Flow and Peculiar Velocities for Generic Motion in Spherically Symmetric Black Holes

A. V. Toporensky[1,2]* and O. B. Zaslavskii[2,3]**

[1]*Sternberg Astronomie, Moscow State University, Moscow, Russia*
[2]*Kazan Federal University, Kremlevskaya 18, Kazan, 420008 Russia*
[3]*Department of Physics and Technology, Kharkov V.N. Karazin National University, 4 Svoboda Square, Kharkov, 61022 Ukraine*

Abstract—In this methodological paper we consider geodesic motion of particles in spherically symmetric black hole space-times. We develop an approach based on splitting the velocity of a freely falling particle to the flow velocity, which depends only on the metric, and a deviation from it (a peculiar velocity). It applies to a wide class of spherically symmetric metrics and is exploited under the horizon of the Schwarzschild black hole. The present work generalizes previous results obtained for pure radial motion. Now, the motion is, in general, nonradial, so that an observer can have a nonzero angular momentum. This approach enables us to give simple physical interpretation of redshifts (blueshifts) inside the horizon including the region near the singularity and agrees with the recent results obtained by direct calculations.

1. INTRODUCTION

The goal of the present paper is to describe geodesic motion in the interior of a spherically symmetric black hole from a viewpoint which is rather rarely presented in scientific and methodological literature concerning black holes, and has been mostly developed in cosmology. The concept of "expansion of space" is well known in methodological literature and is widely used in cosmological textbooks (see, for example, [1]). Though from a purely scientific perspective it is usually considered as a redundant concept, and there are no formal differences in interpretations of Hubble expansion as "expansion of space" and the motion of galaxies through space (see [2] and references therein), visualization of the Hubble flow and peculiar velocities with respect to it can be considered to be pedagogically useful. However, it rises a question of what is specific in cosmology that gives rise to this concept. A recent development gives rather an unexpected answer to this question: such an interpretation is not specific to cosmology and can be developed in other areas of general relativity.

In [3] it was suggested to think of a black hole as a river of space that flows through it. Later, this picture was expanded in [4, 5]. In particular, this allows splitting of a velocity of a test particle to two natural parts. One of them represents motion with a flow while the second one gives a deviation from it due to peculiar movement. This leads to rather a transparent kinematic picture that is useful in a number of applications. For example, it was applied in [5] to the description of the so-called Bañados-Silk-West (BSW) effect [6] and in [7] to the problem of maximizing the time inside a black hole horizon.

Kinematics of particles (including the inner region of a black hole) was considered in [5] for pure radial motion only and for the Gullstrand-Penlevé frame. Also, radial motion of particles in this frame (but without using the notions of a flow and peculiar velocities) was considered in [8] for the Schwarzschild, Bardeen and Reissner-Nordsrtöm black holes. Our goal here is to generalize our approach developed in [5] in two aspects. First, we consider an arbitrary motion that implies nonzero angular momentum. Second, we use synchronous coordinate systems. This allows us to make a direct comparison of our definitions with those usual in cosmology, as well as to consider another (different from the Gullstrand-Penlevé one) famous frame existing only inside a black hole horizon [9]. Relying on this approach, we establish some generic properties of motion near the black hole singularity and give simple physical explanation of the recent results for redshift inside a black hole [10].

*E-mail: `atopor@rambler.ru`
**E-mail: `zaslav@ukr.net`

We use the system of units in which the fundamental constants $G = c = 1$.

2. STATIC SPHERICALLY SYMMETRIC METRIC AND EQUATIONS OF MOTION

Let us consider the class of spherically symmetric metrics

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\omega^2,$$

$$d\omega^2 = \sin^2\theta d\phi^2 + d\theta^2. \tag{1}$$

The horizon radius is $r = r_+$, $f(r_+) = 0$.

Equation (1) implies that $g_{tt} g_{rr} = -1$. This form includes the Schwarschild, Reissner-Nordström, de Sitter and anti-de Sitter metrics, etc. In principle, we can write a more general form with independent g_{tt} and g_{rr}, but this would cause technical difficulties without qualitative changes, so in the present paper we restrict ourselves to Eq. (1).

We are interested in the behavior of test particles. For the spherically symmetric case, the motion occurs in a plane. We choose it to be $\theta = \pi/2$. Then, in the original frame (1), the equations of motion for a free particle with the energy E and angular momentum L read

$$\frac{dt}{d\tau} = \frac{\varepsilon}{f}, \tag{2}$$

$$\frac{dr}{d\tau} = -P, \tag{3}$$

$$\frac{d\phi}{d\tau} = \frac{\mathcal{L}}{r^2}, \tag{4}$$

where $\varepsilon = E/m$, $\mathcal{L} = L/m$, τ is the proper time, m is the particle mass, and

$$P \equiv \sqrt{\varepsilon^2 - f(1 + \frac{\mathcal{L}^2}{r^2})}. \tag{5}$$

We assumed that a particle moves towards the horizon, so r decreases with time.

The original frame is deficient on the horizon where $f = 0$. To make the metric regular near the horizon, we apply the transformation

$$t = \tilde{t} + F(r) - r^*, \tag{6}$$

$$r = \tilde{r}, \tag{7}$$

where

$$r^* = \int^r \frac{d\bar{r}}{f(\bar{r})}, \tag{8}$$

and F is regular near the horizon $r = r_+$. Then, the metric can be rewritten in the new coordinates in the form

$$ds^2 = -f d\tilde{t}^2 + 2 d\tilde{t} dr (1 - fF')$$

$$+ dr^2 [2F' - f(F')^2] + r^2 d\omega^2. \tag{9}$$

Introducing also v according to

$$v = \sqrt{1 - f}, \tag{10}$$

and setting

$$F' = \frac{1 - v}{f} = \frac{1}{1 + v}, \tag{11}$$

we obtain the metric in the following famous form (the Gullstrand-Penleve (GP) metric)

$$ds^2 = -d\tilde{t}^2 + (dr + v d\tilde{t})^2 + r^2 d\omega^2. \tag{12}$$

The transformation of time can be rewritten in the form

$$t = \tilde{t} - \int \frac{v \, dr}{f}. \tag{13}$$

The form (12) is widely used in analog models of General Relativity, as well as in the so-called "river model" of a black hole, where v represents the velocity of a flow ("river"), being a background on which standard Special Relativity considerations take place (see [3] for details).

