ISSN 0202-2893, Gravitation and Cosmology, 2021, Vol. 27, No. 1, pp. 42–46. © Pleiades Publishing, Ltd., 2021.

The Bakry-Émery Ricci Tensor: Application to Mass Distribution in Space-time

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Received October 10, 2019; revised February 3, 2020; accepted August 7, 2020

Abstract—The Bakry-Émery Ricci tensor gives an analogue of the Ricci tensor for a Riemannian manifold with a smooth function. This notion motivates a new version of Einstein's field equation in which the mass becomes part of geometry. This new field equation is purely geometric and is obtained from an action principle which is formed naturally by the scalar curvature associated with the Bakry-Émery Ricci tensor.

DOI: 10.1134/S0202289321010096

1. INTRODUCTION

No well-defined current physical theory claims to model all nature; each intentionally neglects some effects. Roughly, general relativity is a model of nature, especially of gravity, that neglects quantum effects. In fact, general relativity describes gravity in the context of 4-dimensional Lorentzian manifolds by the Einstein field equation. Einstein argued that the stress-energy of matter and electromagnetism influences space-time (M, g) and suggested his field equation as follows [9]:

$$\mathrm{Ric} - \frac{1}{2}Rg = T + E, \qquad (1)$$

where T and E are the stress-energy tensors of matter and the electromagnetic field, respectively. The expressions in the right-hand side of the Einstein equation (1) are physical concepts whilst the left-hand side is completely geometric. In fact, the influence of matter and electromagnetism on space-time is not described directly by the geometry of the space-time manifold (M, g). Hence, a question naturally arises here: how to geometrize matter and electromagnetism?

Attempts at geometrization of electromagnetism, also known as unification of gravity and electromagnetism, have been made ever since the advent of general theory of relativity. Most of these attempts share the idea that Einstein's original theory must in some way be generalized so that some part of geometry describes electromagnetism. Nevertheless, there are relatively few results on geometrization of mass in general relativity or, more generally, applicable to the space-time manifold, and many of them rely on dimension 5 or other geometric structures like Lie algebroids (see [7, 8] and [4]).

In the following, we present some clues which show that replacing the Ricci curvature tensor of space-time manifold (M, g) with the Bakry-Émery Ricci tensor provides us with a good apparatus for geometrization of mass.

Considering the Einstein equation (1), any attempt on geometrization of mass and electromagnetism leads to a condition on the Einstein equation and so, ultimately, on the Ricci curvature. So, any condition on the Ricci curvature gives a tool for geometrization of mass.

Suppose that f is a smooth function on a Riemannian manifold (M, g), then the m-Bakry-Émery Ricci tensor is defined as [11]

$$\mathrm{Ric}_f^m = \mathrm{Ric} + \mathrm{Hess} f - \frac{1}{m} df \otimes df,$$

where Ric stands for the Ricci curvature tensor of (M, g), and m is a nonzero constant that is also allowed to be infinite, in which case we write

$$\mathrm{Ric}_f^\infty = \mathrm{Ric} + \mathrm{Hess} f.$$

The Ricci curvature tensor can describe gravity, and the m-Bakry-Émery Ricci tensor is capable of describing gravity and matter, simultaneously. In fact, the Ricci curvature describes gravity, and f can play the role of a potential for mass distribution in space-time.

We already know that the Einstein metrics play a crucial role in the general theory of relativity. Hence, the key is provided by generalization of the Einstein

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metrics which should be probably capable of solve our concern.

