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Application of Computer Algebra Methods to Investigate the Dynamics of the System of Two Connected Bodies Moving along a Circular Orbit

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Abstract—Computer algebra methods are used to investigate properties of a nonlinear algebraic system that determines the equilibrium orientations for a system of two bodies connected by a spherical hinge that moves along a circular orbit under the action of gravitational torque. To determine the equilibrium orientations for the system of two bodies, the system of 12 stationary algebraic equations is decomposed using linear algebra methods and algorithms for Gröbner basis construction. Depending on the parameters of the problem, the number of equilibria is found by analyzing the real roots of the algebraic equations from the Gröbner basis constructed. Evolution of the conditions for equilibria existence in the dimensionless parameter space of the problem is investigated. The effectiveness of the algorithms for Gröbner basis construction is analyzed depending on the number of parameters for the problem under consideration.

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1. INTRODUCTION

In this work, we apply computer algebra methods to investigate the dynamics of a system of two bodies (satellite and stabilizer) connected by a spherical hinge that moves in a central Newtonian force field along a circular orbit. Determining the equilibria for the system of bodies on a circular orbit is of practical interest for designing composite gravitational orientation systems of satellites that can stay on the orbit for a long time without energy consumption. The dynamics of various composite schemes for satellite–stabilizer gravitational orientation systems was discussed in detail in [1].

In this paper, we analyze the spatial equilibria (equilibrium orientations) of the satellite–stabilizer system in the orbital coordinate system for certain values of the principal central moments of inertia of the bodies. The action of the stabilizer on the satellite provides new equilibrium orientations for the two-body system, as well as introduces dissipation into the system.

Equilibrium orientations of the satellite–stabilizer system are determined by real roots of a system of algebraic equations. To find equilibrium solutions, the system of algebraic equations is decomposed using linear algebra methods and algorithms for Gröbner basis construction. Some classes of equilibrium solutions are obtained explicitly from algebraic equations included in the Gröbner basis. The parameter values that cause the change in the number of equilibrium orientations for the satellite–stabilizer system are found. Symbolic-numerical analysis of the evolution of conditions for equilibria existence in the space of dimensionless parameters is carried out.

The effectiveness of Gröbner basis construction algorithms is analyzed depending on the number of parameters for the problem under consideration. The investigation is carried out using the Maple computer algebra system [9].

The algebraic methods for determining the equilibrium orientations of the two-body system described in this work were successfully used to analyze the dynamics of a satellite–gyrostat system [2, 3], as well as the dynamics of a satellite with an aerodynamic orientation system [4]. In mechanics, computer algebra is widely employed to analyze polynomial systems with the use of symbolic computations. Some computer

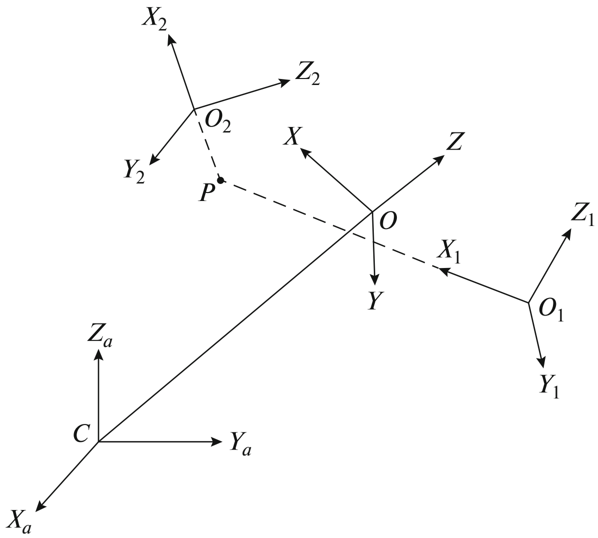

Fig. 1. Basic coordinate systems.

algebra algorithms for solving these problems were described in [13, 15].

2. EQUATIONS OF MOTION

Let us consider the system of two bodies connected by a spherical hinge that moves along a circular orbit [5].

To write equations of motion for two bodies, we introduce the following right-handed Cartesian coordinate systems (see Fig. 1): $CX_aY_aZ_a$ is the absolute coordinate system with the origin at the Earth's center of mass C (the plane CX_aY_a coincides with the equatorial plane and the CZ_a-axis coincides with the Earth's axis of rotation) and $OXYZ$ is the orbital coordinate system. The OZ-axis is directed along the radius vector that connects the Earth's center of mass C with the center of mass of the two-body system O, the OX-axis is directed along the linear velocity vector of the center of mass O, and the OY-axis is directed along the normal to the orbital plane. The coordinate system for the ith body ($i = 1, 2$) is $O_ix_iy_iz_i$, where O_ix_i, O_iy_i, and O_iz_i are principal central axes of inertia for the ith body. The orientation of the coordinate system $O_ix_iy_iz_i$ with respect to the orbital coordinate system is determined using the pitch α_i, yaw β_i, and roll γ_i angles [1]:

$$
\begin{aligned}
a_{11}^{(i)} &= \cos\alpha_i \cos\beta_i,\\
a_{12}^{(i)} &= \sin\alpha_i \sin\gamma_i - \cos\alpha_i \sin\beta_i \cos\gamma_i,\\
a_{13}^{(i)} &= \sin\alpha_i \cos\gamma_i + \cos\alpha_i \sin\beta_i \sin\gamma_i,\\
a_{21}^{(i)} &= \sin\beta_i, \quad a_{22}^{(i)} = \cos\beta_i \cos\gamma_i,\\
a_{23}^{(i)} &= -\cos\beta_i \sin\gamma_i, \qquad\qquad (1)\\
a_{31}^{(i)} &= -\sin\alpha_i \cos\beta_i,\\
a_{32}^{(i)} &= \cos\alpha_i \sin\gamma_i + \sin\alpha_i \sin\beta_i \cos\gamma_i,\\
a_{33}^{(i)} &= \cos\alpha_i \cos\beta_i - \sin\alpha_i \sin\beta_i \sin\gamma_i.
\end{aligned}
$$