For a massive particle, it follows from (2)–(4) and (6) that

$$\frac{d\tilde{t}}{d\tau} = \frac{\varepsilon - Pv}{f}, \tag{14}$$

$$\frac{dr}{d\tilde{t}} = -\frac{Pf}{(\varepsilon - Pv)}, \tag{15}$$

$$\frac{d\phi}{d\tilde{t}} = \frac{f}{\varepsilon - vP} \frac{\mathcal{L}}{r^2}. \tag{16}$$

Thus the four-velocity in coordinates $(\tilde{t}, r, \phi)$ reads

$$u^\mu = \left(\frac{\varepsilon - Pv}{f}, -P, \frac{\mathcal{L}}{r^2} \right). \tag{17}$$

In the present paper we work also in synchronous frames. To obtain such a frame from the GP one, we transform not only the temporal coordinate but also a spatial one. We want to obtain the metric in the form

$$ds^2 = -d\tilde{t}^2 + A d\rho^2 + r^2(\rho, \tilde{t}) d\omega^2 \tag{18}$$

that would generalize the familiar Lemaître frame, well-known for the Schwarzschild metric.

Our goal is achieved with

$$\rho = t + \int^r \frac{d\bar{r}}{fv}. \tag{19}$$

Then,

$$A = 1 - f = v^2. \tag{20}$$

If $f = 1 - r_+/r$, we return to the known formula for the Lemaître form of the Schwarzschild metric (see, e.g., [11]).

The transformation presented here, from the explicitly static form of the metric (1) to the Lemaître one (18), is a special case of a more general procedure described in [12] and Sec. 3.3.3 of [13].

For free motion, using (14) and (3) we have

$$\frac{d\rho}{d\tau} = \frac{\varepsilon v - P}{v f}, \tag{21}$$

$$\frac{d\rho}{d\tilde{t}} = \frac{\varepsilon v - P}{(\varepsilon - Pv)v}. \tag{22}$$

3. DEFINITIONS OF A PECULIAR VELOCITY

Now, we want to describe particle motion by different types of observers. To this end, we attach a tetrad to an observer, where the role of a timelike vector is played by the four-velocity of a reference observer. In particular, if we choose such an observer as a freely falling one, with special values of the energy and angular momentum, this defines a flow. Deviations from it correspond to a peculiar motion. If we choose a suitable definition of a three-velocity $V^{(i)}$ ($i = 1, 2, 3$), such a reference observer will have $V^{(i)} = 0$, whereas a peculiar motion is described by nonzero $V^{(i)}$. In other words, $V^{(i)}$ describes the motion of a particle with respect to the flow.

To choose a correct and physically reasonable definition of $V^{(i)}$, we use the tetrad formalism. Then, the motion of a massive particle is subluminal, so the absolute value of the vector $V^{(i)}$ is less than 1. In this section, we compare two different definitions of such a velocity, which were used independently in different contexts, and show that they coincide.

Let a particle and an observer have the four-velocities u^μ and $\tilde{u}^\mu$, respectively. Then, following Eq. (2) of [14], we can introduce the quantity w^μ according to

$$w^\mu = \tilde{u}^\mu + \frac{u^\mu}{(u, \tilde{u})}. \tag{23}$$

Equation (23) was used, in particular, in Eq. (40) of [15]. Note that our notations of four-velocities differ from those in [14, 15]. The sign at the second term in (23) differs from that in [15] because of our signature $(-, +, +, +)$ instead of $(+, -, -, -)$ in [15]. Here, $(u, u) = (\tilde{u}, \tilde{u}) = -1$ instead of $+1$ in [15].

Meanwhile, there is a standard definition of three-velocity (see Eq. (3.9) in, e.g., [16])

$$V^{(i)} = -\frac{u^\mu h_{(i)\mu}}{h^{\mu(0)} u_\mu}. \tag{24}$$

Let us choose the tetrad of basis vectors. We choose the vector "0" along the four-velocity,

$$h^\mu_{(0)} = \tilde{u}^\mu, \qquad h^{\mu(0)} = -\tilde{u}^\mu. \tag{25}$$

Then, we see that

$$w_\mu h^\mu_{(0)} = 0. \tag{26}$$

Other vectors are orthogonal to "0" by construction, so

$$h_{(i)\mu} h^{\mu(0)} = 0. \tag{27}$$

Then, it follows from (23) and (27) that

$$w_\mu h^\mu_{(i)} = \frac{u^\mu h_{(i)\mu}}{(u, \tilde{u})}. \tag{28}$$

The denominator can be rewritten using (25),

$$(u, \tilde{u}) = h^\mu_{(0)} u_\mu = -h^{\mu(0)} u_\mu. \tag{29}$$

Then, we obtain that

$$w_\mu h^\mu_{(i)} = V_{(i)} = V^{(i)} \tag{30}$$

coincides with (24), as it should be.