A triple (M, g, f) (a Riemannian manifold (M, g) with a smooth function f on M) is said to be $(m-)$quasi-Einstein if it satisfies the equation

$$\mathrm{Ric}_f^m = \mathrm{Ric} + \mathrm{Hess}f - \frac{1}{m}df \otimes df = \lambda g,$$

for some $\lambda \in \mathbb{R}$. The above equation is especially interesting in that when $m = \infty$ it is exactly the gradient Ricci soliton equation; when m is a positive integer, it corresponds to warped product Einstein metrics (see [5]); when f is constant, it gives the Einstein equation. Moreover, it is important to detach that 1-quasi-Einstein metrics satisfying $\triangle e^{-f} + \lambda e^{-f} = 0$ are more commonly called *static metrics* with the cosmological constant λ; for more details see, e.g., [1, 2, 6]. We call a quasi-Einstein metric trivial when f is constant. Following the terminology of Ricci solitons, a quasi-Einstein metric on a manifold M will be called expanding, steady or shrinking, respectively, if $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$.

As we have noticed, the Bakry-Émery Ricci tensor is related to the notion of quasi-Einstein metrics. So, we hope the Bakry-Émery Ricci tensors will provide a good apparatus for geometrization of mass in the next section.

2. NEW FIELD EQUATION

In this section, (M, g) is a compact manifold without boundary, and we regard it as a space-time manifold whose dimension is n, and $2 \leq n$. Each smooth function f on M determines a 1-Bakry-Émery Ricci tensor on M.

Let $\mathcal{M}$ denotes the set of all 1-Bakry-Émery Ricci tensors on a manifold M, then, $\mathcal{M}$ can be written as the set of some pairs (g, f), where g is a semi-Riemannian metric, and f is a smooth function on M. In fact, $\mathcal{M}$ can be considered as an open subset of an infinite-dimensional vector space.

Denote the canonical volume form of a meter g on the oriented manifold M by dV_g. It is well known that critical points of the Einstein-Hilbert functional

$$\mathcal{L}(g) = \int_M R_g dV_g$$

are metrics which satisfy the following equation,

$$\mathrm{Ric} - \frac{1}{2}Rg = 0.$$

The above equation is the Einstein field equation in vacuum. It motivates us to define a new Hilbert–Einstein functional $\mathcal{L} : \mathcal{M} \to \mathbb{R}$,

$$\mathcal{L}(g, f) = \int_M R_{(g,f)} dV_g,$$

where $R_{(g,f)} = \mathrm{tr}(\mathrm{Ric}_f^1)$ is called the scalar curvature of Ric_f^1. It is easy to see that

$$R_{(g,f)} = R + \triangle(f) - |\vec{\nabla} f|^2,$$

where by the notation $\vec{\nabla}$ we mean the gradient of smooth functions. Because the integral of a Laplacian vanishes, our new Hilbert–Einstein functional can be written as follows:

$$\mathcal{L}(g, f) = \int_M [R - |\vec{\nabla} f|^2] dV_g. \tag{2}$$

Equations which the critical points of this functional satisfy will be called the *new field equations*.

For a symmetric 2-covariant tensor s on M, and a smooth function h, set

$$\tilde{g}(t) = g + ts,$$
$$\tilde{f}(t) = f + th.$$

For sufficiently small t, $(\tilde{g}(t), \tilde{f}(t))$ is a 1-Bakry-Émery Ricci tensor on M and is a variation of (g, f). The Bakry-Émery Ricci tensor (g, f) is a critical 1-Bakry-Émery Ricci tensor for the Hilbert action if and only if for any pair (s, h):

$$\frac{d}{dt}\mathcal{L}\big(\tilde{g}(t), \tilde{f}(t)\big)\Big|_{t=0}$$
$$= \frac{d}{dt}\left(\int_M [R_{\tilde{g}(t)} - |\vec{\nabla}\tilde{f}(t)|^2] dV_{\tilde{g}(t)}\right)\Bigg|_{t=0} = 0. \tag{3}$$