Suppose that $(a_i, 0, 0)$ are the coordinates of the spherical hinge P in the body coordinate system $O_ix_iy_iz_i$; A_i, B_i, C_i are principal central moments of inertia; $M = M_1M_2/(M_1 + M_2)$; M_i is the mass of the ith body; p_i, q_i, and r_i are the projections of the absolute angular velocity of the ith body onto the axes Ox_i, Oy_i, and Oz_i; and ω_0 is the angular velocity for the center of mass of the two-body system moving along a circular orbit. Then, using expressions for kinetic energy and force function, which determines the effect of the Earth's gravitational field on the system of two bodies connected by a hinge [1], the equations of motion for this system can be written as Lagrange equations of the second kind by symbolic differentiation in the Maple system:

$$
\begin{aligned}
&A_i\dot p_i + (C_i - B_i)q_ir_i - 3\omega_0^2(C_i - B_i)a_{32}^{(i)}a_{33}^{(i)} = 0,\\
&(B_i + Ma_i^2)\dot q_i - Ma_ia_j(a_{13}^{(i)}a_{13}^{(j)} + a_{23}^{(i)}a_{23}^{(j)} + a_{33}^{(i)}a_{33}^{(j)})\dot q_j\\
&\quad + Ma_ia_j(a_{13}^{(i)}a_{12}^{(j)} + a_{23}^{(i)}a_{22}^{(j)} + a_{33}^{(i)}a_{32}^{(j)})\dot r_j\\
&\quad + Ma_ia_j(a_{13}^{(i)}(r_j(p_ja_{13}^{(j)} - r_ja_{11}^{(j)})\\
&\quad - q_j(q_ja_{11}^{(j)} - p_ja_{12}^{(j)}))\\
&\quad + a_{23}^{(i)}(r_j(p_ja_{23}^{(j)} - r_ja_{21}^{(j)}) - q_j(q_ja_{21}^{(j)} - p_ja_{22}^{(j)})) \qquad (2)\\
&\quad + a_{33}^{(i)}(r_j(p_ja_{33}^{(j)} - r_ja_{31}^{(j)}) - q_j(q_ja_{31}^{(j)} - p_ja_{32}^{(j)})))\\
&\quad + Ma_i\omega_0^2(a_j(a_{13}^{(i)}a_{11}^{(j)} + a_{23}^{(i)}a_{21}^{(j)} + a_{33}^{(i)}a_{31}^{(j)})\\
&\quad + 3a_{33}^{(i)}(a_ia_{31}^{(i)} - a_ja_{31}^{(j)}))\\
&\quad + ((A_i - C_i) - Ma_i^2)r_ip_i - 3\omega_0^2(A_i - C_i)a_{33}^{(i)}a_{31}^{(i)} = 0,\\
&(C_i + Ma_i^2)\dot r_i + Ma_ia_j(a_{12}^{(i)}a_{13}^{(j)} + a_{22}^{(i)}a_{23}^{(j)} + a_{32}^{(i)}a_{33}^{(j)})\dot q_j\\
&\quad - Ma_ia_j(a_{12}^{(i)}a_{12}^{(j)} + a_{22}^{(i)}a_{22}^{(j)} + a_{32}^{(i)}a_{32}^{(j)})\dot r_j\\
&\quad - Ma_ia_j(a_{12}^{(i)}(r_j(p_ja_{13}^{(j)} - r_ja_{11}^{(j)})\\
&\quad - q_j(q_ja_{11}^{(j)} - p_ja_{12}^{(j)}))\\
&\quad + a_{22}^{(i)}(r_j(p_ja_{23}^{(j)} - r_ja_{21}^{(j)}) - q_j(q_ja_{21}^{(j)} - p_ja_{22}^{(j)}))\\
&\quad + a_{32}^{(i)}(r_j(p_ja_{33}^{(j)} - r_ja_{31}^{(j)}) - q_j(q_ja_{31}^{(j)} - p_ja_{32}^{(j)})))\\
&\quad - Ma_i\omega_0^2(a_j(a_{12}^{(i)}a_{11}^{(j)} + a_{22}^{(i)}a_{21}^{(j)} + a_{32}^{(i)}a_{31}^{(j)})\\
&\quad + 3a_{32}^{(i)}(a_ia_{31}^{(i)} - a_ja_{31}^{(j)}))\\
&\quad + ((B_i - A_i) + Ma_i^2)p_iq_i - 3\omega_0^2(B_i - A_i)a_{31}^{(i)}a_{32}^{(i)} = 0.
\end{aligned}
$$

Here,

$$
\begin{aligned}
p_i &= (\dot\alpha_i + \omega_0)a_{21}^{(i)} + \dot\gamma_i,\\
q_i &= (\dot\alpha_i + \omega_0)a_{22}^{(i)} + \dot\beta_i \sin\dot\gamma_i, \qquad (3)\\
r_i &= (\dot\alpha_i + \omega_0)a_{23}^{(i)} + \dot\beta_i \cos\dot\gamma_i.
\end{aligned}
$$

In the first three equations of (2), $i = 1$ and $j = 2$; in the next three equations of (2), $i = 2$ and $j = 1$. In (3), $i = 1, 2$. In (2) and (3), the dot denotes time (t) differentiation.

