ISSN 0202-2893, Gravitation and Cosmology, 2021, Vol. 27, No. 2, pp. 169–177. © Pleiades Publishing, Ltd., 2021.

LRS Bianchi I Model with Bulk Viscosity in $f(R,T)$ Gravity

Siwaphiwe Jokweni[1*], Vijay Singh[1], and Aroonkumar Beesham[1,2***]**

[1]*Department of Mathematical Sciences, University of Zululand, Private Bag X1001, Kwa-Dlangezwa, KwaZulu-Natal 3886, South Africa*

[2]*Faculty of Natural Sciences, Mangosuthu University of Technology, Umlazi, South Africa*

Received October 5, 2020; revised December 30, 2020; accepted January 20, 2021

Abstract—Locally rotationally symmetric Bianchi type-I viscous and nonviscous cosmological models are explored in general relativity (GR) and in $f(R,T)$ gravity. Solutions are obtained by assuming that the expansion scalar is proportional to the shear scalar, which yields a constant value for the deceleration parameter ($q = 2$). Constraints are obtained by requiring physical viability of the solutions. A comparison is made between viscous and nonviscous models, and between the models in GR and in $f(R,T)$ gravity. The metric potentials remain the same in GR and in $f(R,T)$ gravity. Consequently, the geometrical behavior of the $f(R,T)$ gravity models remains the same as in GR. It is found that $f(R,T)$ gravity or bulk viscosity do not affect the behavior of effective matter which acts as a stiff fluid in all models. The individual fluids have a very rich behavior. In one of the viscous models, the matter either follows a semi-realistic EoS or exhibits a transition from stiff matter to phantom, depending on the values of the parameter. In another model, the matter describes radiation, dust, quintessence, phantom, and the cosmological constant for different values of the parameter. In general, $f(R,T)$ gravity diminishes the effect of bulk viscosity.

DOI: 10.1134/S0202289321020079

1. INTRODUCTION

Our universe, at a sufficiently large scale, is homogeneous and isotropic. However, on smaller scales it is neither homogeneous nor isotropic. There are theoretical predictions that the early universe was also highly anisotropic, which has been supported by many observations [1–7]. Among the simplest homogeneous anisotropic models, Bianchi type-I (B-I) models play an outstanding role in understanding the essential features of the early universe. Also, in a universe filled with matter, the initial anisotropy in a B-I universe quickly dies away, and the universe eventually becomes isotropic. Since the present-day universe is isotropic, the prominent features of the B-I models make them a prime candidate for studying the possible effects of anisotropy in the early evolution of the universe. In particular, the locally rotationally symmetric (LRS) B-I space-time is one of the simplified versions of the B-I model. In light of its importance, many researchers have studied the LRS B-I models in various contexts (see [8–12] and references therein).

Though, the present-day universe undergoes an accelerated expansionary evolution, and bulk viscosity plays a very vital role in explaining this phenomenon. However, it does not exclude the existence of a decelerating phase in the early history of our universe. Mak and Harko [13] studied a causal bulk-viscous cosmological fluid in a flat constantly decelerating B-I space-time model and showed that the model leads to a self-consistent thermodynamic description which could characterize a well-determined period of the evolution of our universe. Therefore, decelerating models have their own importance to understand the early evolution of the universe.

On the other hand, although a perfect fluid satisfactorily accounts for the large-scale matter distribution in the universe, a realistic cosmological scenario requires consideration of matter other than a perfect fluid. Some observed physical phenomena such as the large entropy per baryon ratio and the noteworthy degree of isotropy of the cosmic background radiation suggest dissipative effects in cosmology. Entropy producing processes and dissipative effects play a very significant role in the early evolution of the universe. In fluid cosmology, the simplest phenomenon associated with a nonvanishing entropy production is bulk viscosity (for more detail see the review article [14] and references therein).

There are several processes which generates viscous effects (see [15] for a list of some principal processes). The presence of bulk viscosity inaugurates

*E-mail: siwaphiwe14@gmail.com
**E-mail: gtrcosmo@gmail.com
***E-mail: beeshama@unizulu.ac.za

many interesting features in the dynamics of the universe. Initially, it was proposed that neutrino viscosity could smooth out initial anisotropies and result in the isotropic universe that we see today. The presence of bulk viscosity can avert the Big-Bang singularity too. Bulk viscosity can also explain a phenomenological process of particle creation in a strong gravitational field. The back-reaction effects of string creation can be modeled by a bulk-viscous fluid. It has attracted much interest across the field of cosmology, and many investigators have pondered the effects of bulk viscosity in different contexts (see, for example, [16–30] and references therein). Most of these investigations are based on isotropic cosmology. However, in the search for a realistic picture of the early universe, a large number of studies have been done in anisotropic space-times as well [31–46]. The general B-I space-time models have also been studied by many authors [34, 41, 47–55]. More specifically, some authors [56–58] presented LRS B-I bulk viscous cosmological models.

On the other hand, the shortcomings of the ΛCDM model have confronted many authors to seek various alternatives to the fundamental theories of cosmology and astrophysics, which include modifications of general relativity (GR) itself by imposing extra terms in the Einstein-Hilbert action. The modified theories of gravity include higher-derivative theories, Gauss-Bonnet $f(G)$ gravity, $f(R)$ theory, $f(T)$ and $f(R,T)$ gravity theories. In the past decade, $f(R,T)$ gravity has attracted the attention of many researchers wishing to look at many astrophysical and cosmological phenomena in the context of this theory (see [11] for a broad list of references).