We can define the tetrad components of w^μ according to

$$w^{(a)} = h^{(a)\mu} w_\mu. \tag{31}$$

Then we have

$$w^{(a)} = (0, V^{(i)}). \tag{32}$$

4. LOCAL AND NONLOCAL VELOCITIES IN A SYNCHRONOUS FRAME

The concept of a peculiar velocity is widely used in cosmology, where it usually means a velocity with respect to the FLRW frame. Since the flat FLRW metric is

$$ds^2 = -dt^2 + a(t)^2 d\chi^2$$
$$+ a(t)^2 \chi^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{33}$$

(a is the scale factor, and χ is the radial comoving coordinate) the corresponding tetrad is

$$h_{(i)\mu} = \text{diag}(-1, a, a\chi, a\chi \sin \theta), \tag{34}$$

and for the 3-velocity of a particle with 4-velocity u^μ we have according to (24)

$$V_i = \frac{h_{(i)\mu} u^\mu}{-h_{(0)\mu} u^\mu}. \tag{35}$$

In particular, the radial component of a peculiar velocity is

$$V_r = a u^\chi / u^t = a d\chi / dt. \tag{36}$$

The consept of a peculiar velocity is a local one, however, the radial component allows for a nice non-local interpretation [17]. Namely, the rate of change of a proper d istance between the coordinate origin and a distant point $l = a\chi$ is

$$\frac{dl}{dt} = \frac{d(a\chi)}{dt} = \chi\frac{da}{dt} + a\frac{d\chi}{dt} = v_H + V_r, \qquad (37)$$

where $v_H = \chi\dot{a} = (\chi a)\dot{a}/a = lH$ is the velocity of the Hubble flow. The left-hand side of this equation can be considered as a reasonable definition of the velocity of a *distant* object (more precisely, its radial part) which is an intrinsically nonlocal entity. The above equation means that the overall change of a proper distance to a distant point is naturally decomposed into a sum of the velocity of the cosmological flow and the radial part of a peculiar velocity. Note that the summation rule is of the Galileo type, independently of the velocity values. We see that in spite of a nonlocal nature of this equation as a whole, as well as the Hubble flow velocity v_H, the second term in the right-hand side has a local interpretation.

It is also worth noting that the rate of change of a proper distance to a remote point is a natural definition only for one (radial) component of the velocity of a distant point. Obviously, this rate is useless for defining distant tangent velocities. There are several proposals to define them, which give different results, as well as other definitions of a radial velocity, see, e.g., [14, 18, 19]. We will not consider tangent non-local velocities and other definitions of a radial non-local velocity in the present paper.

In [20] it was noticed that the property (37) is not specific to cosmology and is present in any spherically symmetric synchronous system. Indeed, if the value of the "scale factor" a depends on χ as well, and the particle has a comoving coordinate χ_1, we get the same result using the Leibniz integral rule:

$$\frac{dl}{dt} = \frac{d}{dt}\int_0^{\chi_1} a(\chi,t)d\chi = \int_0^{\chi_1} \frac{da(\chi,t)}{dt}d\chi$$

$$+ a(\chi_1,t)\frac{d\chi}{dt} = v_H + V_r. \qquad (38)$$

The radial velocity in the local sense is the same since the derivation of (36) does not use spatial homogeneity. Moreover, both derivations (the local and nonlocal ones) can be valid even if we abandon spherical symmetry, but are still able to introduce a coordinate system in which the free flow is one-dimensional—in this case, non-radial components of the metric (whatever they be) do not enter in the derivation.

In the present paper we mainly describe the properties of peculiar velocities in the Schwarzschild space-time. This concept allows us to get different interpretations of the known physical effects near and inside a black hole horizon. In our previous paper [5] we already used the radial part of the peculiar velocity to reinterpret BSW-like effects, giving a different perspective (particles which are "slow" in a conventional sense appear to be "fast" in the sense of the radial peculiar velocity and *vice versa*). Now we extend our analysis to angular peculiar velocities and show how the known results on a redshift observed inside a horizon can be explained in this approach.

5. A STATIC OBSERVER

Although our ultimate goal in the present paper is to consider 3-velocities with respect to synchronous frames, we start with a more common case of a static observer.

Then, in the original coordinates (t, r, θ, ϕ),

$$e_{(0)\mu} = -(\sqrt{f}, 0, 0, 0), \qquad (39)$$

$$e_{(1)\mu} = (0, 1/\sqrt{f}, 0, 0), \qquad (40)$$

$$e_{\mu(2)} = r(0, 0, 1, 0), \qquad (41)$$

$$e_{\mu(3)} = r\sin\theta(0, 0, 0, 1). \qquad (42)$$

Then, the motion of an observer with respect to this frame within the plane $\theta = \pi/2$ is characterized by the three-velocity with the tetrad components

$$V_{st}^{(1)} = \frac{u^r}{fu^t} = \frac{1}{f}\frac{dr}{dt} = -\frac{P}{\varepsilon}, \qquad (43)$$

$$V_{st}^{(3)} = r\frac{u^\phi}{\sqrt{f}u^t} = \frac{r}{\sqrt{f}}\frac{d\phi}{dt} = \frac{\mathcal{L}\sqrt{f}}{r\varepsilon}. \qquad (44)$$

Now, for $V_{st}^2 = V_{st}^{(1)2} + V_{st}^{(3)2}$ we have

$$V_{st}^2 = \frac{P^2 + \frac{\mathcal{L}^2}{r^2}f}{\varepsilon^2} = \frac{\varepsilon^2 - f}{\varepsilon^2}. \qquad (45)$$

6. OBSERVERS IN THE LEMAÎTRE FRAME

Now, we shall consider freely falling observers. Let $\varepsilon = 1$ and $\mathcal{L} = 0$. We will call this observer the Lemaître frame observer for brevity, though we mostly use the $(\tilde{t}, r)$ coordinates of the GP metric (12) in this section. For such an observer, in the original coordinates (t, r),

$$U^t = \dot{t} = 1/f, \qquad (46)$$

$$U^r = \dot{r} = -\sqrt{1-f} = -v. \qquad (47)$$

Here we have used (1) and (3), (5) with $\varepsilon = 1$, $L = 0$. We denote the corresponding tetrad by $h_{(a)}^\mu$. In the coordinates (t, r, θ, ϕ) it has the form

