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Application of Integrodifferential Splines to Solving an Interpolation Problem

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Abstract—This paper deals with cases when the values of derivatives of a function are given at grid nodes or the values of integrals of a function over grid intervals are known. Polynomial and trigonometric integrodifferential splines for computing the value of a function from given values of its nodal derivatives and/or from its integrals over grid intervals are constructed. Error estimates are obtained, and numerical results are presented.

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In some real-world problems, the task is to construct an approximation to a function from its values at grid nodes and/or the values of its integrals over grid intervals. Polynomial integrodifferential splines were proposed in [1, 2] for solving such problems.

In this paper, we assume that the values of derivatives of a function at grid nodes and the values of its integrals over grid intervals are known. Under these assumptions, the original function is approximated by discontinuous polynomial and trigonometric splines, which are then used to construct a continuously differentiable approximation to the original function.

Nonpolynomial integrodifferential splines were addressed in [3].

Following the definitions given in [2], spline fragments defined on a grid interval are referred to as links.

1. Let us construct discontinuous approximations. Consider a function $U \in C^3[a, b]$. On the interval $[a, b]$, we introduce a nonuniform grid of distinct nodes arranged in increasing order with the variable step $h_k = x_{k+1} - x_k$:

$$\Omega : a = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = b.$$

Assume that the values of the derivative $U'(x_k)$ at the nodes x_k of Ω are given and the values of the integrals $\int_{x_k}^{x_{k+1}} U(t)dt$ $(k = 0, 1, \ldots, n-1)$ are known. The task is to construct a function $\tilde{U}(x)$, $x \in [a, b)$, composed of links of the form

$$\tilde{U}_k(x) = U'(x_k)\omega_{k,1}(x) + U'(x_{k+1})\omega_{k+1,1}(x) + \left(\int_{x_k}^{x_{k+1}} U(t)dt\right)\omega_k^{\langle 1\rangle}(x), \quad x \in [x_k, x_{k+1}), \tag{1}$$

that satisfies the differential and integral compatibility conditions

$$\tilde{U}'_k(x_k) = U'(x_k), \quad \tilde{U}'_k(x_{k+1}) = U'(x_{k+1}), \quad \int_{x_k}^{x_{k+1}} \tilde{U}_k(t)dt = \int_{x_k}^{x_{k+1}} U(x)dx, \quad k = 0, 1, \ldots, n-1,$$

and to estimate the approximation error.

Basis functions $\omega_{k,1}(x)$, $\omega_{k+1,1}(x)$, $\omega_k^{\langle 1\rangle}(x)$, $x \in [x_k, x_{k+1})$, are found using the following conditions:

Case 1. $U(x) - \tilde{U}_k(x) = 0$ if $U(x) = 1, x, x^2$.

Case 2. $U(x) - \tilde{U}_k(x) = 0$ if $U(x) = 1, \sin(x), \cos(x)$.

Let us write the system of equations for Case 2. On the interval $[x_k, x_{k+1})$ the basis splines $\tilde{\omega}_j(x)$, $j = k$, $k+1$, $\tilde{\omega}_k^{\langle 1\rangle}(x)$, are found from the approximation relations

$$
\begin{gathered}
h_k\tilde{\omega}_k^{\langle 1\rangle}(x) = 1,\\
\cos(x_k)\tilde{\omega}_{k,1}(x) + \cos(x_{k+1})\tilde{\omega}_{k+1,1}(x) + \left(\int_{x_k}^{x_{k+1}} \sin(t)dt\right)\tilde{\omega}_k^{\langle 1\rangle}(x) = \sin(x),\\
-\sin(x_k)\tilde{\omega}_{k,1}(x) - \sin(x_{k+1})\tilde{\omega}_{k+1,1}(x) + \left(\int_{x_k}^{x_{k+1}} \cos(t)dt\right)\tilde{\omega}_k^{\langle 1\rangle}(x) = \cos(x).
\end{gathered} \tag{2}
$$

In Case 1, solving the system of equations and making the substitution $x = x_k + th_k$, $t \in [0, 1)$, $x \in [x_k, x_{k+1})$, after some simple algebra, we obtain the basis functions

$$\omega_{k,1}(x_k + th_k) = -\frac{h_k}{2}\left(t^2 - 2t + \frac{2}{3}\right), \quad \omega_{k+1,1}(x_k + th_k) = \frac{h_k}{2}\left(t^2 - \frac{1}{3}\right), \tag{3}$$

$$\omega_k^{\langle 1\rangle}(x_k + th_k) = \frac{1}{h_k}, \tag{4}$$

while, in the trigonometric case, we derive the relations

$$\tilde{\omega}_{k,1}(x_k + th_k) = \frac{1}{\Delta}(\sin(h_k) - h_k\cos(h_k - th_k)), \quad \Delta = -h_k\sin(h_k), \tag{5}$$

$$\tilde{\omega}_{k+1,1}(x_k + th_k) = \frac{1}{\Delta}(-\sin(h_k) + h_k\cos(th_k)), \quad \tilde{\omega}_k^{\langle 1\rangle}(x_k + th_k) = \frac{1}{h_k}. \tag{6}$$

Note that, in the case of splines (5) and (6), the step size h_k must be chosen so that $0 < h_k < \pi$.