Mahanta [59] considered a bulk-viscous LRS B-I model in $f(R,T)$ gravity. The author assumed an expansion scalar proportional to the shear scalar to solve the field equations. However, due to wrong signs considered in the field equations, his solutions are mathematically and physically invalid. Soon after, Shamir [60] presented solutions for the same model but without bulk viscosity. Later on, Sahoo and Reddy [61] considered an LRS B-I model containing bulk-viscous matter in $f(R,T)$ gravity using a different deceleration parameter. Very recently, Yadav et al. [62] discussed the general B-I bulk-viscous model in $f(R,T) = R + \lambda RT$ gravity with a hybrid expansion law of the scale factor.

Our purpose in this paper is to reconsider the model formulated by Mahanta [59] with correct field equations. We eloquently explore the behavior of the model keeping in view the physical viability of the model. Before considering the $f(R,T)$ gravity model, we first discuss the solutions in GR in the presence and absence of bulk viscosity. In this way, we distinguish the outcomes of the $f(R,T)$ gravity model with that of GR and recognize the role of $f(R,T)$ gravity and bulk viscosity.

Also, Mahanta [59], in $f(R,T) = R + 2\lambda T$ model, merely found the expression for the coefficient of bulk viscosity. While, in $f(R,T) = R + 2\lambda T^2$ model, the author also studied the behavior of matter by considering two different forms of the bulk viscosity coefficient. We implement this approach to the $f(R,T) = R + 2\lambda T$ model. Therefore, our solutions are also an extension of Mahanta's work. It is worthwhile to mention that though a single matter content is considered in $f(R,T)$ gravity, due to the coupling between the trace, T and matter, some extra terms appear in the field equations. We treat these additional terms as coupled matter. We study the nature of this additional matter and its contribution to the cosmic evolution.

The work is organized as follows. An LRS B-I space-time model, in the presence and absence of bulk viscosity within the framework of GR, is studied in Section 2 and in its subsections. The $f(R,T) = R + 2\lambda T$ gravity viscous and nonviscous models are explored in Section 3 and in its subsections. The findings are accumulated in the concluding Section 4.

2. THE MODEL IN EINSTEIN'S GRAVITY

The spatially homogeneous and anisotropic LRS B-I space-time metric is given as

$$ds^2 = dt^2 - A^2dx^2 - B^2(dy^2 + dz^2), \qquad (1)$$

where A and B are the scale factors and are functions of the cosmic time t.

The average scale factor and the average Hubble parameter, respectively, are defined as

$$a = (AB^2)^{1/3}, \qquad (2)$$

$$H = \frac{1}{3}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right), \qquad (3)$$

where a dot represents a derivative with respect to t. We consider the energy-momentum tensor of matter as

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij}, \qquad (4)$$

where ρ is the energy density, and p is the thermodynamic pressure of the matter. In comoving coordinates, $u^i = \delta^i_0$, where u_i is the four-velocity of the fluid that satisfies the condition $u_i u^j = 1$.

The Einstein field equations are given by

$$R_{ij} - \frac{1}{2}R g_{ij} = T_{ij}, \qquad (5)$$

where $8\pi G = 1 = c$ are assumed. The field equations (5) for the metric (1), with the consideration of the energy-momentum tensor (4), yield

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\dot{A}\dot{B}}{AB} = \rho, \tag{6}$$

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\ddot{B}}{B} = -p, \tag{7}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -p. \tag{8}$$

These equations contain four unknowns, namely, A, B, p, ρ. Therefore, in order to find exact solutions, one supplementary constraint is required.

Mahanta [59] considered the expansion scalar, $\theta(= 3H)$, to be proportional to the shear scalar σ given by

$$\sigma^2 = \frac{1}{3}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)^2,$$

which leads to

$$A = B^n, \tag{9}$$

where n is an arbitrary constant. From (7) and (8), by the use of (9), one gets

$$\frac{\ddot{B}}{B} + (n+1)\left(\frac{\dot{B}}{B}\right)^2 = 0, \tag{10}$$

which gives

$$B = \beta\left[(n+2)t + c_2\right]^{1/(n+2)}. \tag{11}$$

Consequently,

$$A = \alpha\left[(n+2)t + c_2\right]^{n/(n+2)}. \tag{12}$$

The energy density and pressure become equal,

$$\rho = p = \frac{(1+2n)}{(2+n)^2 t^2}. \tag{13}$$

Hence, the effective matter behaves as stiff fluid. The energy density must be positive for a realistic cosmological scenario which is possible only for $n > -1/2$.

In Section 3 of his paper, Mahanta [59] worked out some geometric parameters, namely, the volume, the expansion scalar and the shear scalar. All these parameters are defined in terms of the metric potentials A and B. We see that the scale factors given in Eqs. (11) and (12) are identical to those of Mahanta's work though we have obtained these metric potentials in GR. In fact, the left-hand side (LHS) of the field equations in GR and in $f(R,T) = R + 2\lambda T$ gravity remains the same, only the right-hand side (RHS) is different. When one simplifies equations (7) and (8) or (19) and (20) in Mahanta's paper, the RHS is canceled out irrespective of the theory or even whatever be the matter content (viscous or nonviscous) of the model. Hence, the metric potentials are independent of $f(R,T)$ gravity and the matter content considered in such formulation. Consequently, all geometric parameters remain independent from $f(R,T)$ gravity and from the matter content. Thus the geometrical behavior of the model remains identical to the model in GR. We refer to [60] for details of geometrical behavior of the model.