To find a derivative in (3), we must compute derivatives of $R_{\tilde{g}(t)}$, $|\vec{\nabla}\tilde{f}(t)|^2$ and dV_{g+ts} for $t = 0$. In [3], it is shown that

$$(R_{\tilde{g}(t)})'(0) = -\langle s, \mathrm{Ric}\rangle + \mathrm{div}(X),$$
$$X \in \mathcal{X}(M), \tag{4}$$

$$(dV_{g+ts})'(0) = \frac{1}{2}\langle g, s\rangle dV_g. \tag{5}$$

By local computations, we find derivatives of $\triangle_{\tilde{g}(t)}(\tilde{f}(t))$ and $|\vec{\nabla}\tilde{f}(t)|^2$ at $t = 0$. By means of a local coordinate system on M with a local frame ∂_i, we can write

$$\frac{d}{dt}\Big|_{t=0} |\vec{\nabla}\tilde{f}(t)|^2 = \left(\tilde{g}^{ij}(t)\frac{\partial\tilde{f}(t)}{\partial x^i}\frac{\partial\tilde{f}(t)}{\partial x^j}\right)'(0)$$
$$= -s^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} + 2g^{ij}\frac{\partial h}{\partial x^i}\frac{\partial f}{\partial x^j}$$
$$= -\langle s, df \otimes df\rangle + 2\langle\vec{\nabla}h, \vec{\nabla}f\rangle.$$

Recall that the integral of the divergence of every vector field on M is zero, consequently. the integral of a Laplacian of any smooth function on M is zero.

Also, for any two smooth functions f and h by a similar argument with [10] we have

$$\int_M \langle \vec{\nabla} h, \vec{\nabla} f \rangle dV_g = -\int_M \triangle(f) h dV_g.$$

Now, we are prepared to find a critical 1-Bakry-Émery Ricci tensor for the Hilbert action:

$$\begin{aligned} 0 &= \frac{d}{dt}\mathcal{L}\big(\tilde{g}(t), \tilde{f}(t)\big)\Big|_{t=0} \\ &= \int_M (R - |\vec{\nabla} f|^2).\frac{1}{2}\langle s, g\rangle dV_g \\ &+ \int_M \big(-\langle s, \mathrm{Ric}\rangle + \mathrm{div}(X) + \langle s, df \otimes df\rangle \\ &- 2\langle \vec{\nabla} h, \vec{\nabla} f\rangle\big) dV_g \\ &= \int_M \langle -\mathrm{Ric} + \frac{1}{2} R g + df \otimes df - \frac{|\vec{\nabla} f|^2}{2} g, s\rangle dV_g \\ &+ \int_M 2\triangle(f) h dV_g. \end{aligned}$$

The above expressions vanish for all pairs (g, f) if and only if

$$\mathrm{Ric} - \frac{1}{2} R g = \left(df \otimes df - \frac{|\vec{\nabla} f|^2}{2} g \right), \quad (6)$$

$$\triangle(f) = 0. \quad (7)$$

Taking a trace from both sides of (6) yields $R = |\vec{\nabla} f|^2$. The scalar curvature R is related to a matter distribution at points of the space-time manifold. Hence, f is related to a matter distribution. Equation (7) can be written as $\mathrm{div}(\vec{\nabla} f) = 0$, so, $\vec{\nabla} f$ can be interpreted as the current of mass which satisfies a conservation law. Also, we can interpret the expression appearing in the right-hand side of (6) as the stress-energy tensor of matter. We need to show that the divergence of

$$T^f := df \otimes df - \frac{|\vec{\nabla} f|^2}{2} g,$$

which retrieves the conservation law of stress-energy tensor.