$$h_{(0)\mu} = (-1, -v/f, 0, 0), \qquad (48)$$

$$h_{(1)\mu} = (v, 1/f, 0, 0), \qquad (49)$$

$$h_{(3)\mu} = r(0, 0, 0, 1). \qquad (50)$$

In the coordinates $(\tilde{t}, r, \theta, \phi)$, we have

$$h_{(0)\mu} = (-1, 0, 0, 0), \qquad (51)$$

$$h_{(1)\mu} = (v, 1, 0, 0). \qquad (52)$$

Let some particle move with the four-velocity u^μ. By the definition (24),

$$V^{(i)} = -\frac{h_{(i)\mu} u^\mu}{h_{(0)\mu} u^\mu}. \qquad (53)$$

After a simple calculation, we obtain that $V^{(1)}$ satisfies

$$\frac{dr}{d\tilde{t}} = -v + V^{(1)}, \qquad (54)$$

thus returning to Eq. (5.3) of [5]. This gives us an intuitive interpretation of a full radial velocity (the rate of change of the radial coordinate r with respect to the Lemaître time $\tilde{t}$) as a sum of the flow velocity taken with negative sign and a radial peculiar velocity. Note the difference between (54) and the general result of this type for synchronous systems (38), which is still valid if we change the notation for the radial spatial variable from χ, usually used in cosmology, to ρ as it is accepted for the Lemaître metric (18). Namely, the general formula (38) with the same form of the right-hand side as in (54) deals with the rate of change of a proper distance l, not the radial coordinate r. However, in the Lemaître frame the proper distance between two points at the same radius is equal simply to the difference of their values of r—we can see from (12) that sections of constant time $\tilde{t}$ are flat. That is why in this particular frame the radial coordinate r (instead of such a nonlocal entity as the proper distance l) appears in the left-hand side of (54), and the Lemaître frame appears to be especially useful in visualizing both inside and outside regions of a black hole in a single picture. The sign of v here should be reversed (in contrast to the cosmological case) since the free flow motion is directed towards lower values of r, while in the cosmological case the Hubble flow makes the radial distances increase.

For a particle in the flow $V^{(1)} = 0$. In the general case, when a peculiar motion is present, Eq. (15) gives us

$$V^{(1)} = \frac{\varepsilon v - P}{\varepsilon - Pv}. \qquad (55)$$

Taking also into account (21) and (22), one can check that $vd\rho/d\tilde{t}$ (the radial peculiar velocity of (38)) coincides with (55), as should be the case.

The inverse formula reads

$$P = \varepsilon \frac{v - V^{(1)}}{1 - vV^{(1)}}. \qquad (56)$$

These formulae relate the radial peculiar velocity with the integrals of motion.

For the component $V^{(3)}$ we obtain from (51), (50), and (53) that

$$V^{(3)} = r\frac{u^\phi}{u^{\tilde{t}}} = r\frac{d\phi}{d\tilde{t}}. \qquad (57)$$

The reason why this formula is so simple is again the fact that the sections of constant time $\tilde{t}$ are flat, and we can use the standard relations of Euclidean geometry.

For free motion, using (16) and (15), we have

$$V^{(3)} = \frac{\mathcal{L}}{r}\frac{1 - v^2}{\varepsilon - vP} = \frac{\mathcal{L}}{r}\frac{f}{\varepsilon - vP}. \qquad (58)$$

From (55) and (58) we can obtain the relation that will be used below:

$$V^{(3)2} = \frac{\mathcal{L}^2/r^2}{1 + \mathcal{L}^2/r^2}(1 - V^{(1)2}), \qquad (59)$$

whence

$$\frac{1}{1 - v_p^2} = \frac{1 + \mathcal{L}^2/r^2}{1 - V^{(1)2}}, \qquad (60)$$

where

$$v_p^2 \equiv V^{(1)2} + V^{(3)2}. \qquad (61)$$

Now it is possible to express the energy ε through the components of the peculiar velocity. Indeed, it follows from (56) that

$$\varepsilon(v - V^{(1)}) = P(1 - vV^{(1)}). \qquad (62)$$

Taking the square and recalling (5), we obtain

$$\varepsilon^2 = \frac{(1 - vV^{(1)})^2}{1 - v_p^2}, \qquad (63)$$

where (60) was used. Thus,

$$\varepsilon = \frac{1 - vV^{(1)}}{\sqrt{1 - v_p^2}}. \qquad (64)$$

Here, we chose the sign to have $\varepsilon > 0$ in the limit $v \to 0$.

Note that independently of the angular velocity, $\varepsilon = 0$ always corresponds to $vV^{(1)} = 1$. We will consider such particles further in the next section. They can exist only inside the horizon where v exceeds the speed of light, so $V^{(1)}$ is subliminal, as it should be for a velocity having a direct physical meaning. Equation (64) gives us also a simple criterion for the

energy to be negative in terms of the flow and radial peculiar velocity: $\varepsilon < 0$ is equivalent to $V^{(1)} > 1/v$.

Also, from (56) and (64), a useful relation follows:

$$P = \frac{v - V^{(1)}}{\sqrt{1 - v_p^2}}. \qquad (65)$$

Then, one can also rewrite (58) as

$$V^{(3)} = \frac{\mathcal{L}}{r}\sqrt{1 - v_p^2}. \qquad (66)$$

To finish this section, we also relate the velocities with respect to the Lemaître and static frames.

As follows from (43)–(45) and (55),

$$V^{(1)} = \frac{v + V_{\text{st}}^{(1)}}{1 + vV_{\text{st}}^{(1)}}. \qquad (67)$$

In the case under discussion, $V_{\text{st}}^{(1)}$ is proportional to $-P$ according to (43), so $V_{\text{st}}^{(1)} < 0$ (the motion is towards the black hole). We can recognize in (67) the Lorentz composition law for collinear velocities v and $V_{\text{st}}^{(1)}$.

Also, taking into account (44) we obtain

$$V^{(3)} = V_{\text{st}}^{(3)} \frac{\sqrt{f}}{1 + vV_{\text{st}}^{(1)}} = V_{\text{st}}^{(3)} \frac{\sqrt{1 - v^2}}{1 + vV_{\text{st}}^{(1)}}, \qquad (68)$$

which represents the Lorentz composition law for perpendicular velocities.

