Hybrid Metric-Palatini Gravity: Black Holes, Wormholes, Singularities, and Instabilities

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Abstract—The hybrid metric-Palatini theory of gravity (HMPG), proposed in 2012 by T. Harko et al., is known to successfully describe both local (solar-system) and cosmological observations. We discuss static, spherically symmetric vacuum solutions of HMPG with the aid of its scalar-tensor theory (STT) representation. This scalar-tensor theory coincides with general relativity with a conformally coupled scalar field (which can be canonical or phantom), therefore the known solutions of this theory are re-interpreted in terms of HMPG. In particular, in the case of zero scalar field potential $V(\phi)$, such that both Riemannian and Palatini Ricci scalars are zero, generic asymptotically flat solutions either contain naked singularities or describe traversable wormholes, and there are only special cases of black hole solutions with extremal horizons. There is also a one-parameter family of solutions with an infinite number of extremal horizons between static regions. Examples of analytical solutions with nonzero potentials $V(\phi)$ are also described, among them black hole solutions with simple horizons which are generic but, for canonical scalars, they require (at least partly) negative potentials. With phantom scalars there are "black universe" solutions that lead beyond the horizon to an expanding universe instead of a singularity. Most of the solutions under consideration turn out to be unstable under scalar monopole perturbations, but some special black hole solutions are stable.

1. INTRODUCTION

General relativity (GR) that has recently celebrated its century, is known to still successfully describe all local observational effects. It is however, unable to completely account for large-scale phenomena, facing the so-called Dark Matter and Dark Energy problems. There are two alternative ways in addressing these problems: one is still to adhere to GR but to introduce so far unobserved forms of matter like WIMPs (weakly interacting massive particles) as Dark Matter, and a cosmological constant or a "quintessence" scalar field, etc., as Dark Energy [1]. An alternative approach is to modify GR itself, considering more general Lagrangian functions (for instance, $f(R)$), introducing new degrees of freedom (e.g., scalar or vector fields), extra dimensions or/and geometric quantities such as torsion and nonmetricity [2, 3].

The hybrid metric-Palatini gravity (HMPG) theory, proposed in [4], is one of such theories. This the-

ory assumes the existence of the Riemannian metric $g_{\mu\nu}$ along with an independent connection $\hat{\Gamma}^\alpha_{\mu\nu}$. The total action reads [4]

$$S = \frac{1}{2\varkappa^2} \int d^4x \sqrt{-g}[R + F(\mathcal{R})] + S_m, \quad (1)$$

where $R = R[g]$ is the scalar curvature derived from $g_{\mu\nu}$, while $F(\mathcal{R})$ is a function of the scalar $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$ obtained with the Ricci tensor $\mathcal{R}_{\mu\nu}$ built in the standard manner from the connection $\hat{\Gamma}^\alpha_{\mu\nu}$; also, $g = \det(g_{\mu\nu})$, $\varkappa^2$ is the gravitational constant, and S_m is the action of nongravitational matter.

Thus HMPG combines the metric and Palatini approaches to gravity and is an extension of $f(R)$ theories. This theory has been shown to agree with the classical gravitational tests in the Solar system [5], rather well describes the dynamic properties of galaxies and galaxy clusters, thus approaching an explanation of the dark matter problem [6], and is able to create models of the accelerating Universe without a cosmological constant [7], see reviews [8, 9] for a more detailed description of HMPG and its achievements. Noether symmetries in HMPG were studied

in [10], while a relationship of HMPG with R^2 gravity in different formulations is discussed in [11]. A further generalization of HMPG, with an arbitrary function of both R and $\mathcal{R}$, is suggested in [12], see recent results obtained in this theory in [13−15].

The present paper continues the study of static, spherically symmetric solutions of HMPG, began in [16], where the simplest case $F(\mathcal{R}) \propto \mathcal{R}$ was considered. In this case, the HMPG theory is equivalent to GR with a conformally invariant scalar field that can be either canonical or phantom; the phantom case, which seems to appear more naturally from HMPG (since then $dF/d\mathcal{R} > 0$), was discussed in [16]. Here we briefly reproduce the results of [16], add a discussion for the canonical ϕ field, and also present two simple analytically solvable cases of fields with nonzero potentials that correspond to more complex $F(\mathcal{R})$ than $F \sim \mathcal{R}$.

The paper is organized as follows. The next section discusses the basic features of the STT representation of HMPG [4, 8]. Section 3 is devoted to static, spherically symmetric solutions in the massless case ($V(\phi) = 0$) for both canonical and phantom ϕ fields. Section 4 discusses analytical solutions with $V(\phi) \neq 0$, also with canonical and phantom fields. In all cases we pay special attention to globally regular solutions and solutions containing Killing horizons, in particular, possible black hole solutions with nonzero potentials. Section 5 is a brief consideration of the stability of all HMPG solutions discussed in this paper under radial (monopole) perturbations. Section 6 contains some concluding remarks.