Theorem 1. *Suppose that f is a smooth function on a Riemannian manifold (M, g) such that $\triangle(f) = 0$, then the symmetric tensor T^f is divergence-free.*

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal base on M and denote its reciprocal base by $\{e^i\}_{i=1}^n$. So, we can write

$$\begin{aligned} \mathrm{div}(df \otimes df)(X) &= \sum_{i=1}^n (\nabla_{e_i} df \otimes df)(e^i, X) \\ &= \sum_{i=1}^n \big((\nabla_{e_i} df) \otimes df + df \otimes (\nabla_{e_i} df)\big)(e^i, X) \\ &= \sum_{i=1}^n \big((\nabla_{e_i} df)(e^i) df(X) + df(e^i)(\nabla_{e_i} df)(X) \\ &= \triangle(f) df(X) + \mathrm{Hess} f(\vec{\nabla} f, X) = \mathrm{Hess} f(\vec{\nabla} f, X). \end{aligned}$$

Also,

$$\begin{aligned} \mathrm{div}(|\vec{\nabla} f|^2 g)(X) &= d(|\vec{\nabla} f|^2)(X) \\ &= X\langle \vec{\nabla} f, \vec{\nabla} f\rangle = 2\langle \nabla_X(\vec{\nabla} f), \vec{\nabla} f\rangle \\ &= 2(\nabla_X df)(\vec{\nabla} f) = 2\mathrm{Hess} f(\vec{\nabla} f, X). \end{aligned}$$

The above computations show that $\mathrm{div}(T^f) = 0$. □

In the case where $\dim M \geq 3$, a simpler form of equation (6) is obtained.

Theorem 2. *Suppose that* $\dim M = n \geq 3$, *then the critical Bakry-Émery tensors of the Hilbert action satisfy the following equations*:

$$\mathrm{Ric} = df \otimes df, \quad (8)$$

$$\triangle(f) = 0. \quad (9)$$

Proof. The second equation is the same as (7). Let us assume that (6) holds, then by taking a trace on both sides of this equation, we have

$$R - \frac{n}{2} R = |\vec{\nabla} f|^2 - \frac{n}{2} |\vec{\nabla} f|^2 \;\Rightarrow\; R = |\vec{\nabla} f|^2.$$

By replacing the above formula in (6), we derive (8). Now, let (8) hold. By computation of traces of the two sides of Eq. (8), we find

$$R = |\vec{\nabla} f|^2.$$

Hence, by addition suitable expression to each side of Eq. (8) we obtain (6). □

Our obtained field equations are purely geometric when (g, f, λ) is a 1-quasi Einstein metric for some $\lambda \in \mathbb{R}$.

Theorem 3. *Suppose that (g, f) is a critical point of the Hilbert–Einstein functional, then (g, f, λ) is a 1-quasi Einstein metric if and only if $\lambda = 0$ (the quasi-Einstein metric is steady) and $\vec{\nabla} f$ is a parallel vector field.*

Proof. Assume that (g, f, λ) is a 1-quasi Einstein metric on M, then

$$\mathrm{Ric}_f^1 = \mathrm{Ric} + \mathrm{Hess} f - df \otimes df = \lambda g.$$

Hence, Eq. (8) implies that

$$\mathrm{Hess} f = \lambda g.$$

By taking traces on both sides of the above formula and using (9), we get

$$0 = \lambda n,$$

so, $\lambda = 0$, which means that the quasi-Einstein metric is steady, and we have

$$\mathrm{Hess} f = 0,$$

which implies that $\vec{\nabla} f$ is a parallel vector field on M.

Invesely, by definition of a steady 1-quasi Einstein metric (g, f) we can write

$$\mathrm{Ric} + \mathrm{Hess} f - df \otimes df = 0.$$

Since $\vec{\nabla} f$ is a parallel vector field, we have $\mathrm{Hess} f = 0$, which implies Eqs. (8) and (9). □

Remark. Denote by θ the 1-form df appearing in the above theorem. In the Riemannian manifold (M, g)

$$\mathrm{Hess} f = \nabla df = 0,$$

means that θ is a parallel 1-form. Moreover, in the generic case (where Ric is nondegenerate), by the Weitzenböck formula we have $\theta = 0$. Hence, in this case f is locally constant, and because of connectedness of M it will be globally constant. So, generic Riemannian metrics are an obstruction to geometrize matter in general relativity.