The Inverse formulas read

$$V_{\text{st}}^{(1)} = \frac{V^{(1)} - v}{1 - vV^{(1)}}, \qquad (69)$$

$$V_{\text{st}}^{(3)} = \frac{V^{(3)}\sqrt{f}}{1 - vV^{(1)}}. \qquad (70)$$

7. LIMITING TRANSITIONS

Let us consider the horizon limit, when $f \to 0$, $v \to 1$. We first consider the case of $\varepsilon > 0$. Then,

$$P = \varepsilon - \frac{f}{2\varepsilon}\left(1 + \frac{\mathcal{L}^2}{r_+^2}\right) + ..., \qquad (71)$$

$$v = 1 - \frac{f}{2} + ... \qquad (72)$$

Then, it follows from (55) and (58) that

$$V^{(1)} \to \frac{1 - \varepsilon^2 + \mathcal{L}^2/r_+^2}{1 + \varepsilon^2 + \mathcal{L}^2/r_+^2}, \qquad (73)$$

$$V^{(3)} \approx \frac{2\mathcal{L}\varepsilon}{r_+(\varepsilon^2 + 1 + \mathcal{L}^2/r_+^2)}. \qquad (74)$$

We remind a reader that the velocity with respect to a static frame at the horizon always has components in this limit (see Eqs. (43), (44))

$$|V_{\text{st}}^{(1)}| \to 1, \qquad (75)$$

$$V_{\text{st}}^{(3)} \to 0. \qquad (76)$$

On the contrary, the velocity with respect to the Lemaître frame at the horizon does depend on the particular motion of the particle in question. In particular, the radial velocity can take any value in the range $0 \le |V^{(1)}| < 1$. For pure radial motion, the left limit is realized for $\varepsilon = 1$, and this corresponds to a particle comoving with the Lemaître frame, so the peculiar velocity is evidently zero. The upper limit is a limiting case for $\varepsilon \to 0$. As for the angular part of the peculiar velocity, near the horizon $V_{\text{st}}^{(3)}$ is small, so a particle hits a horizon radially. However, a small $V_{\text{st}}^{(3)}$ near the horizon is compensated by a significant Lorentz boost (68) with (75). As a result, in the Lemaitre system $V^{(3)}$ is finite and nonzero. It is a bright manifestation of the known relativistic effect according to which a vector, not collinear to the direction of motion, rotates under a Lorentz transformation.

If the metric has an inner horizon, the case of $\varepsilon < 0$ for a particle approaching the horizon is possible as well. The asymptotics for a negative energy are totally different, since now

$$P \to -\varepsilon, \qquad (77)$$

and Eq. (16) gives us $d\phi/d\tilde{t} \to 0$, so that

$$V^{(3)} \to 0. \qquad (78)$$

As for the radial motion, (55) in the limit (77) gives

$$V^{(1)} \to 1, \qquad (79)$$

so that $v_p \to 1$.

Near the singularity, $r \to 0$, $f \to -\infty$, and for pure radial motion

$$v \approx P \approx \sqrt{|f|}, \qquad (80)$$

which gives us from (55) that

$$V^{(1)} \approx \frac{1 - \varepsilon}{v} \approx \frac{1 - \varepsilon}{\sqrt{|f|}} \to 0. \qquad (81)$$

For a nonradial motion with $\mathcal{L} \ne 0$, we have a different asymptopics of P:

$$P \approx \sqrt{|f|}\frac{\mathcal{L}^2}{r^2} \gg v, \qquad (82)$$

and it follows from (55) that

$$V^{(1)} \approx \frac{1}{v} \approx \frac{1}{\sqrt{|f|}} \to 0. \qquad (83)$$

This means that any initial differences in radial motion for different particles disappear near the singularity, and the radial motion of any particle tends to the motion of the frame. On the contrary, if $\mathcal{L} \neq 0$, then

$$V^{(3)} \approx \frac{\mathcal{L}}{|\mathcal{L}|} = \pm 1, \qquad (84)$$

so pure radial motion appears to be unstable—an arbitrary small deviation grows infinitely and results in an ultrarelativistic motion in an angular direction. If initially the directions of the vectors $\mathcal{L}$ are distributed randomly, the corresponding particles have mutual ultrarelativistic relative velocities near a singularity.

8. ANOTHER SYNCHRONOUS SYSTEM INSIDE A HORIZON

In this section we consider another synchronous system (see, e.g., [9] and page 25 of the book [22]), which can be obtained from static coordinates when using them inside the horizon. In this region the initial signature of the metric changes, and the coordinate r becomes timelike, as well as the coordinate t changes its nature and becomes spacelike. Therefore, it is appropriate to make a simple change of notations defining a new time coordinate $T = -r$, a new spatial coordinate $y = t$ and a function $g = -f$. Note that g is a function of the new time inside the horizon,

$$ds^2 = -\frac{dT^2}{g} + gdy^2 + T^2 d\Omega^2. \qquad (85)$$

To turn this form of the metric to an explicitly synchronous form, we need to make an additional time reparametrization,

$$d\hat{t} = \frac{dT}{\sqrt{g}}, \qquad (86)$$

so that the metrics becomes
$$ds^2 = -d\hat{t}^2 + gdy^2 + T^2(\hat{t})d\Omega^2. \qquad (87)$$

The metric coefficients depend only on time, so this is a cosmology-like metrics. Indeed, taken by itself it can be considered as a metric of a Kantowski−Sachs (KS) Universe. The obvious cosmological intuition makes it natural to treat particles with constant y and angular coordinates as being at rest and consider peculiar velocities with respect to them.