2.1. Viscous Model

The energy density of bulk viscous matter remains the same, but the pressure in the energy-momentum tensor (4) for a viscous fluid modifies as

$$\bar{p} = p'_m - \xi\theta, \tag{14}$$

where p'_m is the pressure of matter, and ξ is the coefficient of bulk viscosity.

The field equations for a viscous model remain similar to (6)–(8) except that the pressure p is replaced by the bulk-viscous pressure $\bar{p}$. Therefore, the assumption (9) again leads to the solution (13), i.e., $\rho = \bar{p}$, which is identical to that in the nonviscous model. Hence, the bulk viscosity does not affect the behavior of effective matter, and it acts as stiff matter. However, it is to be noted that the new field equations contain five unknowns, namely, A, B, ρ, p'_m, and ξ. Therefore, to determine exact solutions completely, we require one more constraint other than (9). We have two ways: first, assume an EoS that relates ρ to p'_m, and then determine ξ; and second, assume an explicit form of ξ, determine $\bar{p}$. We shall follow both approaches in the following section.

2.1.1. The behavior of the bulk viscosity coefficient. We assume that matter follows the perfect fluid EoS

$$p'_m = \omega\rho, \tag{15}$$

where $0 \leq \omega \leq 1$ is the EoS parameter.

From (14), the expression for the coefficient of bulk viscosity is obtained as

$$\xi(t) = \frac{(2n+1)(\omega - 1)}{(n+2)^2\, t}. \tag{16}$$

Since we have $n > -1/2$ for the energy density to be positive, the coefficient of bulk viscosity for any kind of matter except stiff matter ($\omega = 1$) remains negative and increases with the evolution of the universe, for example, ultra-relativistic radiation ($\omega = 1/3$), non-relativistic dust ($\omega = 0$) or even vacuum energy ($\omega = -1$). Also, as $\xi \to 0$ when $t \to \infty$, the effect of bulk viscosity disappears at late times. In the case of stiff matter, the coefficient of bulk viscosity vanishes, and the solutions obtained in (13) are recovered.

2.1.2. The behavior of matter. By assuming a perfect fluid EoS, in Sections 3 and 4.1, Mahanta [59] merely obtained an expression for the coefficient of bulk viscosity. However, in Section 4.2, while considering $f(R,T) = \lambda R + \lambda T^2$, the author also considered two different relations between the bulk viscosity coefficient and the expansion scalar to study the properties of matter and viscous fluid. However, in addition to wrong signs in the field equations, there is another flaw in the model of $f(R,T) = \lambda R + \lambda T^2$. The author overdetermined the solutions in this model. One needs two constraints to close the system, but the author used three, i.e., (21) and (61), and the perfect fluid EoS, i.e., $p = \epsilon\rho$, $0 \leq \epsilon \leq 1$. Regardless of overdetermining the solutions, the sign on the RHS of the field equations is also incorrect. Though we are not incorporating this model in the present study, we shall use two assumptions, those considered by Mahanta [59] in his model $f(R,T) = \lambda R + \lambda T^2$. These assumptions are: (i) the coefficient of bulk viscosity is inversely proportional to the expansion scalar, i.e., $\xi\theta = k$, where k is a positive constant; and (ii) the product of the bulk viscosity coefficient and the expansion scalar is directly proportional to the energy density, i.e., $\xi\theta = k_1\rho$, where $k_1 > 0$ is a constant. We consider both in in following cases to examine the nature of matter.

Case (i) $\boldsymbol{\xi\theta = k}$. In this case, the EoS parameter $\omega' = p'_m/\rho$ is

$$\omega' = 1 + \frac{k(2+n)^2 t^2}{1+2n}. \tag{17}$$

At the origin, we have $\omega' = 1$ (stiff matter). Mahanta considered only the case where $k > 0$. If $k > 0$, the EoS parameter starts from $\omega' = 1$ and increases with the evolution. This case corresponds to a semirealistic EoS $p = \varepsilon p$ ($\varepsilon \geq 1$). Many researchers [63–65] studied cosmological models with the semirealistic matter in forward approaches. However, if $k < 0$, the EoS parameter renders an interesting behavior. It exhibits a smooth transition from $\omega' = 1$ (stiff matter) to $\omega' \to -\infty$ (phantom matter). Thus it describes all kinds of known matter (stiff matter, radiation and dust) including the hypothetical form of dark energy (quintessence and phantom) and a cosmological constant as well. Though the model only describes a decelerated universe, the dark energy characteristics anyway do not contradict it because the matter which is showing this characteristic is not the effective matter in this model. We have already seen that the effective matter behaves as a stiff fluid.