Performing similar calculations on the neighboring interval $[x_{k-1}, x_k)$ and combining the results yields the following formulas for the basis splines. On a uniform grid of nodes with $h = h_k$, we have the polynomial and trigonometric "original" splines

$$\omega_1(t) = \begin{cases} -\frac{h}{2}\left(t^2 - 2t + \frac{2}{3}\right), & t \in [0, 1), \\ \frac{1}{2}ht^2 + \frac{1}{3}h + th, & t \in [-1, 0), \\ 0, & t \notin [-1, 1) \end{cases}$$

and

$$\tilde{\omega}_1(t) = \begin{cases} \frac{(-\sin(h) + h\cos(th - h))}{h\sin(h)}, & t \in [0, 1), \\ \frac{(\sin(h) - h\cos(th + h))}{h\sin(h)}, & t \in [-1, 0), \\ 0, & t \notin [-1, 1), \end{cases}$$

respectively (the term "original spline" was introduced in [4]), $\omega_{k,1}(x) = \omega_1\left(\frac{x - x_k}{h}\right)$, and $\tilde{\omega}_{k,1}(x) = \tilde{\omega}_1\left(\frac{x - x_k}{h}\right)$.

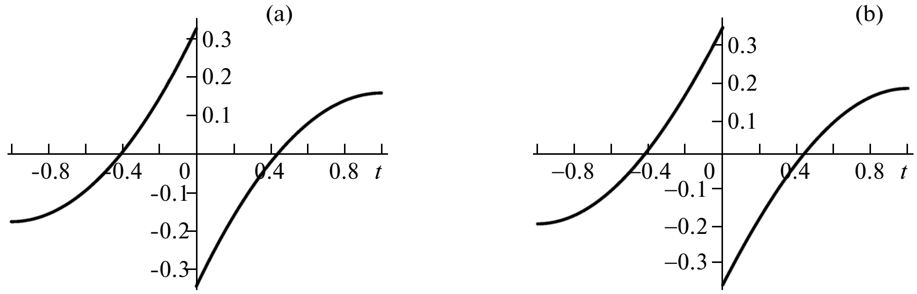

Fig. 1. Plots of the (a) polynomial $\omega_1(t)$ and (b) trigonometric $\tilde{\omega}_1(t)$ splines.

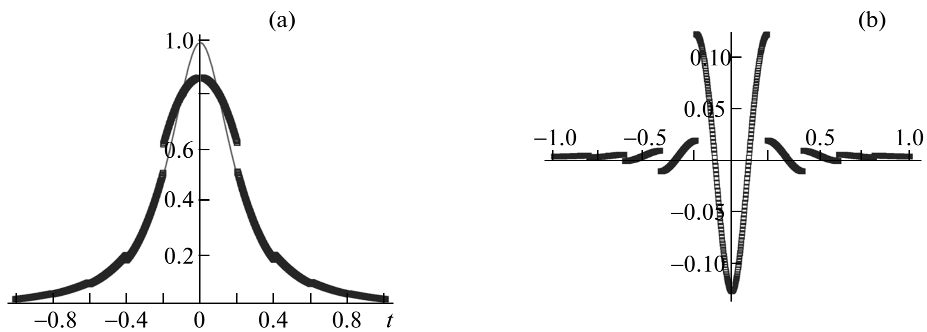

Fig. 2. (a) Runge function and its approximation by polynomial splines with $n = 10$ and (b) the approximation error.

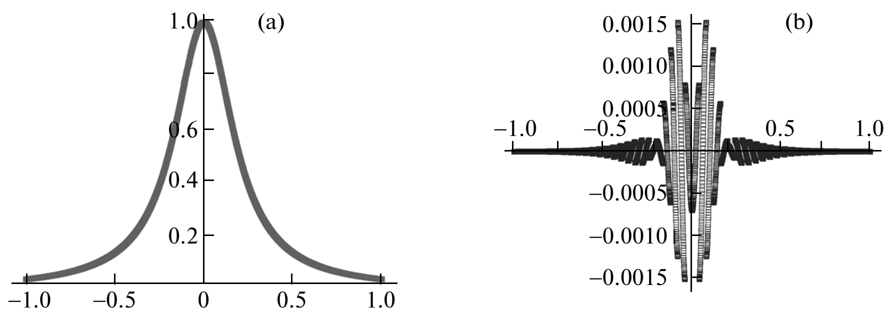

Fig. 3. (a) Runge function and its approximation by polynomial splines with $n = 50$ and (b) the approximation error.

It is easy to see that the polynomial and trigonometric basis functions are discontinuous, although their first derivatives are continuous. The trigonometric and polynomial basis splines are related by the formula $\tilde{\omega}_1(t) = \omega_1(t) + O(h^3)$.

Figure 1 presents the graphs of $\omega_1(t)$ and $\tilde{\omega}_1(t)$ for $h = 1$. It is well known that, on a uniform grid of nodes constructed on the interval $[-1, 1]$, a sequence of interpolating polynomials of increasing degrees does not converge to the Runge function, but convergence occurs in the case of spline approximation on refined grids (see [5, 7]). The Runge function $1/(1 + 25x^2)$ and its approximation by polynomial integrodifferential splines on the interval $[-1, 1)$ for $n = 10$ and $n = 50$ are plotted in Figs. 2a and 3a, respectively. The respective approximation errors are shown in Figs. 2b and 3b.