The geodesic equations give us for the 4-velocity components

$$u^{\hat{t}} = \frac{d\hat{t}}{d\tau} = \frac{P}{\sqrt{g}}, \qquad (88)$$

$$u^y = \frac{dy}{d\tau} = -\frac{\varepsilon}{g}, \qquad (89)$$

$$u^\phi = \frac{d\phi}{d\tau} = \frac{\mathcal{L}}{T^2}. \qquad (90)$$

As usual,

$$u_\mu u^\mu = \frac{\mathcal{L}^2}{T^2} + \frac{\varepsilon^2}{g} - \frac{P2}{g} = -1. \qquad (91)$$

We see from (89) that the coordinate y is constant if $\varepsilon = 0$. We are familiar with this condition from Section 6 where we have shown (see the discussion after Eq. (64)) that such particles have peculiar velocities *with respect to the Lemaître frame* equal to the inverse flow velocity of the Lemaître frame. In the Kantowski-Sachs frame, considered in the present section, the peculiar velocities of particles with $\varepsilon = 0$ are zero by definition.

For the general situation, we fix the tetrad in the coordinates $(\hat{t}, y, \theta, \phi)$ as

$$\hat{h}^\mu_{(0)} = (1, 0, 0, 0), \qquad (92)$$

$$\hat{h}_{(0)\mu} = -(1, 0, 0, 0), \qquad (93)$$

$$h^\mu_{(1)} = (0, \frac{1}{\sqrt{g}}, 0, 0), \qquad (94)$$

$$\hat{h}_{(1)\mu} = (0, \sqrt{g}, 0, 0), \qquad (95)$$

$$\hat{h}_{(3)\mu} = (0, 0, 0, |T|). \qquad (96)$$

A simple calculation using (88)−(90) gives

$$\hat{V}^{(1)} = \frac{\hat{h}_{(1)\mu} u^\mu}{-\hat{h}_{(0)\mu} u^\mu} = \sqrt{g}\frac{dy}{d\hat{t}} = -\frac{\varepsilon}{P}, \qquad (97)$$

$$\hat{V}^{(3)} = \frac{\mathcal{L}\sqrt{g}}{|T|P}. \qquad (98)$$

For the Schwarzschild metric and pure radial motion, our Eq. (97) agrees with Eqs. (18), (A.9) of [23].

The absolute value of the peculiar velocity

$$\hat{V}^2 = \hat{V}^{(1)2} + \hat{V}^{(3)2} \qquad (99)$$

can be presented as

$$\hat{V}^2 = \frac{1}{P^2}\left(\varepsilon^2 + \frac{\mathcal{L}^2}{T^2}g\right) = \frac{P^2 - g}{P^2}, \qquad (100)$$

or, directly in terms of the conserved quantities,

$$\hat{V}^2 = \frac{\frac{\varepsilon^2}{g} + \mathcal{L}^2/T^2}{1 + \varepsilon^2/g + \mathcal{L}^2/T^2}. \qquad (101)$$

If $\varepsilon = 0$, evidently $\hat{V}^{(1)} = 0$, and we have a simple formula for the angular velocity

$$\hat{V}^{(3)} = \frac{\mathcal{L}}{\sqrt{\mathcal{L}^2 + T^2}}. \qquad (102)$$

If $\mathcal{L} = 0$, $\hat{V}^{(3)} = 0$, so that the condition for zero peculiar velocity in terms of the conserved quantities is $\varepsilon = 0$ and $\mathcal{L} = 0$.

It is easy to see that for any nonzero ε near the horizon, where $g \to 0$,

$$|\hat{V}^{(1)}| \to 1, \hat{V}^{(3)} \to 0. \qquad (103)$$

On the other hand, near the singularity $g \to \infty$, and if $\mathcal{L} = 0$, then P diverges as $\sqrt{g}$, and it follows from (97) that

$$\hat{V}^{(1)} \approx -\frac{\varepsilon}{\sqrt{g}} \approx -\frac{\varepsilon}{v} \to 0. \qquad (104)$$

If $\mathcal{L} \neq 0$, then the asymptotic of P near a singularity is different,

$$P \approx \sqrt{g}\left|\frac{\mathcal{L}}{T}\right|, \qquad (105)$$

so that

$$V^{(1)} \approx -\frac{\varepsilon}{\sqrt{g}}\left|\frac{T}{\mathcal{L}}\right| \to 0. \qquad (106)$$

For the angular component we have

$$\hat{V}^{(3)} \to \text{sgn}\mathcal{L} = \pm 1. \qquad (107)$$

The "cosmological" intuition matches with these results completely. Since the radial "scale factor" g diverges, the radial peculiar velocity tends to zero, while the angular peculiar velocity tends to the speed of light since the angular "scale factor" T^2 tends to zero. We see that near the singularity the limiting values of the radial and angular components of the peculiar velocity with respect to the Kantowski-Sachs frame (106), (107) are the same as with respect to the Lemaître frame (83), (84) though the forms of asymptotics are different.

9. REDSHIFTS INSIDE A BLACK HOLE

The redshift measured by an observer freely falling inside a black hole depends rather nontrivially on the angular motion of a photon and that of an observer himself. We start with reminding a formal derivation of the redshift (for a full treatment see [10]), and then explain how different limiting cases can be understood intuitively using the formalism of peculiar velocities.

A photon is characterized by the wave vector k_μ. Let us consider the metric (1). It is assumed that $k_t = -\omega_0$, where ω_0 has the meaning of the frequency measured at infinity. And, $k_\phi = -\omega_0 l$ has the meaning of the angular momentum. A photon is propagating from infinity inwards. The normalization condition $k_\mu k^\mu = 0$ gives us outside the event horizon in the coordinates (t, r, ϕ)

$$k^\mu = \left(\frac{\omega_0}{f}, -Q, \frac{\omega_0 l}{r^2}\right),$$

$$k_\mu = \left(-\omega_0, \frac{-Q}{f}, \omega_0 l\right), \qquad (108)$$

where

$$Q = \omega_0\sqrt{1 - \frac{fl^2}{r^2}}. \qquad (109)$$

Below we will be interested in the properties of particles inside the horizon. The corresponding formulas can be obtained by the substitution $f = -g < 0$.