3. AN EXAMPLE OF SPACE-TIME-MASS MANIFOLD

In this section we give an example of a manifold which satisfies the equations (8) and (9). This example is the Einstein-de Sitter model in general relativity.

Suppose that $(N, \bar{g})$ is a Riemannian manifold of dimension $n \geq 2$ and set $M = N \times (0, \infty)$. Any tangent vector to M at a point (p, t) is of the form (v, λ), in which $v \in T_pN$ and $\lambda \in \mathbb{R}$ is a scalar. The vector field $(0, 1)$ on M is denoted by ∂_t. Considering this vector field as a derivative of $C^\infty(M)$, for a smooth function $h(p, t)$ on M we have $\partial_t(h) = \partial h/\partial t$. Denote the vector fields on N by $X, Y, Z, \cdots$ and consider them as special vector fields on M. Also, denote the second projection map $(p, t) \mapsto t$ on M by t. We can interpret t as time. Note that for the 1-form dt we have $dt(v, \lambda) = \lambda$, so $dt(\partial_t) = 1$.

For some smooth function $a : (0, \infty) \to \mathbb{R}$ define a metric g on M as follows:

$$g = e^{2a(t)} \bar{g} - dt \otimes dt.$$

Consider $f : N \times (0, \infty) \to \mathbb{R}$ which depends only on t and denote it by $f(t)$. Consequently, $df = f'(t)dt$, $\vec{\nabla} f = -f'(t)\partial_t$, and $|\vec{\nabla} f|^2 = -|f'(t)^2|$.

Let $\overline{\nabla}$ and ∇ denote the Levi-Civita connections of N and M, respectively. Routine computations show that

$$\nabla_X, Y = \overline{\nabla}_X, Y + a'(t) g(X, Y)\partial_t,$$
$$\nabla_X, \partial_t = a'(t) Y,$$
$$\nabla_{\partial_t} \partial_t = 0.$$

Also, an easy local computation indicates that

$$\triangle(f) = -na'(t)f'(t) - f''(t). \tag{10}$$

Hence, the equation $\triangle(f) = 0$ implies that $|f'(t)| = ce^{-na(t)}$, for some constant c.

Now, let $\overline{\mathrm{Ric}}$ and Ric denote the Ricci curvature tensors of N and M, respectively. Straightforward computations show that

$$\mathrm{Ric}(X, Y) = \overline{\mathrm{Ric}}(X, Y) + (a''(t) + na'(t)) g(X, Y),$$
$$\mathrm{Ric}(X, \partial_t) = 0,$$
$$\mathrm{Ric}(\partial_t, \partial_t) = -n(a''(t) + a'(t)).$$

Equation (8) for this example becomes

$$\mathrm{Ric} = |f'(t)|^2 dt \otimes dt.$$

Hence, this equation holds if and only if

$$\overline{\mathrm{Ric}}(X, Y) = -(a''(t) + na'(t)) g(X, Y), \tag{11}$$

$$n(a''(t) + a'(t)) = |f'(t)|^2 = c^2 e^{-2na(t)}. \tag{12}$$

The left-hand side of (11) does not depend on t, so $a''(t) + na'(t)$ has to be constant, and N is an Einstein manifold. In the case $a''(t) + na'(t) = 0$ we find the solution $a(t) = (1/n) \ln t$, and for this solution, Eq. (12) also holds for $c = \sqrt{(n-1)/n}$.

So, for an n-dimensional Ricci-flat manifold $(N, \bar{g})$ the meter $g = t^{2/n}\bar{g} - dt \otimes dt$ on M and the function $f(t) = \sqrt{(n-1)/n} \ln t$ satisfiy the field equations (8) and (9). In this model, as t approaches zero, the universe become smaller, and the density of matter increases to infinity. The time $t = 0$ is not in M, and this time is the instant of the Big Bang.

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