Let

$$\|f\|_{[x_k, x_{k+1})} = \sup_{x \in [x_k, x_{k+1})} |f(x)|.$$

Table 1

№	$U(x)$	$\tilde{R}^p$	R^p	$\tilde{R}^t$	R^t
1	$\sin(3x)\cos(5x)$	0.52×10^{-1}	0.95×10^{-1}	0.52×10^{-1}	0.83×10^{-1}
2	$\tan(x)$	0.11×10^{-1}	0.21×10^{-1}	0.12×10^{-1}	0.20×10^{-1}
3	$\cos(2x)$	0.26×10^{-2}	0.30×10^{-2}	0.19×10^{-2}	0.20×10^{-2}
4	$1/(1 + 25\cos(x))$	0.34×10^{-3}	0.64×10^{-3}	0.36×10^{-3}	0.61×10^{-3}
5	$\dfrac{\cos(5x)}{(1+25\cos(x))}$	0.11×10^{-2}	0.18×10^{-2}	0.98×10^{-3}	0.15×10^{-2}
6	$1/(1 + 25x^2)$	0.45×10^{-3}	0.22	0.49×10^{-3}	0.20
7	$\sin(x)$	0.21×10^{-3}	0.38×10^{-3}	0.15×10^{-13}	0
8	x^2	0.20×10^{-14}	0	0.61×10^{-3}	0.67×10^{-3}
9	$\dfrac{\tan(x/2)}{1+\tan^2(x/2)}$	0.10×10^{-3}	0.19×10^{-3}	0.8×10^{-14}	0
10	$\sin(x) + \cos(x)$	0.47×10^{-3}	0.53×10^{-3}	0.49×10^{-13}	0
11	$\sin(x/2) + \cos(x/2)$	0.10×10^{-3}	0.19×10^{-3}	0.8×10^{-14}	0

The following is easy to prove.

Theorem 1. *Let $U \in C^3[x_k, x_{k+1}]$. For $x \in [x_k, x_{k+1})$, consider the expressions*

$$\tilde{U}_k(x) = U'(x_k)\hat{\omega}_{k,1}(x) + U'(x_{k+1})\hat{\omega}_{k+1,1}(x) + \left(\int_{x_k}^{x_{k+1}} U(t)dt\right)\hat{\omega}_k^{\langle 1\rangle}(x),$$

where $\hat{\omega}_{k,1}(x)$, $\hat{\omega}_{k+1,1}(x)$, and $\hat{\omega}_k^{\langle 1\rangle}(x)$ are discontinuous trigonometric or polynomial basis splines. Then the approximation error satisfies the estimate

$$\|\tilde{U}_k(x) - U(x)\|_{[x_k, x_{k+1})} \le h_k^3 K \|LU\|_{C_{[x_k, x_{k+1})}}, \quad K > 0,$$

where LU has the following the form:

(i) *$LU = U'''$ in the case of polynomial basis splines;*

(ii) *$LU = U''' + U'$ in the case of trigonometric basis splines.*

Proof. The functions $U(x)$ and $U'(x_{k+1})$ are represented as Taylor series expansions about the point $x = x_k$. Since $|\omega_{k,1}(x)| \le h_k/3$ and $|\omega_{k+1,1}(x)| \le h_k/3$, in the polynomial case, we have $K \le 3/8$.

In the trigonometric case, the error is found by applying the method proposed in [3]. For $x \in [x_k, x_{k+1})$ the function $U(x)$ is represented as

$$U(x) = 2\int_{x_k}^{x}(U'(t) + U'''(t))\sin^2\frac{x-t}{2}dt + c_1 + c_2\sin(x) + c_3\cos(x),$$

where c_i ($i = 1, 2, 3$) are arbitrary constants.

Taking into account approximation relations (2) and the inequality $\sin(h) < h$ for $0 < h < \pi/2$, in the trigonometric case, we obtain $K \le 1/3$.

Tables 1 and 2 give the actual and theoretical error estimates for the function $U(x)$, $x \in [-1, 1)$, approximated by the polynomial and trigonometric splines with $n = 10$ and $n = 100$, respectively. Here, $\tilde{R}^p = \sup_{[-1,1)}|\tilde{U}^p(x) - U(x)|$ and $\tilde{R}^t = \sup_{[-1,1)}|\tilde{U}^t(x) - U(x)|$, where $\tilde{U}^p(x)$ and $\tilde{U}^t(x)$ are the approximations obtained with the help of polynomial and trigonometric splines, respectively, with Digits = 15 in Maple, while R^p and R^t are the theoretical errors in the case of polynomial and trigonometric spline approximation.

Table 2

№	$U(x)$	$\tilde{R}^p$	R^p	$\tilde{R}^t$	R^t
1	$\sin(3x)\cos(5x)$	0.84×10^{-4}	0.95×10^{-4}	0.82×10^{-4}	0.83×10^{-4}
2	$\tan(x)$	0.18×10^{-4}	0.21×10^{-4}	0.188×10^{-4}	0.200×10^{-4}
3	$\cos(2x)$	0.27×10^{-5}	0.30×10^{-5}	0.199994×10^{-5}	0.2000000×10^{-5}
4	$1/(1 + 25\cos(x))$	0.54×10^{-6}	0.64×10^{-6}	0.57×10^{-6}	0.61×10^{-6}
5	$\dfrac{\cos(5x)}{(1+25\cos(x))}$	0.16×10^{-5}	0.18×10^{-5}	0.14×10^{-5}	0.15×10^{-5}
6	$1/(1 + 25x^2)$	0.68×10^{-6}	0.22×10^{-3}	0.72×10^{-6}	0.20×10^{-3}

The results in the tables suggest that the theoretical and actual error estimates are in agreement and, in some cases, the trigonometric splines are preferable to the polynomial ones.

2. Consider the construction of a continuously differentiable polynomial and trigonometric function $\tilde{\tilde{U}}(x)$, $x \in [a, b)$, for $h_k = h$. Let C_k be real numbers.

Assume that the values of U' are known at the points x_0 and x_n. Let $F^{\langle k\rangle} = \int_{x_k}^{x_{k+1}} U(t)dt$.

