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On Testing the Existence of Universal Denominators for Partial Differential and Difference Equations

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Abstract—We consider the problem of testing the existence of a universal denominator for partial differential or difference equations with polynomial coefficients and prove its algorithmic undecidability. This problem is closely related to finding rational function solutions in that the construction of a universal denominator is a part of the algorithms for finding solutions of such form for ordinary differential and difference equations.

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1. INTRODUCTION

We consider linear partial differential equations of the form

$$\sum_{s\in S} a_s(x_1,\dots,x_m)\frac{\partial^{s_1+\dots+s_m} y(x_1,\dots,x_m)}{\partial x_1^{s_1}\dots\partial x_m^{s_m}} = 0, \qquad (1)$$

as well as linear partial difference equations given by

$$\sum_{s\in S} a_s(x_1,\dots,x_m)y(x_1+s_1,\dots,x_m+s_m) = 0, \qquad (2)$$

where S is a finite subset $\mathbb{Z}^m_{\geq 0}$, $x_1,\dots,x_m$ is a vector of independent variables (further denoted as $\mathbf{x}$ for brevity), and $a_s(x_1,\dots,x_m)$ are polynomial coefficients.

When $m = 1$, i.e., in the case of ordinary differential and difference equations, there exist algorithms that find all polynomial solutions of a given equation of such a type and all solutions in the form of rational functions. These algorithms use the fact that, for a given differential or difference equation, a polynomial $q(x)$ can be found such that any rational solution of this equation can be represented as $\frac{p(x)}{q(x)}$, where $p(x)$ is also a polynomial. If such a $q(x)$ is found, the problem of searching solutions in the form of rational functions reduces to that of searching polynomial solutions by transforming the equation and replacement of the unknown function. The function in this case is referred to as a *universal denominator.*

As for partial differential or difference equations, there are no similar algorithms even in the case of two variables. Moreover, the problems of testing existing of both polynomial and rational solutions turn out to be algorithmically undecidable [1, 2]. Nevertheless, construction of universal denominators for such equations is an important problem since reduction of the problem of searching solutions in the form of rational functions to searching polynomial solutions is useful by itself.

The problem of searching for universal denominators for partial differential and difference equations was considered by M. Kauers and C. Schneider in [3, 4]. The authors divide all polynomials of several variables $p(x_1,\dots,x_m)$ into periodic and aperiodic ones according to the following rule:

1. a polynomial $p(x_1,\dots,x_m)$ is periodic if the set of $d\in\mathbb{Z}^m$ satisfying the condition

$$\mathtt{gcd}(p(x_1,\dots,x_m), p(x_1+d_1,\dots,x_m+d_m)) \neq 1$$

contains an infinite number of elements;

2. otherwise, $p(x_1,\dots,x_m)$ is an aperiodic polynomial.

Note that any polynomial can uniquely be factored into the product of its periodic and aperiodic parts.

При этом любой полином можно однозначно разложить в произведение его периодической и непериодической частей. The algorithm presented in [4] finds a "partial" universal denominator for a difference equation with polynomial coefficients that contains all aperiodic multipliers and covers a wide class of periodic multipliers of denominators of all rational solutions of the equation. With its help, one can find universal denominators for ordinary difference equations, since all polynomials in one variable are aperiodic. In addition, if we have a system of several equations (rather than one equation) in one unknown function, then, in some cases (depending on the equation structure), the universal denominator can also be found by means of this algorithm. However, in the general case of $m \geq 2$, the algorithm fails to find a universal denominator. Moreover, as shown in [4], for some difference equations, it does not exist at all. For example, equation $y(x_1+1, x_2) - y(x_1, x_2+1) = 0$ has an infinite set of solutions of the form $\frac{1}{(x_1+x_2)^k}$, $k \in \mathbb{N}$, which have no common denominator.

2. RELATION TO THE DIOPHANTINE EQUATIONS

We will use symbol δ to denote both the differential and difference operators of the form

$$\delta_i = \begin{cases} x_i \frac{\partial}{\partial x_i}, & \text{in the differential case,} \\ x_i \Delta_i, & \text{in the difference case,} \end{cases} \tag{3}$$

$$\delta^n = \delta_1^{n_1} \dots \delta_m^{n_m} \quad \text{for} \quad n \in \mathbb{Z}_{\geq 0}^m.$$

This will allow us to formulate assertions that are valid for both differential and difference equations.