Case (ii) $\boldsymbol{\xi\theta = k_1\,\rho}$. The EoS parameter in this case takes a constant value

$$\omega' = 1 + k_1. \tag{18}$$

Hence, if $k_1 > 0$, matter in this case also follows the semi-realistic EoS. On the other hand, if $k_1 < 0$, the model renders a variety of matter depending on the values of k_1, e.g., $\omega' = 1/3$ (radiation) for $k_1 = -2/3$, $\omega' = 0$ (dust) for $k_1 = -1$, $\omega' = -1/3$ (quintessence) for $k_1 = -4/3$, $\omega' = -1$ (cosmological constant) for $k_1 = -2$, and $\omega' < -1$ (phantom) when $k_1 < -1$. If $k_1 = 0$, we have $\omega' = 1$ (stiff matter), which implies $\xi = 0$ as $\theta = 1/t \neq 0$. Hence, in the absence of bulk viscosity, the solutions obtained in (13) are recovered.

3. THE MODEL IN $f(R,T)$ GRAVITY

It is vital to note that ρ and p in Section 2 are the effective energy density and pressure, respectively, while in $f(R,T)$ gravity both physical quantities no longer epitomize the effective energy density and pressure. The coupling between geometry and matter in $f(R,T)$ gravity adds some additional terms visible on the RHS of the field equations. These terms must be treated as matter that can be called coupled matter. Therefore, to distinguish between the main matter and coupled matter, we replace p with p_m and ρ with ρ_m, which represent the primary or main matter. The notations for the energy density and pressure of the coupled matter are defined in Section 3.1.

The field equations in $f(R,T) = R + 2f(T)$ gravity with the system of units $8\pi G = 1 = c$ are obtained as

$$R_{ij} - \frac{1}{2}Rg_{ij} = T_{ij} + 2\left(T_{ij} + p_m g_{ij}\right)f'(T) + f(T)g_{ij}, \tag{19}$$

where a prime stands for a derivative with respect to the trace, T. For $f(T) = \lambda T$, i.e., $f(R,T) = R + 2\lambda T$, where $T = g^{ij}T_{ij} = \rho_m - 3p_m$, Eq. (19) simplifies as

$$R_{ij} - \frac{1}{2}Rg_{ij} = (1+2\lambda)T_{ij} + \lambda(\rho_m - p_m)g_{ij}, \tag{20}$$

which for the metric (1) and the energy-momentum tensor (4) yield

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\dot{A}\dot{B}}{AB} = (1+3\lambda)\rho_m - \lambda p_m, \tag{21}$$

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\ddot{B}}{B} = -(1+3\lambda)p_m + \lambda\rho_m, \tag{22}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -(1+3\lambda)p_m + \lambda\rho_m. \tag{23}$$

This is the correct set of the field equations. One can see that the terms on the RHS of these equations are different from Eqs. (18)–(20) in [59].

Using (11) and (12) in (21) and (22), provided that $\lambda \neq -1/2$, we obtain

$$\rho_m = p_m = \frac{(2n+1)}{(1+2\lambda)(n+2)^2t^2}. \qquad (24)$$

It is the correct expression for the energy density and pressure, which is different from the incorrect one obtained in Eq. (26) by Mahanta [59]. The primary matter acts as stiff matter. The energy density and pressure decrease with the evolution. The energy density ought to be positive for any physically viable cosmological model, which is possible either

$$\lambda > -1/2; \quad \text{if} \quad n > -1/2,$$
$$\text{or} \quad \lambda < -1/2; \quad \text{if} \quad n < -1/2. \qquad (25)$$

Thus, while the solutions in GR are valid only for $n > -1/2$, $f(R,T)$ gravity makes them valid for $n < -1/2$ as well.

It is worthwhile to mention here that we have obtained the expression (24) without bulk viscosity, but Mahanta [59] considered bulk viscous matter to obtain his expression (24). It is to be noted that $\bar{P}$ in Eqs. (22)–(24) in Mahanta's paper is just a symbol, P with an overhead bar. One may readily verify that there is no use of Eq. (14) to calculate the expression (26) in his work. Hence, for viscous or nonviscous models, one gets the same expressions for the energy density and pressure. Thus the energy density and pressure obtained in (24) remain independent of bulk viscosity. We shall consider the bulk-viscous model in Section 3.2.

3.1. The Behavior of Coupled Matter

As elucidated above, ρ_m and p_m do not represent the effective matter in this model of $f(R,T)$ gravity. The terms containing λ in Eqs. (21)–(23) can be assumed to be associated with the coupled matter. By separating these terms, the equations can be expressed as

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\dot{A}\dot{B}}{AB} = \rho_m + \rho_f, \qquad (26)$$

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\ddot{B}}{B} = -(p_m + p_f), \qquad (27)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -(p_m + p_f), \qquad (28)$$

where $p_f = \lambda\,(3\rho_m - p_m)$, and $p_f = \lambda(3p_m - \rho_m)$, respectively, represent the energy density and pressure of the coupled matter and are obtained as

$$\rho_f = p_f = \frac{2\lambda(2n+1)}{(1+2\lambda)(n+2)^2t^2}. \qquad (29)$$

Hence, the coupling terms contribute as stiff matter. The energy density and pressure decrease with the evolution. For a physically viable model, the energy density must be positive, which is corroborated under the constraints

$$-1/2 < \lambda < 0; \quad \text{if} \quad n < -1/2,$$
$$\text{or} \quad \lambda < -1/2 \quad \text{or} \quad \lambda > 0; \quad \text{if} \quad n > -1/2. \qquad (30)$$

These constraints, in view of (25), agree with $\lambda > 0$ and $n > -1/2$ only. Thus, in general, $f(R,T)$ gravity makes the model physically viable also for $n < -1/2$ when $\lambda < -1/2$, but if we treat the coupling terms as matter, then the model becomes physically viable only for $\lambda > 0$ and $n > -1/2$.