2.1. Consider expressions with polynomial basis functions:

$$\tilde{\tilde{U}}_k(x) = C_k\omega_{k,1}(x) + C_{k+1}\omega_{k+1,1}(x) + F^{\langle k\rangle}\omega_k^{\langle 1\rangle}, \quad x \in [x_k, x_{k+1}),$$
$$\tilde{\tilde{U}}_{k-1}(x) = C_{k-1}\omega_{k-1,1}(x) + C_k\omega_{k,1}(x) + F^{\langle k-1\rangle}\omega_{k-1}^{\langle 1\rangle}, \quad x \in [x_{k-1}, x_k).$$

We have $\tilde{\tilde{U}}'_{k-1}(x_k) = \tilde{\tilde{U}}'_k(x_k)$.

We find coefficients C_k such that

$$\tilde{\tilde{U}}_{k-1}(x_k-) = \tilde{\tilde{U}}_k(x_k+), \quad k = 1, 2, \dots, n-1.$$

In the case of polynomial basis functions, this yields the system of equations

$$-\frac{h}{6}C_{k-1} - \frac{2h}{3}C_k - \frac{h}{6}C_{k+1} = f_k, \quad k = 1, \dots, n-1, \tag{7}$$

where

$$C_0 = U'(x_0), \quad C_n = U'(x_n),$$
$$f_k = (F^{\langle k-1\rangle} - F^{\langle k\rangle})/h, \quad k = 2, \dots n-2,$$
$$f_1 = (F^{\langle 0\rangle} - F^{\langle 1\rangle})/h + hU'(x_0)/6,$$
$$f_{n-1} = (F^{\langle n-2\rangle} - F^{\langle n-1\rangle})/h + hU'(x_n)/6.$$

Since the matrix of the system of equations is diagonally dominant, the system has a unique solution, which can be found, for example, by applying the tridiagonal matrix algorithm (see [8]).

Next, an approximation is constructed using the formulas

$$\tilde{\tilde{U}}_k(x) = C_k\omega_{k,1}(x) + C_{k+1}\omega_{k+1,1}(x) + F^{\langle k\rangle}\omega_k^{\langle 1\rangle}, \quad x \in [x_k, x_{k+1}), \quad k = 0, 1, \dots, n-1.$$

We introduce a function $\tilde{\tilde{U}}(x)$ alternatively defined as $\tilde{\tilde{U}}(x) = \tilde{\tilde{U}}_k(x)$, $x \in [x_k, x_{k+1})$, $k = 0, 1, \dots, n-1$. By construction, $\tilde{\tilde{U}}$ is a continuously differentiable piecewise polynomial function on $[a, b)$.

2.2. In the case of trigonometric basis functions, the system of equations becomes

$$A_kC_{k-1} + B_kC_k + A_kC_{k+1} = f_k, \quad k = 1, \dots, n-1, \tag{8}$$

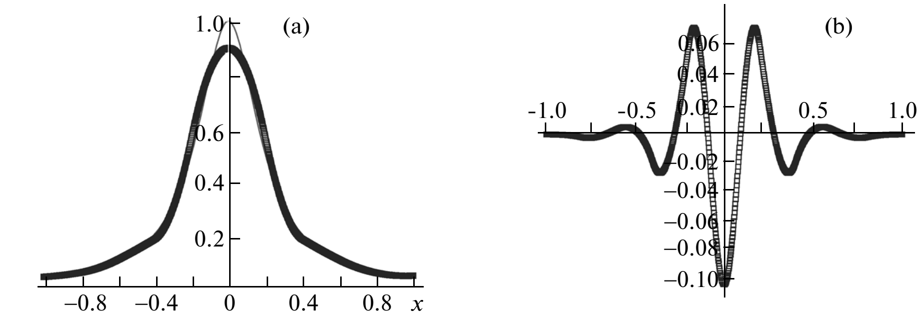

Fig. 4. (a) Runge function and its continuously differentiable polynomial approximation with $n = 10$ and (b) the approximation error on $[-1, 1)$.

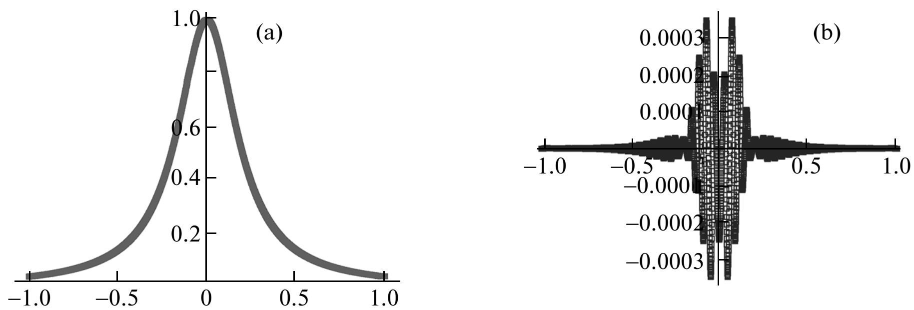

Fig. 5. (a) Runge function and its continuously differentiable polynomial approximation with $n = 50$ and (b) the approximation error on $[-1, 1)$.

where

$$B_k = 2\frac{-\sin(h) + h\cos(h)}{h\sin(h)}, \quad A_k = \frac{\sin(h) - h}{h\sin(h)},$$

$$f_k = (F^{\langle k-1\rangle} - F^{\langle k\rangle})/h, \quad k = 2, \ldots, n-2,$$

$$f_1 = \frac{F^{\langle 0\rangle} - F^{\langle 1\rangle}}{h} - U'(x_0)\frac{\sin(h) - h}{h\sin(h)},$$

$$f_{n-1} = \frac{F^{\langle n-2\rangle} - F^{\langle n-1\rangle}}{h} - U'(x_n)\frac{\sin(h) - h}{h\sin(h)}.$$

Note that

$$B_k = -\frac{2h}{3} + O(h^3), \quad A_k = -\frac{h}{6} + O(h^3).$$

Example 1. Suppose that a uniform grid of nodes with $n = 10$ has been constructed on $[-1, 1]$. We want to approximate the Runge function. Solving the system of equations for C_k yields an approximation to this function:

$$\tilde{\tilde{U}}_k(x) = C_k\omega_{k,1}(x) + C_{k+1}\omega_{k+1,1}(x) + F^{\langle k\rangle}\omega_k^{\langle 1\rangle}, \quad x \in [x_k, x_{k+1}), \quad k = 0, 1, \ldots, n-1,$$

where $\omega_k^{\langle 1\rangle}$, $\omega_{k,1}(x)$, and $\omega_{k+1,1}(x)$ are polynomial basis functions.