3. EQUILIBRIUM ORIENTATIONS

By assuming $\alpha_i = \alpha_{i0} = \text{const}$, $\beta_i = \beta_{i0} = \text{const}$, and $\gamma_i = \gamma_{0i} = \text{const}$ in (2) and (3), as well as by introducing the designations $a_{ij}^{(1)} = a_{ij}$ and $a_{ij}^{(2)} = b_{ij}$, for $A_i \neq B_i \neq C_i$, we obtain the equations

$$\begin{gathered} a_{22}a_{23} - 3a_{32}a_{33} = 0, \\ m_1(a_{23}a_{21} - 3a_{33}a_{31}) + a_{23}b_{21} - 3a_{33}b_{31} = 0, \\ n_1(a_{22}a_{21} - 3a_{32}a_{31}) - a_{22}b_{21} + 3a_{32}b_{31} = 0, \\ b_{22}b_{23} - 3b_{32}b_{33} = 0, \\ b_{23}a_{21} - 3b_{33}a_{31} + m_2(b_{23}b_{21} - 3b_{33}b_{31}) = 0, \\ -b_{22}a_{21} + 3b_{32}a_{31} + n_2(b_{22}b_{21} - 3b_{32}b_{31}) = 0, \end{gathered} \tag{4}$$

which allow us to determine the equilibrium orientation for the system of two bodies connected by a spherical hinge in the orbital coordinate system. In (4), we introduce the following designations: $m_1 = ((A_1 - C_1) - Ma_1^2)/Ma_1a_2$, $m_2 = ((A_2 - C_2) - Ma_2^2)/Ma_1a_2$, $n_1 = ((B_1 - A_1) - Ma_1^2)/Ma_1a_2$, and $n_2 = ((B_2 - A_2) - Ma_2^2)/Ma_1a_2$. Taking into account the expressions for the direction cosines from (1), system (4) can be regarded as a system of six equations in six unknowns α_i, β_i, and γ_i ($i = 1, 2$).

Another (more convenient) way of closing equations (4) is to add six orthogonality conditions for the direction cosines:

$$\begin{gathered} a_{21}^2 + a_{22}^2 + a_{23}^2 - 1 = 0, \\ a_{31}^2 + a_{32}^2 + a_{33}^2 - 1 = 0, \\ a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0, \\ b_{21}^2 + b_{22}^2 + b_{23}^2 - 1 = 0, \\ b_{31}^2 + b_{32}^2 + b_{33}^2 - 1 = 0, \\ b_{21}a_{31} + b_{22}b_{32} + b_{23}b_{33} = 0. \end{gathered} \tag{5}$$

Equations (4) and (5) form a closed algebraic system of equations in 12 unknown direction cosines that determine the equilibrium orientations of the two-body system. For this system, the following problem is formulated: for given m_1, n_1, m_2, and n_2, determine all twelve direction cosines. The other six direction cosines (a_{11}, a_{12}, a_{13} and b_{11}, b_{12}, b_{13}) can be obtained from the orthogonality conditions.

In this work, we focus on symbolic methods for analyzing equilibrium solutions given by algebraic equations (4) and (5).

4. ANALYZING THE EQUILIBRIUM ORIENTATIONS OF THE TWO-BODY SYSTEM BY COMPUTER ALGEBRA METHODS

Let us analyze solutions Change to "of" the system of algebraic equations (4) and (5) in detail.

To solve the algebraic system (4) and (5), various algorithms for Gröbner basis construction are used [8]. Construction of the Gröbner basis is an algorithmic procedure that completely reduces the problem with a system of polynomials in many variables to the consideration of a polynomial in one variable.

The investigation is carried out using the Groebner[Basis] package for Maple 17 [9, 10]. This package uses a set of five algorithms to compute Gröbner bases for monomials of various orders and various codomains of polynomial coefficients. The type of the algorithm is specified by the option `method=meth`. This option executes the fastest universal direct method available.

Construction of the Gröbner basis for the system (4) and (5) of 12 second-order algebraic equations, whose coefficients depend on four parameters, is a very complicated algorithmic problem.

To analyze the system (4) and (5), the following three algorithms were applied. First, `method=mapleF4` executes the F4 algorithm developed by D.S. Faugere and written in the Maple language. The F4 algorithm supports all orders of monomials and types of polynomial coefficients from any field, as well as computations in Ore noncommutative algebras. Second, `method=fglm` executes the FGLM algorithm for Gröbner basis construction, developed by Faugere, Gianni, Lazard, and Mora [11]. Third, `method=walk` executes the Walk algorithm developed by Collart, Kalkbrener, and Mall, which supports all commutative fields and orders of monomials; it converts the Gröbner basis from one monomial order to another [12].

Using the *plex* option (lexicographic ordering by variables), it is generally impossible to construct the Gröbner basis for 12 polynomials f_i ($i = 1, 2, \ldots, 12$) that are the left-hand sides of the equations in the system (4) and (5) in 12 variable direction cosines a_{ij} and b_{ij} ($i = 2, 3, j = 1, 2, 3$).

Below, we present results of Gröbner basis construction for the system (4) and (5) in some special cases.

The instruction Maple–`infolevel[GroebnerBasis] := 2` gives information on the type of the algorithm used and its run time. All computations were carried out on a computer with Intel Core i7 2.4 GHz and 8 GB RAM.

Case 1: $m_1 = n_1 = m_2 = n_2 = m$. In this case, the F4 algorithm with the *tdeg* option (ordering by powers) was used to construct the Gröbner basis. The time for computing the Gröbner basis with the F4 algorithm and *tdeg* order was approximately two hours (7124.35 s) for a 3.668-Kb file containing the polynomials of the basis. We failed to recompute the resulting basis for the *plex* order with the FGLM algorithm because of a program interruption caused by exceeding the allowable memory limit for the Maple system. The error message was as follows: Error, (in Groebner:-Basis) object too large.