Under the horizon, we have in the coordinates (T, y, ϕ)

$$k^\mu = \left(Q, \frac{\omega_0}{g}, \frac{\omega_0 l}{T^2}\right), \qquad (110)$$

$$k_\mu = \left(-\frac{Q}{g}, \omega_0, \omega_0 l\right), \qquad (111)$$

where now $k_y = \omega_0$ is conserved, $\omega_0 > 0$, and

$$Q = \omega_0\sqrt{1 + \frac{g}{T^2}l^2}. \qquad (112)$$

It follows from (109)–(112) that

$$\omega = \frac{PQ}{g} - \omega_0\frac{\varepsilon}{g} - \omega_0 l\frac{\mathcal{L}}{r^2}. \qquad (113)$$

This general formula leads to a number of different asymptotics near a singularity, depending on the motion of an observer and the photon observed, resulting in infinite redshifts, infinite blueshifts or finite redshifts. They are summarized in [10]. Here we interpret these results from the viewpoint of peculiar velocities.

Using (64)–(66), we obtain from (113)

$$\omega = \frac{Q(v - V^{(1)}) - \omega_0(1 - vV^{(1)})}{g\sqrt{1 - v_p^2}}$$
$$- \frac{\omega_0 l}{r}\frac{V^{(3)}}{\sqrt{1 - v_p^2}}. \qquad (114)$$

Let $l = 0$. Then $Q = \omega_0$ and

$$\frac{\omega}{\omega_0} = \frac{(v - V^{(1)}) - (1 - vV^{(1)})}{g\sqrt{1 - v_p^2}}. \qquad (115)$$

Substituting (10), we obtain

$$\frac{\omega}{\omega_0} = \frac{(1 + V^{(1)})}{(1 + v)\sqrt{1 - v_p^2}}. \qquad (116)$$

If the motion is purely radial, $V^1 = V, v_p^2 = V^2$, we return to our known formula (11.3) from [5] (Eq. (78) of its arxiv version)

$$\frac{\omega}{\omega_0} = \frac{\sqrt{1 + V}}{(1 + v)\sqrt{1 - V}}. \qquad (117)$$

The peculiar velocity notation gives us the possibility to interpret some results on redshifts observed by a freely falling observer near the singularity. Indeed, the combined redshift is

$$(1 + z) = (1 + z_g)(1 + z_d), \qquad (118)$$

where

$$1 + z_g = \frac{1}{1 + v}, \qquad 1 + z_d = \frac{\sqrt{1 + V}}{\sqrt{1 - V}}. \qquad (119)$$

The gravitational part z_g is the shift observed by a particle in the flow (with a zero peculiar velocity), which has been calculated in the Lemaître frame in [20], and z_d is the Doppler shift due to a nonzero peculiar velocity. In this interpretation Eq. (118) represents an evident combination of the Lorentz redshift (the part which depends on V) and the gravitational redshift (the part which depends on v). Near a singularity, depending on the direction of the photon, we have either an infinite redshift or an infinite blueshift.

The same decomposition can be written also with respect to the KS frame (87), (92)–(96). We continue to consider a radial photon and a radially falling observer ($l = 0$, $\mathcal{L} = 0$). Now the redshift measured by an observer in the flow ($\varepsilon = 0$) has an even simpler form since only the first term in the main formula (113) is nonzero: $\omega/\omega_0 = 1/\sqrt{g}$. For an observer with nonzero ε we get

$$\frac{\omega}{\omega_0} = \frac{P - \varepsilon}{g}. \qquad (120)$$

Using (5) with $\mathcal{L} = 0$ and (97), it can be rewritten in the form

$$\frac{\omega}{\omega_0} = \frac{1}{\sqrt{g}} \frac{\sqrt{1 + \hat{V}^{(1)}}}{\sqrt{1 - \hat{V}^{(1)}}}, \qquad (121)$$

giving again a combination of gravitational and Lorentz redshifts, but with respect to the KS frame now.

Now we consider a situation with a nonradial photon ($l \neq 0$). It is easier to work in the KS frame. For a radially falling observer near the singularity we have $g \to \infty$, and recalling the definitions of P and Q, we see that the first term in (113) dominates, giving an infinite blueshift with the asymptotic

$$\frac{\omega}{\omega_0} \approx \frac{l}{|T|}. \qquad (122)$$

Let us now consider the case of a nonradial motion of an observer as well. In this case the infinite gravitational blueshift (122) is combined with an infinite Doppler redshift since the angular peculiar velocity approaches 1. Indeed, near the singularity, the term with ε in (101) is negligible, $V^{(1)} \to 0$ according to

(104), so Eq. (102) becomes a good approximation. It can be rewritten as

$$\mathcal{L} = \frac{|T|\hat{V}^{(3)}}{\sqrt{(1 - \hat{V}^{(3)})(1 + \hat{V}^{(3)})}}. \qquad (123)$$

Taking into account that now $T \to 0$, for a finite $\mathcal{L}$ the velocity component $|\hat{V}^{(3)}| \to 1$ in agreement with (107). Let for definiteness $\mathcal{L} > 0$. Then $\hat{V}^{(3)} \to +1$, and

$$\sqrt{1 - \hat{V}^{(3)}} \approx |T|/(\mathcal{L}\sqrt{2}). \qquad (124)$$

Let $l > 0$. Taking into account the Doppler factor, we find that the ratio

$$\frac{\omega}{\omega_0} = \frac{l}{|T|}\sqrt{\frac{1 - \hat{V}^{(3)}}{1 + \hat{V}^{(3)}}} \approx \frac{l}{|T|}\frac{|T|}{\mathcal{L}\sqrt{2}\sqrt{2}} = \frac{l}{2L} \quad (125)$$

is finite and coincides with the result obtained earlier in Eq. (47) of [10].