3.2. Bulk-viscous Model

The gravitational field equations with bulk-viscous matter remain the same as given in (21)–(23) or (26)–(28), except that the pressure p_m is replaced with

$$\bar{p}_m = p'_m - \xi\theta. \qquad (31)$$

Now we shall repeat the same procedure that we have followed in Section 2.1. First, to examine the behavior of the bulk viscosity coefficient,we consider normal matter follows the prefect fluid EoS. Second, by considering the relations for bulk viscosity assumed in cases (i) and (ii) of Section 2.1.2, we shall study the behavior of normal matter.

3.2.1. The behavior of the bulk viscosity coefficient. Using the prefect fluid EoS $p'_m = \omega\rho_m$, where $0 \le \omega \le 1$, we obtain

$$\xi(t) = \frac{(1+2n)(\omega-1)}{(1+2\lambda)\,(2+n)^2\,t}. \qquad (32)$$

Since $n > -1/2$ and $\lambda > 0$ for a physically viable model, with any kind of matter except a stiff fluid, ξ remains negative, which increases with the evolution and vanishes at late times. For stiff matter ($\omega = 1$), the bulk viscosity coefficient vanishes, and the solutions reduce to those of the nonviscous model, as discussed above. Therefore, the behavior of the bulk viscosity coefficient is similar to that in the model in GR. Hence, $f(R,T) = R + 2\lambda T$ gravity plays no significant role, except that a large value of λ diminishes the effect of bulk viscosity.

3.2.2. The behavior of matter

Case (i), where $\xi\theta = k$. The EoS parameter of matter, $\omega'_m = p'_m/\rho_m$, gives

$$\omega'_m = 1 + \frac{k(n+2)^2(1+2\lambda)t^2}{1+2n}. \qquad (33)$$

In view of the restrictions $n > -1/2$ and $\lambda > 0$, the above EoS parameter for $k > 0$ represents semi-realistic matter, whereas for $k < 0$ it shows a transition from $\omega'_m = 1$ to $\omega'_m \to -\infty$ as $t \to 0$, which is similar to the model in GR. Hence, this also indicates that $f(R,T)$ gravity plays no significant role in this model. However, a large value of λ makes the growth (when $k > 0$) or reduction (when $k < 0$) of ω'_m much faster. At the origin of evolution, $\omega'_m = 1$. If $k = 0$, the solutions reduce to the model without bulk viscosity.

Case (ii), when $\xi\theta = k_1 \rho$. The EoS in this case has a constant value of ω'_m,

$$\omega'_m = 1 + k_1, \quad (34)$$

which is identical to (18). Hence, there is no role of $f(R,T)$ gravity in this case.

4. CONCLUSION

Mahanta [59] studied an LRS Bianchi-I model in $f(R,T)$ gravity with bulk-viscous matter. The signs in the field equations in his all three models of $f(R,T)$ are incorrect. This minor but serious error makes the model studied by him mathematically, and hence physically, invalid. However, the positive aspect is that the wrong signs do not affect the metric potential. Consequently, the geometrical parameters, namely, the volume, the expansion scalar, the Hubble parameter and the shear scalar are mathematically correct. However, the author skipped the physical interpretation of these geometrical parameters. Later on, Shamir [60] also studied some models without bulk viscosity in the same formulation. He discussed the geometrical behavior of the model. To obtain the solutions, the authors in both said works have assumed an expansion scalar proportional to the shear scalar, which returns a constant value of the deceleration parameter, $q = 2$. Hence, the model can describe only a decelerated expansion of the universe.

In this paper, we have reconsidered the $f(R,T) = R + 2\lambda T$ model studied by Mahanta [59]. A comparison of the outcomes in the modified gravity model with the outcomes of the model in GR helps us to understand the role of modified gravity. So, before considering the $f(R,T)$ gravity model, we have studied viscous and nonviscous models in GR. A part of our work is also an extension of Shamir's work. Shamir has discussed the geometric behavior, we have not repeated it here. However, we have shown that these parameters are independent of $f(R,T)$ gravity. Also, while the authors in [59, 60] ignored the testing of physical viability of the models, we have obtained constraints for a physically realistic cosmological scenario. Mahanta [59] in Sections 3 and 4.1 merely obtained the expressions of the coefficient of bulk viscosity. Extending his work, we have also studied the behavior of normal matter for two different forms of the bulk viscosity coefficient considered by him in the model $f(R,T) = R + \lambda T^2$.

The model in GR has been found to be physically viable only for $n > -1/2$. The effective matter behaves as stiff matter, irrespective of a viscous or viscosity-free model. In the viscous model, the bulk viscosity coefficient with perfect fluid (except for stiff matter) is found to be a negative and increasing function of cosmic time. In the case of stiff matter, the coefficient of bulk viscosity vanishes. In the reverse approach, with the first assumption $\xi\theta = k$ for $k > 0$, the matter follows a semi-realistic EoS, while for $k < 0$ the EoS of matter exhibits a transition from a stage of stiff matter to phantom. With the second assumption $\xi\theta = k_1\rho$, the EoS of matter becomes constant ($\omega = 1 + k_1$), which also renders semi-realistic matter for $k_1 > 0$, whereas for $k_1 < 0$ the EoS can describe a variety of kinds of matter including radiation, dust, quintessence, phantom, and cosmological constant for different choices of k_1. If $k = 0 = k_1$, the solutions reduce to the model without viscosity.