Case 2: $m_1 = n_1 = m; m_2 = n_2 = 1$. This case is simpler than case 1. Here, the Gröbner basis was also constructed using the F4 algorithm with the *tdeg* option. The time for computing the Gröbner basis with the F4 algorithm and *tdeg* order was 5503.5 s for a 3375-Kb file containing the polynomials of the basis. We also failed to recompute the resulting basis for the *plex* order with the FGLM algorithm because of a program interruption caused by exceeding the allowable memory limit. The Maple system terminated after 5952 s.

Case 3: $m_1 = m, n_1 = m_2 = n_2 = 1$. This case is even simpler than cases 1 and 2. The Gröbner basis was constructed using the Walk algorithm with the *plex* option for 12 polynomials f_i $(i = 1, 2, \ldots, 12)$, which were the left-hand sides of the equations in the system (4) and (5) in 12 variables a_{ij} and b_{ij} $(i = 2, 3, j = 1, 2, 3)$:

```
G:=map(factor,Groebner[Basis]([f1,
… f12], plex(b21, …, a33))).
```

The procedure for Gröbner basis construction involved two steps. First, the F4 algorithm was used (its run time was 3017.5 s). Then, the Walk algorithm was executed. The total computation time was 12 324 s. The number of polynomials in the resulting Gröbner basis was 170, and the number of rows in the basis exceeded one million. Nevertheless, in this basis, we selected the following polynomial that depends only on one variable $x = a_{33}^2$:

$$P(x) = P_1(x)P_2(x)P_3(x)P_4(x)P_5(x)P_6(x) = 0. \qquad (6)$$

Here,

$$\begin{gathered} P_1(x) = x(x-1), \\ P_2(x) = 16(m^2-1)x^2 \\ -40(m^2-1)x + 49m^2 - 25 = 0, \\ P_3(x) = 64m^2(4m-1)x^2 \\ -32(2m^2-1)(4m-1)x + 7m + 5 = 0, \\ P_4(x) = 64m^2(4m+1)x^2 \\ -32(2m^2-1)(4m+1)x + 7m - 5 = 0, \\ P_5(x) = 18(m+1)^4(m-1)x^2 \\ +12(m+1)^2(m-1)(m^2-3m-1)x \\ +(7m+5)(7m^2-1)(5m^2-2) = 0, \\ P_6(x) = m^2x - m^2 + 1 = 0. \end{gathered}$$

To find equilibrium solutions, we had to consider six individual cases with zero quadratic polynomials, $P_i = 0, (i = 1, 2, \ldots, 6)$. As a result, we obtained the values of a_{33} that depend only on the parameter m. The value of the direction cosine a_{32} can be found from the second polynomial in the Gröbner basis, while a_{31} is obtained from orthogonality conditions (5). The values of the other direction cosines can be found from the system of equations (4).

It should be noted that, in case 3, the two-step algorithm for computing the Gröbner basis with the lexicographical ordering option (first, it uses the *tdeg* order and, then, recomputes the resulting basis for the *plex* order by using the FGLM algorithm) proved to be more efficient. The computation time with the F4 algorithm and *tdeg* order was 630.125 s for 490 polynomials in the basis. The recomputation of this basis for the *plex* order by using the FGLM algorithm took 1823.4 s for 170 polynomials in the basis. The total run time of the two-step algorithm was 2453.525 s, which is five times faster than that of the Walk algorithm for the same case. Among all methods described above, the elimination block ordering *lexdeg* with explicit specification of the method (`method=walk`) showed the maximum computation time.

Equilibrium solutions for the system of two bodies in the orbital plane for $\beta_i = 0$, $\gamma_i = 0$, and $\alpha_i \neq 0$ were considered in [6, 7]. In the planar case, the system (4) and (5) has simple solutions of four types:

$$\begin{gathered} \sin\alpha_1 = 0, \quad \sin\alpha_2 = 0, \\ \cos\alpha_1 = 0, \quad \cos\alpha_2 = 0, \\ \cos\alpha_1 = 0, (\sin\alpha_1 + m_2\sin\alpha_2) = 0, \\ \cos\alpha_2 = 0, (m_1\sin\alpha_1 + \sin\alpha_2) = 0. \end{gathered} \qquad (7)$$

In [7], planar oscillations of the two-body system were analyzed, all equilibrium orientations (7) were determined, and sufficient conditions for the stability of the equilibrium orientations were obtained using the energy integral as a Lyapunov function.

The results presented above suggest that, in the general case, the system of algebraic equations (4) and (5) cannot be solved by direct application of the Gröbner basis construction methods. Below, we consider a combined approach to this problem.