Figures 4 and 5 show the Runge function $U(x) = 1/(1 + 25x^2)$, its continuously differentiable polynomial approximation $\tilde{\tilde{U}}(x)$, and the error $\tilde{\tilde{U}}(x) - U(x)$ for $n = 10$ and 50, respectively (the computations with polynomial splines were performed in Maple with Digits = 10).

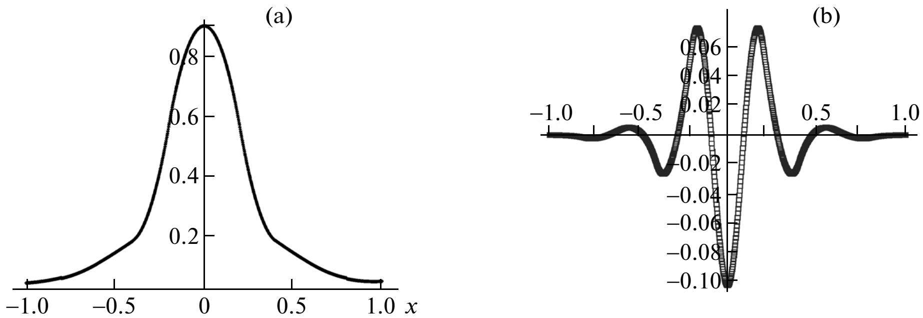

Fig. 6. (a) Continuously differentiable approximation of the Runge function by trigonometric splines with $n = 10$ and (b) the approximation error.

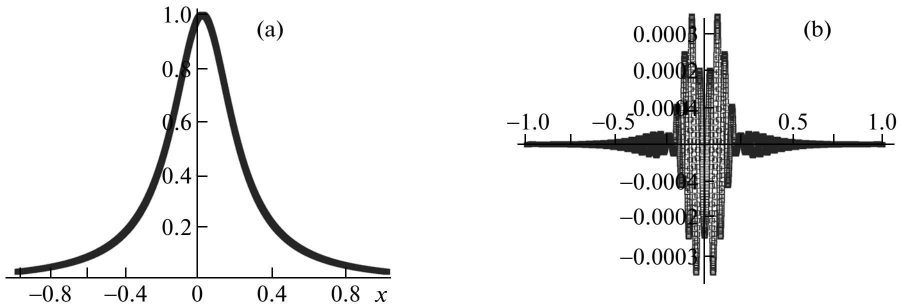

Fig. 7. (a) Continuously differentiable approximation of the Runge function by trigonometric splines with $n = 50$ and (b) the approximation error.

In the case of trigonometric basis splines, after computing C_k, a continuously differentiable spline was constructed using the formulas

$$\tilde{\tilde{U}}_k(x) = C_k\tilde{\omega}_{k,1}(x) + C_{k+1}\tilde{\omega}_{k+1,1}(x) + F^{\langle k\rangle}\tilde{\omega}_k^{\langle 1\rangle}(x), \quad x \in [x_k, x_{k+1}), \quad k = 0, 1, \dots, n-1,$$

where $\tilde{\omega}_{k,1}(x)$, $\tilde{\omega}_{k+1,1}(x)$, and $\tilde{\omega}_k^{\langle 1\rangle}(x)$ are trigonometric basis splines (the computations were performed in Maple with Digits = 20).

Figures 6 and 7 present the approximations to the Runge function by continuously differentiable trigonometric splines and the approximation error for $n = 10$ and $n = 50$, respectively.

3. Let us estimate the errors of the continuously differentiable polynomial splines. Let M denote the $(n-1) \times (n-1)$ matrix of system (7), and let $\Delta_{n-1} = \det(M)$.

System (7) is brought to the form

$$C_{k-1} + 4C_k + C_{k+1} = \tilde{f}_k, \quad k = 1, \dots, n-1, \tag{9}$$

where

$$C_0 = U'(x_0), \quad C_n = U'(x_n), \quad \tilde{f}_k = 6(F^{\langle k\rangle} - F^{\langle k-1\rangle})/h^2, \quad k = 2, \dots, n-2,$$
$$\tilde{f}_1 = 6(F^{\langle 1\rangle} - F^{\langle 0\rangle})/h^2 - U'(x_0'), \quad \tilde{f}_{n-1} = 6(F^{\langle n-1\rangle} - F^{\langle n-2\rangle})/h^2 - U'(x_n).$$

After changing to the unknowns $S_i = C_i - U'(x_i)$, system (9) is written as

$$\begin{gathered} S_0 = 0, \\ S_{i-1} + 4S_i + S_{i+1} = Q_i, \\ Q_i = 6(F^{\langle i\rangle} - F^{\langle i-1\rangle})/h^2 - U'(x_{i-1}) - 4U'(x_i) - U'(x_{i+1}), \quad i = 1, \dots, n-1, \\ S_n = 0. \end{gathered} \tag{10}$$

Lemma 1. *The coefficients C_i determined by* (9) *satisfy the relation* $|C_i - U'(x_i)| \le q$, *where* $q = Kh^2\|U'''\|_{[x_0, x_n]}$ *with* $K > 0$.