As far as the $f(R,T)$ gravity model is concerned, Shamir [60] has studied the behavior of effective matter only. However, in the case of $f(R,T)$ gravity, some extra terms appear on the right hand side of the field equations. These terms can be treated as representing some additional matter due to the coupling between matter and geometry. Therefore, by considering matter and geometry coupling terms as coupled matter, we have examined its behavior. Since the metric potential remains identical to that in the model of GR, the effective matter (both for viscous and nonviscous models) acts as stiff matter in $f(R,T)$ gravity as well.

In general, the solutions in $f(R,T)$ gravity are physically viable for $\lambda > -1/2$ and $n > -1/2$ or $\lambda < -1/2$ and $n < -1/2$. However, when the coupling terms are treated as matter, then a physically viable model is possible only for $\lambda > 0$ and $n > -1/2$. The primary matter as well as the coupled matter act as stiff matter. Thus, the behavior of the bulk-viscous model in $f(R,T)$ gravity is almost similar to that of the model in GR. The only difference is that $f(R,T) = R + 2\lambda T$ gravity for large values of λ diminishes the effect of viscous matter.

Many researchers explored cosmological models with stiff matter in the forward approach in different contexts (see, for example, [40, 48, 66, 67] and references therein). While these works utilize simplified assumptions of the EoS of stiff matter to get exact solutions, it is a natural outcome of the present study. The stiff matter cosmological models are interesting in the sense that for such models the speed of light is equal to the speed of sound [68, 69]. A realistic example of a distribution of stiff fluid is a polytropic fluid inside a star. The existence of such realistic objects in the universe makes the studies of stiff matter models prominent.

It is also worthwhile to mention here that Mahanta [59] considered three models of $f(R,T)$, namely, $f(R,T) = R + 2\lambda T$, $f(R,T) = \lambda R + \lambda T$ and $f(R,T) = R + \lambda T^2$. The sign in the field equations for all three models is incorrect. The first two forms are, in fact, not different since the first one is a particular case of the second one. Consequently, both forms would produce similar results. Moreover, the second model is formulated in a way that the coupling terms are treated as a variable cosmological constant, $\Lambda = (\rho - p)/2$. As we have seen, the energy density and pressure of effective matter as well as coupled matter become equal. Resultantly, Λ vanishes in such a formulation, and the solutions reduce to the model in GR. Consequently, even if one considers the correct sign in the field equations, the outcomes would be identical to the model in GR. Therefore, we have not studied this form explicitly.

Finally, we would like to point out that apart from the wrong signs in the field equations, Mahanta [59], in his model $f(R,T) = R + \lambda T^2$, overdetermined the solutions. We see that Eqs. (58)–(60) have five unknowns, namely, H_1, H_3, ρ, P and ξ. Therefore, only three assumptions would be required to close the system, but the author used four, namely, (27), (28), (61) or (65) along with the EoS $P = \epsilon\rho$. We have not considered this model in the present study for the sake of keeping our paper of mandate length. Shamir [60] has studied this form without bulk viscosity. We shall consider this model with bulk viscosity somewhere else.

ACKNOWLEDGMENTS

The authors are grateful to an anonymous referee for constructive criticism which led to an improvement in the presentation. Vijay Singh expresses his sincere thank to the University of Zululand, South Africa, for providing a postdoctoral fellowship and necessary facilities."

FUNDING

This work is based on the research supported wholly/in part by the National Research Foundation of South Africa (Grant Numbers 118511).

REFERENCES

1. C. B. Netterfield et al., “Improved measurement of the angular power spectrum of temperature anisotropy in the cosmic microwave background from two new analyses of BOOMERANG observations,” Astrophys. J. **571**, 604–614 (2002); astro-ph/0104460.
2. D. N. Spergel et al., “First year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Determination of cosmological parameters,” Astrophys. J. Suppl. **148**, 175–194 (2003); astro-ph/0302209.
3. L. Anderson et al., “The clustering of galaxies in the SDSS-III Baryon Oscillation Spectroscopic Survey: Baryon acoustic oscillations in the data release 9 spectroscopic galaxy sample,” Mon. Not. Roy. Astron. Soc. **427**, 3435 (2013); arXiv: 1203.6594.
4. C. L. Bennett et al. “Nine-year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Final maps and results,” Astrophys. J. Supp. Ser. **208**, 20 (2013); arXiv: 1212.5225.
5. G. Hinshaw et al., “Nine-year Wilkinson Microwave Anisotropy Probe (WMAP) observations: cosmological parameter results,” Astrophys. J. Supp. Ser. **208**, 19 (2013); arXiv: 1212.5226.
6. P. A. R. Ade et al., “Planck 2015 results. XIII. Cosmological parameters,” Astron. Astrophys. **594**, A13 (2016); arXiv: 1502.01589.
7. N. Aghanim et al., “Planck 2018 results—VI. Cosmological parameters,” Astron. Astrophys. **641**, A6 (2020); arXiv: 1807.06209.
8. J. A. Belinchon, “Perfect fluid LRS Bianchi I with time varying constants,” Astrophys. Space Sci. **301**, 161 (2006); gr-qc/0411005.
9. V. Singh and A. Beesham, “LRS Bianchi I model with constant deceleration parameter,” Gen. Relativ. Grav. **51**, 165 (2019); arXiv: 1912.05850.
10. V. Singh and A. Beesham, “LRS Bianchi I model with perfect fluid equation-of-state,” Int. J. Mod. Phys. D **28**, 1950056 (2019); arXiv: 1912.05305.
11. V. Singh and A. Beesham, “Plane symmetric model in $f(R,T)$ gravity,” Eur. Phys. J. Plus. **135**, 319 (2020); arXiv: 2003.08665.
12. V. Singh and A. Beesham, “LRS Bianchi I model with constant expansion rate in $f(R,T)$ gravity,” Astrophys. Space Sci. **365**, 125 (2020); arXiv: 2003.04602.
13. M. K. Mak and T. Harko, “Bianchi type I universes with causal bulk viscous cosmological fluid,” Int. J. Mod. Phys. D **11**, 447 (2002).
14. Ø. Grøn, “Viscous inflationary universe models,” Astrophys. Space Sci. **173**, 191 (1990).
15. G. P. Singh and A. Beesham, “Bulk viscosity and particle creation in Brans-Dicke theory,” Austr. J. Phys. **52**, 1039 (1999).
16. D. Pavon, J. Bafaluy, and D. Jou, “Causal Friedmann-Robertson-Walker cosmology,” Class. Quant. Grav. **8**, 247 (1991).
17. A. Beesham, “Cosmological models with a variable cosmological term and bulk viscous models,” Phys. Rev. D. **48**, 3529 (1993).