2. BASIC FEATURES OF HMPG AND ITS SCALAR-TENSOR REPRESENTATION

Variation of (1) with respect to the independent connection $\hat{\Gamma}^\alpha_{\mu\nu}$ leads to the conclusion [8, 17] that $\hat{\Gamma}^\alpha_{\mu\nu}$ is the Levi-Civita connection corresponding to a metric conformal to $g_{\mu\nu}$, namely $h_{\mu\nu} = \phi g_{\mu\nu}$, with the conformal factor $\phi = F_{\mathcal{R}} \equiv dF/d\mathcal{R}$. It shows that this theory actually contains, in addition to $g_{\mu\nu}$, only one dynamic degree of freedom expressed in the scalar field ϕ. As shown in [4, 8], the whole theory admits a reformulation as a scalar-tensor theory with the gravitational part of the action

$$S_g = \int d^4x\sqrt{-g}$$
$$\times \left[(1+\phi)R - \frac{3}{2\phi}(\partial\phi)^2 - V(\phi)\right], \quad (2)$$

where[1] the potential $V(\phi)$ is related to $f(\mathcal{R})$ by

$$V(\phi) = \mathcal{R}F_{\mathcal{R}} - F(\mathcal{R}). \quad (3)$$

The theory with the action (2) evidently belongs to the Bergmann−Wagoner−Nordtvedt class of STT [18−20] in which the gravitational action is

$$S_g = \int d^4x\sqrt{-g}$$
$$\times \left[f(\phi)R + h(\phi)(\partial\phi)^2 - V(\phi)\right], \quad (4)$$

with arbitrary functions $f(\phi)$, $h(\phi)$, and $V(\phi)$. In our case, V is given by (3), while

$$f(\phi) = 1 + \phi, \quad h(\phi) = -\frac{3}{2\phi}. \quad (5)$$

The general action (4) admits a well-known transformation [19] to the Einstein conformal frame in which the scalar field is minimally coupled to the metric (while the formulation (4) is called the Jordan conformal frame). The transformation reads [19]

$$\bar{g}_{\mu\nu} = f(\phi)g_{\mu\nu}, \quad \frac{d\phi}{d\bar{\phi}} = f(\phi)|D(\phi)|^{-1/2},$$
$$D(\phi) = f(\phi)h(\phi) + \frac{3}{2}\left(\frac{df}{d\phi}\right)^2, \quad (6)$$

and results in

$$S_g = \int d^4x\sqrt{-\bar{g}}\left[\bar{R} + n\bar{g}^{\mu\nu}\bar{\phi}_{,\mu}\bar{\phi}_{,\nu} - \frac{V(\phi)}{f^2(\phi)}\right], \quad (7)$$

where bars mark quantities obtained from or with the transformed metric $\bar{g}_{\mu\nu}$, while the factor $n = \text{sign}D(\phi)$ distinguishes canonical scalar fields ($n = +1$) with positive kinetic energy from so-called phantom fields ($n = -1$) with negative kinetic energy.

In the theory (2) we have $D = -3/(2\phi)$ and $n = -\text{sign}\phi$, so that

$$\phi = -\tanh^2\frac{\bar{\phi}}{\sqrt{6}} \quad (n = +1, \ -1 < \phi < 0), \quad (8)$$

$$\phi = \tan^2\frac{\bar{\phi}}{\sqrt{6}} \quad (n = -1, \ \phi > 0). \quad (9)$$

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[1] Unlike [4, 8, 17] etc., we are using the metric signature $(+ - - -)$, hence the plus sign before $(\partial\phi)^2 = g^{\mu\nu}\phi_\mu\phi_\nu$ corresponds to a canonical field and a minus to a phantom field. We will also safely omit the factor $1/(2\varkappa^2)$ at the gravitational part of the action since only vacuum configurations, where $S_m = 0$, will be considered. The Ricci tensor is defined as $R_{\mu\nu} = \partial_\nu\Gamma^\alpha_{\mu\alpha} - \dots$, so that, for example, the scalar curvature is positive in de Sitter space-time. We also use the units in which $c = G = 1$ (c being the speed of light and G the Newtonian gravitational constant).

Thus, depending on the sign of ϕ, the theory splits into canonical and phantom sectors, and the emergence of the latter looks more natural since in this case all three metrics $g_{\mu\nu}$, $\bar{g}_{\mu\nu}$ and $h_{\mu\nu} = \phi g_{\mu\nu}$ have the same signature. Let us also note that values of ϕ smaller than -1 lead to a negative effective gravitational constant and are thus manifestly nonphysical.

The substitution $\phi = -n\chi^2/6$ converts the action (2) to the form

$$S_g = \int d^4x \sqrt{-g}\Big[(1 - n\chi^2/6)R$$
$$+ n(\partial\chi)^2 - W(\chi)\Big], \qquad (10)$$

with $W(\chi) = V(\phi)$. The action (10) describes GR where the source of gravity is a conformally coupled scalar field (as mentioned in [8]), which has the usual sign of kinetic energy if $\phi < 0$ ($n = 1$) and is of phantom nature if $\phi > 0$ ($n = -1$). Conformally coupled scalar fields have been considered in a great number of studies, beginning with those of Penrose [21] (a massless conformally invariant field) and Chernikov and Tagirov [22] (massive conformally coupled fields). The theory (10) with a phantom scalar was also discussed in [23] as a possible alternative to GR in astrophysical and cosmological applications.