Proof. As is well known (see [5]),

$$|S_i| \le \max_k |Q_k|.$$

Representing $U(t)$, $U'(x_{i-1})$, and $U'(x_{i+1})$ in expression (10) for Q_i as Taylor series expansions about the point x_i and combining like terms, we obtain

$$Q_i = \frac{6h^2}{4!}(U'''(\zeta) - U'''(\xi)) - \frac{h^2}{2}U'''(\tau) - \frac{h^2}{2}U'''(\varsigma),$$

where $\varsigma, \zeta \in [x_i, x_{i+1}]$, $\tau, \xi \in [x_{i-1}, x_i]$.

From this, $|Q_i| \le Kh^2\|U'''\|_{[x_{i-1}, x_{i+1}]}$, where $K = \frac{3}{2}$. Set $q = \max_i |Q_i|$. Thus, the lemma is proved for $K \le \frac{3}{2}$.

Let $\tilde{M} = \begin{pmatrix} 4 & 1 & 0 & \dots & 0 & 0 \\ 1 & 4 & 1 & \dots & 0 & 0 \\ 0 & 1 & 4 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$,

$\tilde{\Delta}_{n-1} = \det(\tilde{M})$, and C_n^k be the number of combinations of n elements taken k at a time.

For convenience, we introduce $m = n - 1$.

The determinant $\tilde{\Delta}_{n-1}$ of the system of equations is easy to calculate.

Lemma 2. *The following assertions hold:*

1. $\Delta_m = 4^m - C_{m-1}^1 4^{m-2} + C_{m-2}^2 4^{m-4} + \dots$.

2. $\operatorname{cond}(\tilde{M}) \le 3$.

Proof. 1. Decomposing $\tilde{\Delta}_m$ in terms of the elements of the first row yields the relation

$$\tilde{\Delta}_m = 4\tilde{\Delta}_{m-1} - \tilde{\Delta}_{m-2}.$$

Next, the required relation is derived in a similar manner to solving Problem 221 in [6].

2. By applying Gershgorin's theorem, it is easy to see that $\operatorname{cond}(\tilde{M}) \le 3$.

Theorem 2. *Let* $U \in C^3[a, b]$ *and* $\tilde{\tilde{U}}$ *be a continuously differentiable approximation constructed with the help of polynomial basis splines. Then*

$$\|\tilde{\tilde{U}} - U\|_{[a, b)} \le K_0 h^3 \|U'''\|_{[a, b)}, \tag{11}$$

where $K_0 \le 15/8$.

Proof. We have

$$\tilde{\tilde{U}}_k(x) - U(x) = C_k\omega_{k,1}(x) + C_{k+1}\omega_{k+1,1}(x) + F^{\langle k\rangle}\omega_k^{\langle 1\rangle}(x) - U(x), \quad x \in [x_k, x_{k+1}).$$

In view of Theorem 1 and Lemma 1,

$$\left|\tilde{\tilde{U}}(x) - U(x)\right| \le \left|U'_k\omega_{k,1}(x) + U'_{k+1}\omega_{k+1,1}(x) + F^{\langle k\rangle}\omega_k^{\langle 1\rangle}(x) - U(x)\right|$$

$$+ \left|(C_k - U'_k)\omega_{k,1}(x) + (C_{k+1} - U'_{k+1})\omega_{k+1,1}(x)\right| \le (3/8 + 3/2)h^3 \max_{[x_{k-1}, k_{k+1}]} |U'''|, \quad x \in [x_k, x_{k+1}),$$

which yields inequality (11) with $K_0 \le 15/8$.

4. Let us estimate the errors for continuously differentiable trigonometric splines. System (8) is reduced to the form

$$\tilde{A}_k C_{k-1} + \tilde{B}_k C_k + \tilde{A}_k C_{k+1} = \tilde{f}_k, \quad k = 1, \ldots, n-1, \tag{12}$$

where

$$\tilde{B}_k = 12\frac{\sin(h) - h\cos(h)}{h^2\sin(h)}, \quad \tilde{A}_k = 6\frac{-\sin(h) + h}{h^2\sin(h)},$$

$$\tilde{f}_k = 6(F^{\langle k-1\rangle} - F^{\langle k\rangle})/h^2, \quad k = 2, \ldots, n-2,$$

$$\tilde{f}_1 = 6\frac{F^{\langle 0\rangle} - F^{\langle 1\rangle}}{h^2} - 6U'(x_0)\frac{\sin(h) - h}{h^2\sin(h)},$$

$$\tilde{f}_{n-1} = 6\frac{F^{\langle n-2\rangle} - F^{\langle n-1\rangle}}{h^2} - 6U'(x_n)\frac{\sin(h) - h}{h^2\sin(h)}.$$

Note that $\tilde{B}_k = 4 + O(h^2)$ and, for $0 < h \le 1$, we have

$$|\tilde{B}_k| \le 4.3, \tag{13}$$

while $\tilde{A}_k = 1 + O(h^2)$ and, for $0 < h \le 1$,

$$|\tilde{A}_k| \le 1.15. \tag{14}$$

We have $|\tilde{B}_k| - 2|A_k| \ge 2 + 0.46h^2$ for $0 < h \le 1$.

The following result is easy to prove by analogy with Theorem 2.

Lemma 3. *For C_k satisfying system* (12), *it is true that*

$$|C_k - U'(x_k)| \le \tilde{q}, \tag{15}$$

where $\tilde{q} = Kh^2\|U' + U'''\|_{[x_0, x_n]}$ *with* $K > 0$.