5. COMBINED APPROACH TO THE PROBLEM

Let us consider the second, third, fifth, and sixth equations of system (4) with respect to the variables a_{21}, a_{31}, b_{21}, and b_{31}. These equations form a homogeneous subsystem

$$\begin{gathered} (m_1a_{23})a_{21} - (3m_1a_{33})a_{31} + (a_{23})b_{21} - (3a_{33})b_{31} = 0, \\ (n_1a_{22})a_{21} - (3n_1a_{32})a_{31} - (a_{22})b_{21} + (3a_{32})b_{31} = 0, \\ (b_{23})a_{21} - (3b_{33})a_{31} + (m_2b_{23})b_{21} - (3m_2b_{33})b_{31} = 0, \\ -(b_{22})a_{21} + (3b_{32})a_{31} + (n_2b_{22})b_{21} - (3n_2b_{32})b_{31} = 0. \end{gathered} \qquad (8)$$

The determinant of system (8) is

$$\Delta = (m_1n_1m_2n_2 + 1)a_{11}b_{11} - (m_1m_2 + n_1n_2)(a_{22}b_{33} - a_{32}b_{23}) \times (a_{33}b_{22} - a_{23}b_{32}) + (m_1n_1 + m_2n_1)(a_{22}b_{32} - a_{32}b_{22})(a_{23}b_{33} - a_{33}b_{23}).$$

If $\Delta \neq 0$, then $a_{21} = a_{31} = b_{21} = b_{31} = 0$ and system (8) is divided into two subsystems:

$$\begin{aligned} a_{22}a_{23} - 3a_{32}a_{33} &= 0, & b_{22}b_{23} - 3b_{32}b_{33} &= 0,\\ a_{22}^2 + a_{23}^2 &= 1, & b_{22}^2 + b_{23}^2 &= 1,\\ a_{32}^2 + a_{33}^2 &= 1, & b_{32}^2 + b_{33}^2 &= 1,\\ a_{22}a_{32} + a_{23}a_{33} &= 0, & b_{22}b_{32} + b_{23}b_{33} &= 0. \end{aligned} \tag{9}$$

System (9) can be solved using Gröbner basis construction methods. Let us construct the Gröbner basis by using the FGLM algorithm with the *plex* option for eight polynomials f_i $(i = 1, 2, \ldots, 8)$, which are the left-hand sides of the equations in system (9) in eight variables a_{ij} and b_{ij} $(i = 2, 3, j = 2, 3)$. The total computation time is 0.05 s. The resulting Gröbner basis includes 18 polynomials. Let us write out some polynomials that depend on the variables a_{22} and a_{33} from this basis: $a_{22}(a_{22}^2 - 1)$, $a_{33}(a_{33}^2 - 1)$, and $a_{22}^2 - a_{33}^2$. From the Gröbner basis constructed, we can find the solutions of system (9):

$$\begin{aligned} a_{22}^2 &= 1, & a_{23} &= 0, & a_{32} &= 0, & a_{33}^2 &= 1,\\ b_{22}^2 &= 1, & b_{23} &= 0, & b_{32} &= 0, & b_{33}^2 &= 1; \end{aligned} \tag{10}$$

$$\begin{aligned} a_{22} &= 0, & a_{23}^2 &= 1, & a_{32}^2 &= 1, & a_{33} &= 0,\\ b_{22} &= 0, & b_{23}^2 &= 1, & b_{32}^2 &= 1, & b_{33} &= 0. \end{aligned} \tag{11}$$

In total, we have eight different solutions: four solutions for $a_{22}^2 = 1$ and $a_{33}^2 = 1$ (solutions (10)) and four solutions for $a_{23}^2 = 1$ and $a_{32}^2 = 1$ (solutions (11)). The condition $\Delta \neq 0$, e.g., for the solution with $a_{22} = a_{33} = b_{22} = b_{33} = 1$, holds for the following constraints on the system parameters: $m_1m_2(n_1n_2 - 1) + (m_1 - n_1)n_2 + m_2n_1 \neq 0$. For $\Delta = 0$, we have the case of parametric solutions to system (8) [14].

Let us now consider the first and second equations in system (4). They form a homogeneous subsystem with respect to the variables a_{23} and a_{33}:

$$\begin{aligned} (a_{22})a_{23} - (3a_{32})a_{33} &= 0,\\ (m_1a_{21} + b_{21})a_{23} - 3(m_1a_{31} + b_{31})a_{33} &= 0. \end{aligned} \tag{12}$$

If the determinant of system (12) is $\Delta_1 = a_{22}b_{31} - a_{32}b_{21} - m_1(a_{21}a_{32} - a_{22}a_{31}) \neq 0$, then $a_{23} = 0$ and $a_{33} = 0$.

Equations 4 and 5 of system (4) also form a homogeneous subsystem with respect to b_{23} and b_{33}:

$$\begin{aligned} (b_{22})b_{23} - (3b_{32})b_{33} &= 0,\\ (m_2b_{21} + a_{21})b_{23} - 3(m_2b_{31} + a_{31})b_{33} &= 0. \end{aligned} \tag{13}$$

If the determinant of system (13) is $\Delta_2 = b_{22}a_{31} - b_{32}a_{21} - m_2(b_{21}b_{32} - b_{22}b_{31}) \neq 0$, then $b_{23} = 0$ and $b_{33} = 0$. If $\Delta_1 \neq 0$ and $\Delta_2 \neq 0$, then $a_{23} = 0$, $a_{33} = 0$, $b_{23} = 0$, $b_{33} = 0$, and system (4) takes the form

$$\begin{aligned} n_1(a_{21}a_{22} - 3a_{31}a_{32}) - b_{21}a_{22} + 3b_{31}a_{32} &= 0,\\ n_2(b_{21}b_{22} - 3b_{31}b_{32}) - a_{21}b_{22} + 3a_{31}b_{32} &= 0,\\ a_{21}^2 + a_{22}^2 - 1 = 0, \quad b_{21}^2 + b_{22}^2 - 1 &= 0,\\ a_{31}^2 + a_{32}^2 - 1 = 0, \quad b_{31}^2 + b_{32}^2 - 1 &= 0,\\ a_{21}a_{31} + a_{22}a_{32} = 0, \quad b_{21}b_{31} + b_{22}b_{32} &= 0. \end{aligned} \tag{14}$$