We will also use the notation

$$\mathbf{x}^{\langle n \rangle} = \begin{cases} \mathbf{x}^n, & \text{in the differential case,} \\ \mathbf{x}^{\overline{n}}, & \text{in the difference case,} \end{cases}$$

where $n \in \mathbb{Z}^m$, $\mathbf{x}^n = x_1^{n_1} \dots x_m^{n_m}$, $\mathbf{x}^{\overline{n}} = x_1^{\overline{n_1}} \dots x_m^{\overline{n_m}}$. The notation $x^{\overline{k}}$, $k \in \mathbb{Z}$, in the given case is meant to be the so-called "increasing" power,

$$x^{\overline{k}} = \begin{cases} x(x+1)\dots(x+k-1), & \text{if} \quad k > 0, \\ 1, & \text{if} \quad k = 0, \\ \frac{1}{(x-1)(x-2)\dots(x-|k|)}, & \text{if} \quad k < 0. \end{cases}$$

Note that, both in the differential and difference cases, operator δ and monomials of the form $\mathbf{x}^{\langle n \rangle}$ satisfy the following equation:

$$P(\delta_1, \dots, \delta_m)\mathbf{x}^{\langle n \rangle} = P(n_1, \dots, n_m)\mathbf{x}^{\langle n \rangle}, \quad P \in \mathbb{Z}[x_1, \dots, x_m]. \tag{4}$$

From (4), it follows that the equation

$$P(\delta_1, \dots, \delta_m)\mathbf{x}^{\langle n \rangle} = 0, \quad P \in \mathbb{Z}[x_1, \dots, x_m]$$

has a monomial solution $\mathbf{x}^{\langle n \rangle}$, $n \in \mathbb{Z}^m$ if and only if n is a solution of the Diophantine equation $P(n_1, \dots, n_m) = 0$. At the same time, from the Davis—Putnam—Robinson—Matiyasevich theorem, which solves (in the negative sense) Hilbert's tenth problem, it is known that there does not exists an algorithm to check whether an arbitrary Diophantine equation has integer solutions [5].

This relationship between the equations

$$P(\delta_1, \dots, \delta_m)y(\mathbf{x}) = 0,$$

which are a particular case of equations (1) and (2), and Hilbert's tenth problem has already been used for proving algorithmic undecidability of various problems related to partial differential and difference equations [1, 2, 6].

3. UNDECIDABILITY THEOREM

In what follows, $o(\mathbf{x}^n)$ will denote a function that can be represented as a sum of monomials the multi-exponents of which are lexicographically less than n; i.e.,

$$o(\mathbf{x}^n) = \sum_{s \prec n,\, s \in \mathbb{Z}^m} c_s \mathbf{x}^s, \quad n \in \mathbb{Z}^m,$$

where $s \prec n$ means that s is lexicographically less than n (it is assumed that coefficients c_s are constant and that the sum on the right-hand side is finite).

The lemma below is proved in [2].

Lemma 1. If a rational function of the form $\frac{x^\nu + o(x^\nu)}{x^\mu + o(x^\mu)}$ is a solution of the equation $P(\delta_1, \dots, \delta_m)y(\mathbf{x}) = 0$, where $P \in \mathbb{Z}[n_1, \dots, n_m]$, then $x^{\langle \nu - \mu \rangle}$ is also its solution.

Using this lemma and the Davis—Putnam—Robinson—Matiyasevich theorem, we will prove that the problem of testing existence of a universal denominator for the equations under consideration is algorithmically undecidable.

Theorem 1. *Given an arbitrary linear homogeneous partial differential or difference equation, there does not exist an algorithm that verifies whether there exist a universal denominator for it.*

Proof.

Let $P(n_1, \dots, n_m) \in \mathbb{Z}[n_1, \dots, n_m]$ and

$$Q(n_1, \dots, n_{m+2}) = P(n_1, \dots, n_m)^2 + (n_{m+1} + n_{m+2})^2.$$

Let the Diophantine equation $P(n_1, \dots, n_m) = 0$ have no integer solutions. Then, equation $Q(n_1, \dots, n_{m+2}) = 0$ has no integer solutions either. From equation (4), we find that differential and difference equations of the form

$$Q(\delta_1, \dots, \delta_{m+2})y(x_1, \dots, x_{m+2}) = 0 \tag{5}$$

have no monomial solutions of the form $y(\mathbf{x}) = \mathbf{x}^{\langle n \rangle}$, $n \in \mathbb{Z}^{m+2}$. Then, from Lemma 1, it follows that equation (5) has no rational solutions; hence, there exists a universal denominator $q(\mathbf{x}) = 1$ for it.

Let the Diophantine equation $P(n_1,\dots,n_m)=0$ have an integer solution $(n_1,\dots,n_m)$. Then, equation $Q(n_1,\dots,n_{m+2})=0$ has an infinite number of solutions $(n_1,\dots,n_m,k,-k)$, where k is an arbitrary integer. From equation (4), we find that differential and difference equations of form (5) have an infinite number of solutions of the form $y(\mathbf{x}) = x_1^{\langle n_1\rangle}\dots x_m^{\langle n_m\rangle}x_{m+1}^{\langle k\rangle}x_{m+2}^{\langle -k\rangle}$, $k\in\mathbb{N}$, and, hence, have no universal denominator, since there does not exist a polynomial that is divided by x_{m+2}^{k} (in the differential case) or by $(x_{m+2}-1)(x_{m+2}-2)\dots(x_{m+2}-k)$ (in the difference case) for all possible $k\in\mathbb{N}$.