Proof. Passing to the unknown $\tilde{S}_i = C_i - U'(x_i)$ in (12) gives

$$\begin{gathered} \tilde{S}_0 = 0, \\ \tilde{A}_i\tilde{S}_{i-1} + \tilde{B}_i\tilde{S}_i + \tilde{A}_i\tilde{S}_{i+1} = \tilde{Q}_i, \\ Q_i = 6(F^{\langle i\rangle} - F^{\langle i-1\rangle})/h^2 - \tilde{A}_i U'(x_{i-1}) - \tilde{B}_i U'(x_i) - \tilde{A}_i U'(x_{i+1}), \quad i = 1, \ldots, n-1, \\ \tilde{S}_n = 0. \end{gathered} \tag{16}$$

Applying the relation

$$U(x) = \frac{1}{2}\int_{x_k}^{x}(U'(t) + U'''(t))\left(\sin\frac{x-t}{2}\right)^2 dt + c_1 + c_2\sin(x) + c_3\cos(x), \tag{17}$$

where $x \in [x_k, x_{k+1}]$ and c_1,c_2, and c_3 are arbitrary constants, we obtain

$$U'(x_{k-1}) = \frac{1}{4}\int_{x_k}^{x_{k-1}}(U'(t) + U'''(t))\sin(x_{k-1} - t)dt + c_2\cos(x_{k-1}) - c_3\sin(x_{k-1}), \tag{18}$$

$$U'(x_{k+1}) = \frac{1}{4}\int_{x_k}^{x_{k+1}}(U'(t) + U'''(t))\sin(x_{k+1} - t)dt + c_2\cos(x_{k+1}) - c_3\sin(x_{k+1}), \tag{19}$$

$$U'(x_k) = 0. \tag{20}$$

Table 3

№	$U(x)$	$\sup_{[-1,1)}\lvert U-\tilde{\tilde{U}}^p\rvert$	$\sup_{[-1,1)}\lvert U-\tilde{\tilde{U}}^t\rvert$	R^p	R^t
1	$\sin(3x)\cos(5x)$	0.25×10^{-1}	0.25×10^{-1}	0.47	0.27
2	$\tan(x)$	0.75×10^{-2}	0.74×10^{-2}	0.11	0.66×10^{-1}
3	$\cos(2x)$	0.15×10^{-2}	0.10×10^{-2}	0.15×10^{-1}	0.66×10^{-2}
4	$1/(1+25\cos(x))$	0.23×10^{-3}	0.23×10^{-3}	0.32×10^{-2}	0.20×10^{-2}
5	$\frac{\cos(5x)}{(1+25\cos(x))}$	0.74×10^{-3}	0.69×10^{-3}	0.90×10^{-2}	0.49×10^{-2}
6	$1/(1+25x^2)$	0.10	0.10	1.09	0.64
7	$\sin(x)$	0.11×10^{-3}	0.22×10^{-18}	0.19×10^{-2}	0
8	x^2	0.20×10^{-19}	0.37×10^{-3}	0	0.22×10^{-2}
9	$\frac{\tan(x/2)}{1+\tan^2(x/2)}$	0.11×10^{-3}	0.17×10^{-18}	0.94×10^{-3}	0

Table 4

№	$U(x)$	$\sup_{[-1,1)}\lvert U-\tilde{\tilde{U}}^p\rvert$	$\sup_{[-1,1)}\lvert U-\tilde{\tilde{U}}^t\rvert$	R^p	R^t
1	$\sin(3x)\cos(5x)$	0.17×10^{-4}	0.16×10^{-4}	0.47×10^{-3}	0.27×10^{-3}
2	$\tan(x)$	0.98×10^{-5}	0.10×10^{-4}	0.11×10^{-3}	0.66×10^{-4}
3	$\cos(2x)$	0.13×10^{-5}	0.98×10^{-6}	0.15×10^{-4}	0.66×10^{-5}
4	$1/(1+25\cos(x))$	0.32×10^{-6}	0.31×10^{-6}	0.32×10^{-5}	0.20×10^{-5}
5	$\frac{\cos(5x)}{(1+25\cos(x))}$	0.91×10^{-6}	0.31×10^{-6}	0.90×10^{-5}	0.49×10^{-5}
6	$1/(1+25x^2)$	0.38×10^{-4}	0.38×10^{-4}	0.11×10^{-2}	0.64×10^{-3}

Applying the mean-value theorem and using approximating relations (2) and formulas (17)–(20), for $|\tilde{Q}_i|$ (see (16)) we have

$$|\tilde{Q}_i| \le 0.75\max_{[x_0,\,x_n]}|U' - U'''|h^2.$$

Setting $\tilde{q} = \max_i|\tilde{Q}_i|$ yields inequality (15) with $K \le 0.75$. The lemma is proved.

Theorem 3. *Let $U \in C^3[a, b]$ and $\tilde{\tilde{U}}$ be a continuously differentiable approximation constructed with the help of trigonometric splines. Then, for $h \le 1$,*

$$\|\tilde{\tilde{U}} - U\|_{[a,\,b)} \le K_0 h^3 \|U''' + U'\|_{[a,\,b]}, \tag{21}$$

where $K_0 \le 1.1$.

Proof. Applying Theorem 1 and Lemma 3, by analogy with Theorem 2, we obtain (21) with a constant $K_0 \le 1.1$.