System (14) can be solved using Gröbner basis construction methods. Let us construct the Gröbner basis by using the FGLM algorithm with the lexicographic ordering option for eight polynomials f_i $(i = 1, 2, \ldots, 8)$, which are the left-hand sides of the equations in system (14) in eight variables a_{ij} and b_{ij} $(i = 2, 3, j = 1, 2)$:

```
G:=map(factor,Groebner[Basis]([f1, …
f8], plex(b21,…,a32))).
```

The total computation time is 0.125 s. The resulting Gröbner basis includes 27 polynomials. From this basis, we write out the polynomial that depends only on one variable a_{32}:

$$P(a_{32}) = P_7(a_{32})P_8(n_1, n_2, a_{32}) = 0. \tag{15}$$

Here,

$$\begin{aligned} P_7(a_{32}) &= a_{32}(a_{32}^2 - 1),\\ P_8(a_{32}) &= 64n_1^2(4n_1^2n_2 - 1)(n_1^2n_2 - 1)a_{32}^4\\ &\quad - 32(2n_1^2 - 1)(4n_1^2n_2 - 1)(n_1^2n_2 - 1)a_{32}^2\\ &\quad + ((4n_1^2n_2^2 + 2)^2 - 9(n_1 + n_2)^2) = 0. \end{aligned}$$

In addition, this Gröbner basis implies the following relationships:

$$a_{21}^2 = a_{32}^2, a_{22}^2 = a_{31}^2, b_{21}^2 = b_{32}^2, b_{22}^2 = b_{31}^2. \tag{16}$$

Taking into account relationships (16), system (14) can be rewritten as follows:

$$\begin{aligned} 4n_1a_{21}a_{22} - b_{21}a_{22} - 3b_{22}a_{21} &= 0,\\ 4n_2b_{21}b_{22} - a_{21}b_{22} - 3a_{22}b_{21} &= 0,\\ a_{21}^2 + a_{22}^2 - 1 = 0, \quad b_{21}^2 + b_{22}^2 - 1 &= 0. \end{aligned} \tag{17}$$

The solution of system (17) reduces to the solution of the biquadratic equation from (15).

Using relationships (15) and (16), we can write all solutions of system (14).

1. The solutions obtained under the condition $P_7(a_{32}) = 0$ $(a_{32} = 0)$ are

$$\begin{aligned} a_{32} &= 0, & a_{31}^2 &= 1, & a_{22}^2 &= 1, & a_{21} &= 0,\\ b_{22}^2 &= 1, & b_{21} &= 0, & b_{32} &= 0, & b_{31}^2 &= 1. \end{aligned} \tag{18}$$

Solutions (18) exist if $\Delta_1 = (b_{31} - m_1 a_{31}) \neq 0$ and $\Delta_2 = (a_{31} - m_2 b_{31}) \neq 0$ ($|m_2| \neq 1$).

2. The solutions obtained under the condition $P_7(a_{32}) = 0$ ($a_{32}^2 = 1$) are

$$\begin{aligned} &a_{32}^2 = 1, \quad a_{31} = 0, \quad a_{21}^2 = 1, \quad a_{22} = 0, \\ &b_{21}^2 = 1, \quad b_{22} = 0, \quad b_{31} = 0, \quad b_{32}^2 = 1. \end{aligned} \tag{19}$$

Solutions (19) exist if $\Delta_1 = (b_{21} + m_1 a_{21}) \neq 0$ ($|m_1| \neq 1$) and $\Delta_2 = (a_{21} + m_2 b_{21}) \neq 0$ ($|m_2| \neq 1$).

3. In the third case ($P_8(n_1, n_2, a_{32}) = 0$), system (14) has the following solutions at $\Delta_1 \neq 0$ and $\Delta_2 \neq 0$:

$$\begin{gathered} a_{32}^2 = \frac{2n_1^2 - 1}{4n_1^2} \\ \pm \frac{(8n_1^3 n_2 - 5n_1(n_1 + n_2) + 2)\sqrt{(4n_1 n_2 - 1)(n_1 n_2 - 1)}}{8n_1^2(4n_1 n_2 - 1)(n_1 n_2 - 1)}, \\ a_{31}^2 = 1 - a_{32}^2, \quad a_{33} = 0, \\ a_{21}^2 = a_{32}^2, \quad a_{22}^2 = a_{31}^2, \quad a_{23} = 0, \\ b_{32}^2 = \frac{n_2^2 - 1}{4n_2^2} \\ \pm \frac{(8n_2^3 n_1 - 5n_2(n_1 + n_2) + 2)\sqrt{(4n_1 n_2 - 1)(n_1 n_2 - 1)}}{8n_2^2(4n_1 n_2 - 1)(n_1 n_2 - 1)}, \\ b_{31}^2 = 1 - b_{32}^2, \quad b_{33} = 0, \\ b_{21}^2 = b_{32}^2, \quad b_{22}^2 = b_{31}^2, \quad b_{23} = 0. \end{gathered} \tag{20}$$

Solutions (20) exist if $n_1 n_2 < 1/4$ and $n_1 n_2 > 1$.