Suppose that there exists an algorithm capable of determining whether there exists a universal denominator for an arbitrary linear homogeneous partial differential or difference equation with polynomial coefficients. Then, by means of this algorithm, one can verify whether there exists an integer solution for an arbitrary Diophantine equation, since, as shown above, equation

$P(n_1,\dots,n_m)=0$, $P\in\mathbb{Z}[n_1,\dots,n_m]$, has a solution $n\in\mathbb{Z}^m$ if and only if equation (5) has no universal denominator. Since, according to the Davis—Putnam—Robinson—Matiyasevich theorem, the problem of testing existence of solutions of Diophantine equations is algorithmically undecidable, the problem of testing existence of a universal denominator is also undecidable. □

Note that Theorem 1 does not give an answer to the question of whether the problem of testing existence of a universal denominator is decidable if the equation is a priori known to have rational solutions. Let us prove undecidability of this problem.

Theorem 2. *There is no algorithm that determines whether there exits a universal denominator for a given linear homogeneous partial differential or difference equation with polynomial coefficients and a nonzero rational solution of this equation.*

Proof. Suppose that such an algorithm exists. Then, by using it, one can verify whether there exists an integer solution of an arbitrary Diophantine equation $P(n_1,\dots,n_m)=0$, $P\in\mathbb{Z}[n_1,\dots,n_m]$. To do this, it will suffice to check existence of a universal denominator for the equation

$$Q(\delta_1,\dots,\delta_{m+2})y(\mathbf{x})=0, \tag{6}$$

where

$$Q(n_1,\dots,n_{m+2}) = (P(n_1,\dots,n_m)^2+(n_{m+1}+n_{m+2})^2)^2 \times((n_1+1)^2+n_2^2+\dots+n_{m+2}^2).$$

In this case, rational function $y(\mathbf{x})=x_1^{\langle -1\rangle}$ is a solution of equation (6), since $n_1=-1$, $n_2=0$, ..., $n_{m+2}=0$ is a solution of the Diophantine equation $Q(n_1,\dots n_{m+2})=0$.

If the Diophantine equation $P(n_1,\dots,n_m)=0$ has an integer solution $(n_1,\dots,n_m)$, then equation $Q(n_1,\dots,n_{m+2})=0$ has an infinite number of solutions $(n_1,\dots,n_m,k,-k)$, and equation (6) has an infinite number of solutions of the form $y(\mathbf{x}) = x_1^{\langle n_1\rangle}\dots x_m^{\langle n_m\rangle}x_{m+1}^{\langle k\rangle}x_{m+2}^{\langle -k\rangle}$, $k\in\mathbb{N}$, and, hence, has no universal denominator.

If the Diophantine equation $P(n_1,\dots,n_m)=0$ has no integer solutions, then equation $Q(n_1,\dots,n_{m+2})=0$ has only one solution $n_1=-1$, $n_2=0,\dots,n_{m+2}=0$. Then, equation (6) has only one rational solution $y(\mathbf{x})=x_1^{\langle -1\rangle}$ (up to a constant multiplier). Indeed, suppose that there exists another solution $y_1(\mathbf{x})=\dfrac{\mathbf{x}^c+o(\mathbf{x}^c)}{\mathbf{x}^d+o(\mathbf{x}^d)}$, $n,d\in\mathbb{Z}_{\geq 0}^{m+2}$. Then,

- if $c-d=(-1,0,\dots,0)$, then equation (6) also has solution $y_2(\mathbf{x}) = y_1(\mathbf{x})-y(\mathbf{x}) = \dfrac{\mathbf{x}^{c'}+o(\mathbf{x}^{c'})}{\mathbf{x}^{d'}+o(\mathbf{x}^{d'})}$, where $c'-d' \neq (-1,0,\dots,0)$; by Lemma 1, $\mathbf{x}^{\langle c'-d'\rangle}$ is also a solution of equation (6) and, hence, $c'-d'$ is a solution of the Diophantine equation $Q(n_1,\dots,n_{m+2})=0$, which brings us at the contradiction;
- if $c-d\neq(-1,0,\dots,0)$, then, by Lemma 1, monomial $\mathbf{x}^{\langle c-d\rangle}$ is a solution of equation (6), and $c-d$ is a solution of equation $Q(n_1,\dots,n_{m+2})=0$, which brings us at the contradiction.

Thus, equation (6) has universal denominator $q(\mathbf{x})=x_1$ (in the differential case) or $q(\mathbf{x})=x_1-1$ (in the difference case). □

Note that these theorems state impossibility of constructing an algorithm for testing existence of a universal denominator for equations in an arbitrary number of variables, i.e., if m is not a priori given or bounded somehow. At the same time, these problems remain undecidable even if m is a fixed (sufficiently large) number, because a similar assertion on undecidability of the problem of testing existence of solutions of Diophantine equations is true [5].

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