5. For the function $U(x)$, $x \in [-1, 1)$, Tables 3 and 4 present the quantities

$$\sup_{[-1,\,1)}|U(x) - \tilde{\tilde{U}}^p(x)| \quad \text{and} \quad \sup_{[-1,\,1)}|U(x) - \tilde{\tilde{U}}^t(x)|, \quad \text{for } n = 10, \quad n = 100,$$

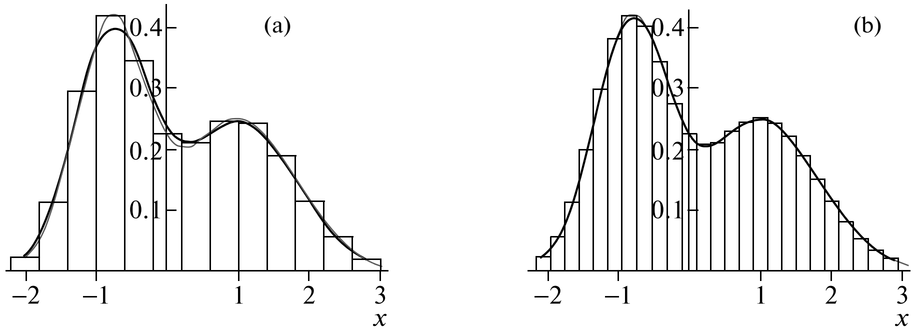

Fig. 8. (a) $n = 12$ and (b) $n = 24$.

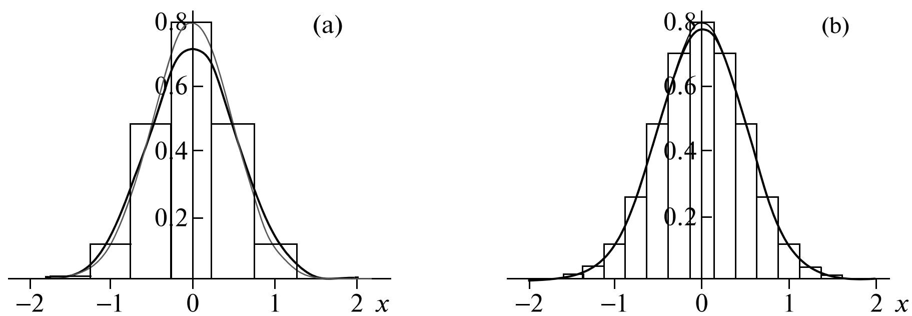

Fig. 9. (a) $n = 8$ and (b) $n = 16$.

where $\tilde{\tilde{U}}^p(x)$ is the approximation obtained with the help of polynomial splines and $\tilde{\tilde{U}}^t(x)$ is the approximation obtained with the help of trigonometric splines in Maple with Digits = 20. The last two columns give the theoretical errors R^p and R^t in the case of polynomial and trigonometric splines, respectively.

Conclusions. For small n, the approximation errors in the case of polynomial and trigonometric splines can differ. In some cases, the trigonometric splines are superior as applied to the approximation of trigonometric functions. For weakly oscillating functions, the trigonometric splines are not advantageous. For sufficiently large n, these differences become insignificant. The theoretical error estimates agree with the actual ones.

6. In [9] an approximating function describing an experimental distribution law was found with the help of Chebyshev–Hermite polynomials. The stability of the solution to the system of equations was improved by applying regularization.

Let us solve the same problem with the help of the polynomial splines proposed above. The integrals $F^{\langle k \rangle}$ are approximately evaluated using the trapezoidal rule and the first derivatives are replaced by the values of the function at grid nodes with an $O(h^3)$ error (see [10]).

It was noted in [11] that, in practice, the probability approach as applied to measurement error estimation assumes primarily that an analytical model is known for the error distribution law and that distributions encountered in metrology are fairly diverse. Additionally, it was noted that, according to a certain study, nearly half of the distributions were exponential, one-fifth consisted of various bimodal distributions, and the others were flattened.

Suppose that the bimodal distribution density

$$f = (f_1 + f_2)/2,$$

where

$$f_i = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-(x-\alpha_i)^2/(2\sigma_i^2)}, \quad i = 1, 2, \quad \sigma_1 = 0.5, \quad \sigma_2 = 0.8, \quad \alpha_1 = -0.8, \quad \alpha_2 = 1,$$

is approximated by the polynomial splines on the interval [−3, 2]. Figure 8 shows the histogram, the density, and its approximation constructed in Maple with Digits = 5. Note that $|f - \tilde{\tilde{f}}^p| \le 0.24 \times 10^{-2}$ for $h = 0.2$, while $|f - \tilde{\tilde{f}}^p| \le 0.12 \times 10^{-1}$ for $h = 0.4$.

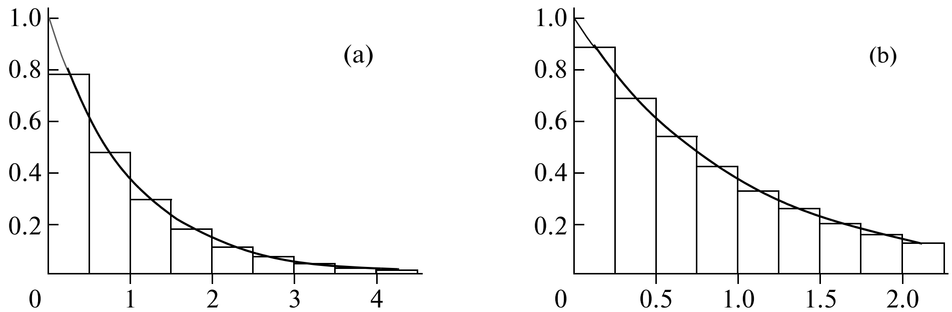

Fig. 10. (a) $h = 0.5$ and (b) $h = 0.25$.

Let

$$f_3 = \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\alpha)^2/(2\sigma^2)}, \quad \sigma = 0.5, \quad \alpha = 0.$$

Figure 9 depicts the histogram, the density f_3, and its polynomial approximation on [–2, 2] constructed in Maple with Digits = 5.

Figure 10 shows the exponential distribution density f_4 and its polynomial approximation on [0, 2] constructed in Maple with Digits = 5.

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