improvement of the controls in the iterative process is satisfied:

$$\|\Delta z^{h+1}\| < \|\Delta z_*^h\|, \quad h = 1, 2, 3\dots. \quad (A.13)$$

The deviations $\Delta z^h$ formed at every iteration step are chosen at our discretion. Indeed, specifying $\rho$ in Eq. (3.33), we determine which part of the complete mismatch $\Delta z^h$ at $\rho = 1$ will be compensated at the $h$th iteration step.

If $\rho \longrightarrow 0$, it follows directly from (A.3) that $\Delta\hat{x}(t_k^h) \longrightarrow 0$. Then balance equation (3.30) implies that $\Delta\tilde{x}(t_k^h) \longrightarrow 0$; therefore, the generating deviation tends to zero: $\Delta z^h \longrightarrow 0$. Thus, choosing $\rho^h$ at every iteration step, we choose the value $\Delta z^h$. As a result, the sequence of quasi-optimal controls tends to the quasi-optimal control with limiting values $u_j = \pm M_j^0$, $j = \overline{1, m}$. At some iteration $h = N$, where $N$ is not fixed, with the switching times $\nu_j^{p,N}$ and the terminal time $t_k^N$ found, the prescribed accuracy of smoothing the quasi-optimal control values is achieved, i.e., constraints (2.2) are satisfied. The theorem is proved.

**Corollary 1.** It follows directly from (A.11) that the local convergence rate is quadratic.

**Corollary 2.** The radius of local convergence is given by $R_{\text{conv}} = \|\Delta z_*^h\|$.

**Corollary 3.** Since $\Delta z^{h+1} = \sigma z^h$, a sufficient condition adequate for (A.12) under which the computational process converges is

$$\|\sigma z^h\| < \|\Delta z^h\|, \quad h = 1, 2, 3, \dots; \quad (A.14)$$

i.e., the norm of the computational error must be less than the norm of the true (exact) deviation.

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## REFERENCES

1. L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes* (Nauka, Moscow, 1976; Gordon and Breach, New York, 1986).
2. A. A. Agrachev and R. V. Gamkrelidze, "The geometry of maximum principle," Proc. Steklov Inst. Math. **273**, 1–22 (2011).
3. R. Gabasov and F. M. Kirillova, "Real-time optimal control and observation," Izv. Ross. Akad. Nauk, Teor. Sist. Upr., No. 3, 90–111 (2006).
4. A. A. Milyutin, A. V. Dmitruk, and N. P. Osmolovskii, *Maximum Principle in Optimal Control* (Mosk. Gos. Univ., Moscow, 2004) [in Russian].
5. A. V. Arguchintsev, V. A. Dykhta, and V. A. Srochko, "Optimal control: Nonlocal conditions, computational methods, and the variational principle of maximum," Russ. Math. **53** (1), 1–35 (2009).
6. A. A. Milyutin, *Maximum Principle in the General Optimal Control Problem* (Fizmatlit, Moscow, 2001) [in Russian].
7. S. A. Kalmykov, Yu. I. Shokin, and Z. Kh. Yuldashev, *Interval Analysis Methods* (Nauka, Novosibirsk, 1986) [in Russian].
8. B. T. Polyak and S. A. Nazin, "Estimation of parameters in linear multidimensional systems under interval uncertainty," J. Autom. Inf. Sci. **38** (2), 19–33 (2006).
9. L. T. Ashchepkov, "Linear interval equations with symmetric solution sets," Comput. Math. Math. Phys. **48** (4), 531–538 (2008).
10. S. P. Shary, "Solvability of interval linear equations and data analysis under uncertainty," Autom. Remote Control **73** (2), 310–322 (2012).
11. Yu. M. Gusev, V. N. Efanov, V. G. Krymskii, and V. Yu. Rutkovskii, "Analysis and synthesis of linear interval dynamical systems (state of the problem)," I: Izv. Ross. Akad. Nauk Tekh. Kibern., No. 1, 3–23 (1991); II: No. 2, 3–30 (1991).
12. V. M. Aleksandrov, "Solution of optimal control problems on the basis of the quasi-optimal control method," in *Models and Optimization Methods: Proceedings of the Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences* (Nauka, Novosibirsk, 1988), Vol. 10, pp. 18–54 [in Russian].
13. V. M. Aleksandrov, "Approximate solution of optimal control problems," in *Problems in Cybernetics* (Nauka, Moscow, 1984), Vol. 41pp. 143–206 [in Russian].
14. V. M. Aleksandrov, "Real-time computation of optimal control," Comput. Math. Math. Phys. **51** (10), 1351–1372 (2012).
15. V. M. Aleksandrov, "Specification of an initial approximation and a method for computing optimal control," Sib. Elektron. Mat. Izv. **11**, 87–118 (2014); http://semr.math.nsc.ru/v11/p87-118.pdf.
16. V. M. Aleksandrov, "A numerical method for solving a linear time-optimal control problem," Comput. Math. Math. Phys. **38** (6), 881–893 (1998).
17. V. A. Srochko, *Iterative Methods for Solving Optimal Control Problems* (Fizmatlit, Moscow, 2000) [in Russian].

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