If $l < 0$, the velocities of an observer and a photon are pointed in opposite directions. Then, instead of (125), we have

$$\frac{\omega}{\omega_0} = \frac{|l|}{|T|}\sqrt{\frac{1 + \hat{V}^{(3)}}{1 - \hat{V}^{(3)}}} \approx \frac{2\mathcal{L}|l|}{T^2} \to \infty. \qquad (126)$$

This agrees with Eq. (48) of [10].

10. CONCLUSIONS

In the present paper we have considered the 3-velocity of an object falling freely into a black hole with respect to two different freely falling frames. By direct analogy with the cosmological terminology, we call this 3-velocity a peculiar velocity. We have determined the dependence of peculiar velocity components of freely falling objects on the integrals of motion and considered their asymptotics near horizons and a singularity. We have developed rather a general approach that can, in principle, be applied both in cosmology and the physics of black holes, including their interiors. Now, the concept of a peculiar velocity is exploited to include nonradial motions. This, in particular, enabled us to give a simple qualitative explanation of the phenomenon of red/blue shifts inside a black hole, especially near the singularity. It agrees with the results of direct calculations done earlier. We have also shown how the general formula for the frequency shift in a radial fall admits a simple decomposition to the gravitational and kinematic parts for two considered frames. Since we considered geodesic motion in a fixed static spherically symmetric metric (without demanding that this metric is a solution of GR equations), our results are valid for any black hole solutions of the form (1) in any metric theory with geodesic motion of particles and photons.

FUNDING

REFERENCES

1. W. Rindler, *Essential Relativity* (Springer-Verlag, New York, 1977), p. 213.

2. T. Davis, Ch. Lineweaver, and J. Webb, "Solution to the tethered galaxy problem in an expanding universe and the observation of receding blueshifted objects," Am. J. Phys. **71**, 358−364 (2003); astro-ph/0104349.

3. A. J. S. Hamilton and J. P. Lisle, 'The river model of black holes," Am. J. Phys. **76**, 519 (2008); arXiv:gr-qc/0411060.

4. A. J. S. Hamilton and G. Polhemus, "Stereoscopic visualization in curved spacetime: seeing deep inside a black hole," New J. Phys. **12**, 123027 (2010); arXiv:1012.4043.

5. A. V. Toporensky and O. B. Zaslavskii, Zero-momentum trajectories inside a black hole and high energy particle collisions," JCAP **12**, 063 (2019); arXiv: 1808.05254.

6. M. Bañados, J. Silk, and S. M. West, "Kerr black holes as particle accelerators to arbitrarily high energy," Phys. Rev. Lett. **103**, 111102 (2009); arXiv: 0909.0169.

7. A. V. Toporensky and O. B. Zaslavskii, "On strategies of motion under the black hole horizon," Int. J. Mod. Phys. D **29**, 2030003 (2020); arXiv: 1905.02150.

8. I. Perez-Roman, N. Bretón, "The region interior to the event horizon of the regular Hayward black hole," Gen. Rel. Grav. **50**, 64 (2018); arXiv: 1805.00906.

9. I. D. Novikov, "Note on the space-time metric inside the Schwarzschild singular sphere," Sov. Astron. **5**, 423 (1961).

10. O. B. Zaslavskii, 'Redshift/blueshift inside the Schwarzschild black hole," Gen. Rel. Grav. **52**, 37 (2020); arXiv: 1910.00669.

11. L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics, Vol. 2: The Classical Theory of Fields* (Nauka, Moscow, 1988; Pergamon, Oxford, 1971).

12. K. A. Bronnikov, I. G. Dymnikova, and E. Galaktionov, "Multi-horizon spherically symmetric space-times with several scales of vacuum energy," Class. Quantum Grav. **29**, 095025 (2012); arXiv: 1204.0534.

13. K. A. Bronnikov and S. G. Rubin, *Black Holes, Cosmology, and Extra Dimensions* (World Scientific, 2013).

14. V. J. Bolós, "Intrinsic definitions of relative velocity in general relativity," Commun. Math. Phys. **273**, 217−236 (2007); gr-qc/0506032.

15. A. Radosz, P. Gusin, A. T. Augousti, and F. Formalik, "Inside spherically symmetric black holes or how a uniformly accelerated particle may slow down," Eur. Phys. J. C **79**, 876 (2019).

16. J. M. Bardeen, W. H. Press, and S. A. Teukolsky, Rotating black holes: locally nonrotating frames, energy extraction, and scalar synchrotron radiation," Astroph. J. **178**, 347 (1972).

17. T. M. Davis and Ch. H. Lineweaver, "Expanding confusion: common misconceptions of cosmological horizons and the superluminal expansion of the Universe," Publ. Astron. Soc. Australia, **21**, 97 (2004); astro-ph/0310808.

18. M. Chodorowski, "The kinematic component of the cosmological redshift," MNRAS **413**, 585 (2011); arXiv: 0911.3536.

19. E. D. Emtsova and A. V. Toporensky, Velocities of distant objects in general relativity revisited," Grav. Cosmol. **26**, 50 (2020); arXiv: 1901.03602.

20. A. V. Toporensky, O. B. Zaslavskii, and S. Popov, Unified approach to redshift in cosmological/black hole spacetimes and synchronous frame, Eur. J. Phys. **39**, 015601 (2018); arXiv:1704.08308.

21. J. P. S. Lemos and D. L. F. G. Silva, "Maximal extension of the Schwarzschild metric: From Painlev́-Gullstrand to Kruskal–Szekeres, arXiv: 2005.14211.

22. V. P. Frolov and I. D. Novikov, *Physics of Black Holes* (Kluwer Academic, Dordrecht, 1998).

23. A. T. Augousti, P. Gusin, B. Kuśmierz, J. Masajada, and A. Radosz, "On the speed of a test particle inside the Schwarzschild event horizon and other kinds of black holes," Gen. Rel. Grav. **50**, 131 (2018).