Next, as in the previous case, we consider the first and third equations of system (4) with respect to a_{22} and a_{32}, as well as the fourth and sixth equations of (4) with respect to b_{22} and b_{32}. If $\Delta_3 = a_{33}b_{21} - a_{23}b_{31} + n_1(a_{23}a_{31} - a_{21}a_{33}) \neq 0$ is the determinant of the first subsystem with respect to a_{22} and a_{32}, then $a_{22} = 0$ and $a_{32} = 0$. If $\Delta_4 = a_{21}b_{33} - b_{23}a_{31} + n_2(b_{23}b_{31} - b_{21}b_{33}) \neq 0$ is the determinant of the second subsystem with respect to b_{22} and b_{32}, then $b_{22} = 0$ and $b_{32} = 0$. If $\Delta_3 \neq 0$ and $\Delta_4 \neq 0$, then $a_{22} = 0$, $a_{32} = 0$, $b_{22} = 0$, $b_{32} = 0$, and system (4) takes the form

$$\begin{aligned} &m_1(a_{23}a_{21} - 3a_{33}a_{31}) + b_{21}a_{23} - 3b_{31}a_{33} = 0, \\ &m_2(b_{23}b_{21} - 3b_{33}b_{31}) + a_{21}b_{23} - 3a_{31}b_{33} = 0, \\ &a_{21}^2 + a_{23}^2 - 1 = 0, \quad b_{21}^2 + b_{23}^2 - 1 = 0, \\ &a_{31}^2 + a_{33}^2 - 1 = 0, \quad b_{31}^2 + b_{33}^2 - 1 = 0, \\ &a_{21}a_{31} + a_{23}a_{33} = 0, \quad b_{21}b_{31} + b_{23}b_{33} = 0. \end{aligned} \tag{21}$$

As in the previous case, system (21) is solved by constructing the Gröbner basis for the system of polynomials (22) with the FGLM algorithm and *plex* option for eight polynomials f_i ($i = 1, 2, \ldots, 8$), which are the left-hand sides of the equations in system (21) in eight variables a_{ij} and b_{ij} ($i = 2, 3$, $j = 1, 3$). The total computation time is 0.172 s. The resulting Gröbner basis includes 27 polynomials.

From this basis, we write out the polynomial that depends only on one variable a_{33}:

$$P(a_{33}) = P_9(a_{33})P_{10}(m_1, m_2, a_{33}) = 0. \tag{22}$$

Here,

$$\begin{gathered} P_9(a_{33}) = a_{33}(a_{33}^2 - 1), \\ P_{10}(a_{33}) = 256m_1^2(2m_1 m_2 + 1)^2 a_{33}^4 \\ - 8(2m_1 m_2 + 1)(64m_1^3 m_2 + 32m_1(m_1 - m_2) - 25)a_{33}^2 \\ + ((8m_1 m_2 - 5)^2 - 36(m_1 - m_2)^2) = 0. \end{gathered}$$

In addition, this Gröbner basis implies the following relationships:

$$a_{21}^2 = a_{33}^2, \quad a_{23}^2 = a_{31}^2, \quad b_{21}^2 = b_{33}^2, \quad b_{23}^2 = b_{31}^2. \tag{23}$$

Using relationships (22) and (23), we can write all solutions of system (21).

4. The solutions obtained under the condition $P_9(a_{33}) = 0$ ($a_{33} = 0$) are

$$\begin{aligned} &a_{33} = 0, \quad a_{31}^2 = 1, \quad a_{23}^2 = 1, \quad a_{21} = 0, \\ &b_{23}^2 = 1, \quad b_{21} = 0, \quad b_{33} = 0, \quad b_{31}^2 = 1. \end{aligned} \tag{24}$$

These solutions exist if $\Delta_3 = (b_{31} - n_1 a_{31}) \neq 0$ ($|n_1| \neq 1$) and $\Delta_4 = (a_{31} - n_2 b_{31}) \neq 0$ ($|n_2| \neq 1$).

5. The solutions obtained under the condition $a_{33}^2 = 1$ are

$$\begin{aligned} &a_{33}^2 = 1, \quad a_{31} = 0, \quad a_{21}^2 = 1, \quad a_{23} = 0, \\ &b_{21}^2 = 1, \quad b_{23} = 0, \quad b_{31} = 0, \quad b_{33}^2 = 1. \end{aligned} \tag{25}$$

These solutions exist if $\Delta_3 = (b_{21} + n_1 a_{21}) \neq 0$ ($|n_1| \neq 1$) and $\Delta_4 = (a_{21} - n_2 b_{21}) \neq 0$ ($|n_2| \neq 1$).

6. For $P_{10}(m_1, m_2, a_{33}) = 0$, system (21) has the following solutions at $\Delta_3 \neq 0$ and $\Delta_4 \neq 0$:

$$\begin{gathered} a_{33}^2 = \frac{1}{2} - \frac{32m_1 m_2 + 25}{64m_1^2(2m_1 m_2 + 1)} \pm \\ \pm \frac{(8m_1^2 - 5)\sqrt{64m_1 m_2(m_1 m_2 + 1) + 25}}{64m_1^2(2m_1 m_2 + 1)}, \\ a_{31}^2 = 1 - a_{33}^2, \quad a_{32} = 0, \\ a_{21}^2 = a_{33}^2, \quad a_{23}^2 = a_{31}^2, \quad a_{22} = 0, \\ b_{33}^2 = \frac{1}{2} - \frac{32m_1 m_2 + 25}{64m_2^2(2m_1 m_2 + 1)} \pm \\ \pm \frac{(8m_2^2 - 5)\sqrt{64m_1 m_2(m_1 m_2 + 1) + 25}}{64m_2^2(2m_1 m_2 + 1)}, \end{gathered} \tag{26}$$

$$b_{31}^2 = 1 - b_{33}^2, \quad b_{32} = 0, \quad b_{21}^2 = b_{33}^2,$$
$$b_{23}^2 = b_{31}^2, \quad b_{22} = 0, \quad (2m_1m_2 \neq -1).$$

The expression $64m_1m_2(m_1m_2 + 1) + 25$ is non-negative for any m_1, m_2.