18. J. Triginer and D. Pavon, "Particle production in a viscous cosmological fluid," Gen. Rel. Grav. **26**, 513 (1994).
19. W. Zimdahl, "Bulk viscous cosmology," Phys. Rev. D. **53**, 5483 (1996).
20. A. Beesham, "Bulk viscosity and cosmological evolution," Nuovo. Cimento. B **111**, 1481 (1996).
21. A. I. Arbab, "Cosmological models with variable cosmological and gravitational constants and bulk viscous models," Gen. Rel. Grav. **29**, 61 (1997).
22. L. P. Chimento, A. S. Jakubi, V. Mendez, and R. Maartens, "Cosmological solutions with nonlinear bulk viscosity," Class. Quantum Grav. **14**, 2263 (1997); gr-qc/9710029.
23. V. R. Gavrilov, V. N. Melnikov, and R. Triay, "Bianchi type I universe with viscous fluid," Class. Quantum Grav. **14**, 2203 (1997); gr-qc/9710029.
24. T. Singh and A. Beesham, "Causal viscous cosmological models with variable G and Λ," Gen. Rel. Grav. **32**, 607 (2000).
25. W. Zimdahl, D. J. Schwartz, A. B. Balakin, and D. Pavon, "Cosmic antifriction and accelerated expansion," Phys. Rev. D **64**, 063501 (2001); astro-ph/0009353.
26. M. Cataldo, N. Cruz, and S. Lepe, "Viscous dark energy and phantom evolution," Phys. Lett. B **619**, 5 (2005); hep-th/0506153.
27. J. C. Fabris, S. V. B. Goncalves, and R. de Sa Ribeiro, "Bulk viscosity driving the acceleration of the Universe," Gen. Rel. Grav. **38**, 495 (2006); astro-ph/0503362.
28. M. Szydlowski and O. Hrycyna, "Dissipative or conservative cosmology with dark energy?" Annals Phys. **322**, 2745 (2007).
29. C. P. Singh, S. Kumar, and A. Pradhan, "Early viscous universe with variable gravitational and cosmological constants," Class. Quantum Grav. **24**, 455 (2007).
30. R. Colistete Jr., J. C. Fabris, R. Tossa, and W. Zimdahl, "Bulk viscous cosmology," Phys. Rev. D **76**, 103516 (2007); arXiv: 0706.4086.
31. S. R. Roy and S. Prakash, "A gravitationally non-degenerate viscous fluid cosmological model in general relativity," Ind. J. Pure App. Math. **8**, 723 (1977).
32. A. Barnejee and N. O. Santos, "Spatially homogeneous cosmological models," Gen. Rel. Grav. **16**, 217 (1984).
33. R. Bali and D. R. Jain, "Viscous fluid universe filled with stiff fluid in general relativity," Astrophys. Space Sci. **185**, 211 (1991).
34. A. Burd and A. Coley, "Viscous fluid cosmology," Class. Quantum Grav. **11**, 83 (1994).
35. R. J. Hoogen and A. A. Coley, "Qualitative analysis of causal anisotropic viscous-fluid cosmological models," Class. Quantum Grav. **12**, 2335 (1995).
36. R. Bali and V. C. Jain, "Some viscous fluid cosmological models in general relativity," Astrophys. Space Sci. **254**, 13 (1997).
37. A. Pradhan and S. K. Singh, "Bianchi type I magnetofluid cosmological models with variable cosmological constant revisited," Int. J. Mod. Phys. D **13**, 503 (2004).
38. B. Saha, "Bianchi type I universe with viscous fluid," Mod. Phys. Lett. A **28**, 2127 (2005).
39. T. Singh and R. Chaubey, "Bianchi Type-V universe with a viscous fluid and Λ-term," Pramana J. Phys. **68**, 721 (2007).
40. R. Bali and P. Kumawat, "Anisotropic cosmological models with bulk viscosity for variable G and Λ," Phys. Lett. B **665**, 331 (2008).
41. S. Chakraborty and A. Roy, "Bulk viscous Bianchi Type V cosmological models with decaying cosmological term Λ," Astrophys. Space Sci. **313**, 389 (2008).
42. J. P. Singh and P. S. Baghel, "Bulk viscous fluid hypersurface homogeneous cosmological models with time varying G and Λ," Int. J. Theor. Phys. **49**, 2734 (2010).
43. S. Ram and M. K. Verma, "Bianchi type V universe with bulk viscous matter and time varying gravitational and cosmological constants," Astrophys. Space Sci. **330**, 151 (2010).
44. P. S. Baghel and J. P. Singh, "Anisotropic cosmological models with conformally invariant scalar fields and constant deceleration parameter," Res. Astron. Astrophys. **12**, 1457 (2012).
45. S. Ram, M. K. Singh, and M. K. Verma, "Anisotropic Bianchi type II viscous fluid model with time dependent gravitational and cosmological constant," J. Math. Phys. **3**, 9 (2012).
46. M. K. Singh, M. K. Verma, and S. Ram, "Anisotropic Bianchi type II viscous fluid model with time dependent gravitational and cosmological constant," Int. J. Phys. **1**, 77 (2013).
47. A. Barnejee and N. O. Santos, "Anisotropic cosmological model with viscous fluid," J. Math. Phys. **24**, 2689 (1983).
48. A. Barnejee, S. B. Duttachoudhury, and A. K. Sanyal, "Bianchi type I cosmological model with a viscous fluid," J. Math. Phys. **26**, 3010 (1985).
49. H. F. M. Goenner and F. Kowalewski, "Exact anisotropic viscous fluid solutions of Einstein's equations," Gen. Rel. Grav. **21**, 467 (1989).
50. V. Romano and D. Pavon, "Causal dissipative Bianchi cosmology," Phys. Rev. D **47**, 1396 (1993).
51. A. I. Arbab, "Bianchi type I viscous universe with variable G and Λ," Gen. Rel. Grav. **30**, 1401 (1998).
52. A. Beesham, "Anisotropic viscous cosmology with variable G and Λ," Gen. Rel. Grav. **32**, 471 (2000).
53. J. A. Belinchon, "Full causal bulk viscous LRS Bianchi I with time varying constants," Astrophys. Space. Sci. **299**, 343 (2005); gr-qc/0410065.
54. B. Saha and V. Rikhvitsky, "Bianchi type I universe with viscous fluid and a Λ term: A qualitative analysis," Physica D **219**, 168 (2006); gr-qc/0410056.
55. C. P. Singh and S. Kumar, "Viscous fluid cosmology in Bianchi type-I space-time," Int. J. Theor. Phys. **48**, 925 (2009).

56. W. H. Huang, "Anisotropic cosmological models with energy density dependent bulk viscosity," J. Math. Phys. **31**, 1456 (1990).
57. A. Pradhan and S. Otarod, "A new class of bulk viscous universe with time dependent deceleration parameter and Λ-term," Astrophys. Space Sci. **311**, 413 (2007).
58. X. X. Wang, "Bianchi type-III string cosmological model with bulk viscosity and magnetic field," Chin. Phys. Lett. **20**, 1674 (2003).
59. K. L. Mahanta, "Bulk viscous cosmological models in $f(R,T)$ theory of gravity," Astrophys. Space Sci. **353**, 683 (2014).
60. M. F. Shamir, "Locally rotationally symmetric Bianchi type I cosmology in $f(R,T)$ gravity," Astrophys. Space Sci. **75**, 354 (2015); gen-ph/1507.08175.
61. P. K. Sahoo and R. Reddy, "LRS Bianchi type-I bulk viscous cosmological models in $f(R,T)$ gravity," Astrophysics **61**, 134 (2018).
62. A. K. Yadav, P. K. Sahoo, and V. Bhardwaj, "Bulk viscous Bianchi-I embedded cosmological model in $f(R,T) = f_1(R) + f_2(R)f_3(T)$ gravity," Mod. Phys. Lett. A **34**, 1950145 (2019).
63. H. A. Buchdahl and W. J. Land, "The relativistic incompressible sphere," Austr. Math. Soc. **8**, 6 (1968).
64. J. M. Whittaker, "An interior solution in general relativity," Proc. R. Soc. Lond. A **306**, 1 (1968).
65. J. Ibanez and J. L. Sanz, "New exact static solutions to Einstein equations for spherically symmetric perfect fluid distributions," J. Math. Phys. **23**, 164 (1982).
66. R. Bali and K. Sharma, "Tilted Bianchi type I cosmological models for barotropic perfect fluid in general relativity," Astrophy. Space Sci. **293**, 367 (2004).
67. K. S. Adhav, V. G. Mete, A. S. Nimkar, and A. M. Pund, "Kantowski-Sachs cosmological model in general theory of relativity," Int. J. Theor. Phys. **47**, 2314 (2008).
68. Ya. B. Zeldovich, "On the upper limit on the density of neutrinos, gravitons, and baryons in the universe," Sov. Phys. J. Exp. Theor. Phys. **14**, 1143 (1962).
69. Ya. B. Zeldovich, "A hypothesis unifying the structure and the entropy of the Universe," Mon. Not. R. Astron. Soc. **160**, 1 (1972).