Thus, in the general case, we have obtained the families of equilibrium solutions (10), (11), (18), (19), (20), (24), (25), and (26) to the original system of 12 algebraic equations (4) and (5) under the constraints $\Delta \neq 0$, $\Delta_1 \neq 0$, $\Delta_2 \neq 0$, $\Delta_3 \neq 0$, and $\Delta_4 \neq 0$. The conditions for satisfying these constraints can be verified numerically for each set of system parameters.

In conclusion, it should be noted that the combination of the computer algebra and linear algebra methods makes it possible to investigate a wide class of equilibrium orientations for the system of two bodies connected by a spherical hinge on a circular orbit under certain constraints imposed on the parameters of the problem. Based on the results presented in this paper, using symbolic computation methods, it has been shown that, on a circular orbit, the two-body system can have both planar and spatial configurations in equilibrium orientation.

Traditionally, to find steady-state configurations, numerical methods for finding roots of systems of nonlinear algebraic equations are employed, e.g., iteration methods (simple iteration method and Seidel method), Newton method, descent method, secant method, and Steffensen's method [16–20]. Disadvantages of this approach are well known. Localizing the solution that guarantees the convergence of the method and finding all real roots are the most difficult problems that arise when solving systems of nonlinear equations numerically. When the coefficients of algebraic equations depend on the parameters of the problem, as in our case, numerical methods fail to determine the behavior of the roots in the parameter space.

In this work, we have used another approach based on symbolic computation methods to solve the system of nonlinear algebraic equations. It has allowed us to reduce the system of 12 second-order algebraic equations to a polynomial in one variable, find its analytical solution, and investigate the conditions for the existence of the solutions in the parameter space of the problem.

REFERENCES

1. Sarychev, V.A., Questions of orientation of artificial satellites, *Itogi Nauki Tekh., Ser.: Issled. Kosm. Prostranstva,* Moscow: V. I. N. i T. I., 1978, vol. 11.
2. Gutnik, S.A. and Sarychev, V.A., Symbolic-numerical methods of studying equilibrium positions of a gyrostat satellite, *Program. Comput. Software,* 2014, vol. 40, no. 3, pp. 143–150.
3. Gutnik, S.A. and Sarychev, V.A., Application of computer algebra methods for investigation of stationary motions of a gyrostat satellite, *Program. Comput. Software,* 2017, vol. 43, no. 2, pp. 90–97.
4. Gutnik, S.A. and Sarychev, V.A., A symbolic investigation of the influence of aerodynamic forces on satellite equilibria, *Lect. Notes Comput. Sci.,* 2016, vol. 9890, pp. 243–254.
5. Sarychev, V.A. and Gutnik, S.A., Equilibrium orientations of a system of two bodies connected by a spherical joint, *Tezisy dokladov XL Akademicheskikh chtenii po kosmonavtike* (Abstr. 40th Academic Readings on Astronautics), Moscow: MVTU im. N. E. Baumana, 2016.
6. Sarychev, V.A., Investigation of the dynamics of a gravitational stabilization system, *Sb. Iskusstv. Sputniki Zemli* (Collect. Artif. Earth Satellites), Moscow: Izd. Akad. Nauk SSSR, 1963, no. 16, pp. 10–33.
7. Sarychev, V.A., Relative equilibrium orientations of two bodies connected by a spherical joint on a circular orbit, *Kosm. Issled.,* 1967, vol. 5, no. 3, pp. 360–364.
8. Buchberger, B., Theoretical basis for the reduction of polynomials to canonical forms, *SIGSAM Bulletin,* 1976, pp. 19–29.
9. Char, B.W., Geddes, K.O., Gonnet, G.H., Monagan, M.B., and Watt, S.M., *Maple Reference Manual,* Watcom Publications Limited, 1992.
10. Maple online help. http://www.maplesoft.com/support/help.
11. Faugere, J., Gianni, P., Lazard, P., and Mora, T., Efficient computation of zero-dimensional Gröbner bases by change of ordering, *J. Symbolic Comput.,* 1993, vol. 16, pp. 329–344.
12. Collart, S., Kalkbrener, M., and Mall, D., Converting bases with the Gröbner walk, *J. Symbolic Comput.,* 1997, vol. **3**, no. 4, pp. 465–469.
13. Chen, C. and Maza, M.M., Semi-algebraic description of the equilibria of dynamical systems, *Computer Algebra in Scientific Computing (CASC),* Gerdt, V.P., Koepf, W., Mayr, E.W., and Vorozhtsov, E.V., Eds., Springer, 2011, vol. 6885, pp. 101–125.
14. Gantmakher, F.R., *Teoriya matrits* (Matrix Theory), Moscow: Fizmatlit, 2010.
15. Batkhin, A.B., Parameterization of the discriminant set of a polynomial, *Program. Comput. Software,* 2016, vol. 42, no. 2, pp. 8–21.
16. Bakhvalov, N.S., Zhidkov, N.P., and Kobel'kov, G.G., *Chislennye metody* (Numerical Method), Moscow: BINOM. Laboratoriya znanii, 2011.
17. Kalitkin, N.P., *Chislennye metody* (Numerical Methods), Moscow: Nauka, 1978.
18. Krylov, V.I., Bobkov, V.V., and Monastyrskii, P.I., *Vychislitel'nye metody* (Computational Methods), Moscow: Nauka, 1978, vol. 1.
19. Amosov, A.A., Dubinskii, Yu.A., and Kopchenova, N.P., *Vychislitel'nye metody dlya inzhenerov* (Computational Methods for Engineers), Moscow: Vysshaya shkola, 1994.
20. Turchak, L.I. and Plotnikov, P.V., *Osnovy chislennykh metodov* (Fundamentals of Numerical Methods), Moscow: Fizmatlit, 2005.

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