In the massless case, $V(\phi) \equiv W(\chi) = 0$, the field equations due to (2) or (10) imply that all vacuum solutions (such that $S_m = 0$) have both zero Ricci scalars, $R = \mathcal{R} = 0$ (see [17]). A general inverse result is also valid [16]:

Let there be a vacuum solution with $R \equiv 0$ and a non-constant scalar field in a theory (4), with $V \equiv 0$, then this STT reduces either to GR with a conformally coupled scalar field (which may be canonical or phantom) or to pure conformal scalar field theory.

The transition (6) is well known as a method of finding exact or approximate solutions to the field equations due to (4) since the equations due to (7) are simpler than those due to (4). An Einstein-frame solution having been found, its Jordan-frame counterpart is easily produced by a transformation inverse to (6).

There is, however, an important subtle point: if the function $f(\phi)$ in (4) turns to zero or infinity at some value of ϕ, it may happen that a singularity in the Einstein-frame manifold $\mathbb{M}_E$ with the metric $\bar{g}_{\mu\nu}$ transforms into a regular surface in the Jordan-frame manifold $\mathbb{M}_J$ with the metric $g_{\mu\nu}$ (or vice versa), and $\mathbb{M}_J$ should then be continued beyond this surface. Such a phenomenon, termed *conformal continuation* [24, 25], has been observed in special cases of a number of scalar-vacuum and scalar-electrovacuum solutions, in particular, those of GR with conformally

coupled scalar fields [26, 27] and in the Brans-Dicke theory [30, 31] (the so-called cold black holes).

All static, spherically symmetric solutions with $V \equiv 0$ are well known, but since they admit a new interpretation in terms of HMPG, it makes sense to discuss them from this viewpoint, it is done in Section 3. A large number of scalar-vacuum solutions with $V \not\equiv 0$ are also known (see, e.g., [34−38] and references therein), and we will discuss some of them in the context of HMPG in Section 4.

A question of interest is: suppose we have found a solution of STT with some $V(\phi)$, then, what is the corresponding HMPG? In other words, given $V(\phi)$, can we determine $F(\mathcal{R})$?

For the case $V(\phi) \equiv 0$, Eq. (3) gives simply $F(\mathcal{R}) = \text{const} \cdot \mathcal{R}$. For $V(\phi) \not\equiv 0$, since $\phi = F_{\mathcal{R}}$, the relation (3) is a Clairaut equation (see, e.g., [39]) whose solution consists of a regular family that contains only linear functions,

$$F(\mathcal{R}) = H\mathcal{R} - V(H), \quad H = \text{const}, \qquad (11)$$

and the so-called singular solution which is an envelope of the regular family and may be presented in a parametric form:

$$F(\mathcal{R}) = \phi\mathcal{R} - V(\phi),$$
$$\mathcal{R} = dV/d\phi. \qquad (12)$$

This issue is discussed in more detail in [17].

3. SOLUTIONS FOR $V(\phi) \equiv 0$

In the case $V(\phi) \equiv 0$, solutions to the Einstein-minimally coupled scalar equations can be written in a unified form for canonical and phantom scalars using the harmonic coordinate condition [26]

$$\alpha(u) = 2\beta(u) + \gamma(u), \qquad (13)$$

in terms of the general static, spherically symmetric metric in $\mathbb{M}_E$

$$ds_E^2 = e^{2\gamma}dt^2 - e^{2\alpha}du^2 - e^{2\beta}d\Omega^2,$$
$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2. \qquad (14)$$

The solution reads

$$\bar{\phi} = \bar{C}u + \bar{\phi}_0, \quad \gamma(u) = -hu,$$

$$e^{-\beta(u)-\gamma(u)} = s(k, u) := \begin{cases} k^{-1}\sinh ku, & k > 0 \\ u, & k = 0 \\ k^{-1}\sin ku, & k < 0, \end{cases}$$

$$h, \ k, \ \bar{C}, \ \bar{\phi}_0 = \text{const}, \qquad (15)$$

where, without loss of generality, the radial coordinate u is defined at $u > 0$ ($u = 0$ corresponds to flat spatial

infinity), while the integration constants h, k, and $\bar{C}$ (the scalar charge) are constrained by the relation

$$2k^2 \text{sign} k = 2h^2 + n\bar{C}^2, \qquad (16)$$

where, as before, $n = +1$ corresponds to a canonical field and Fisher's solution [40], and $n = -1$ to its phantom counterpart [41, 42] (sometimes called the "anti-Fisher" solution). The metric (14) now reads

$$ds_E^2 = \text{e}^{-2hu}dt^2 - \frac{\text{e}^{2hu}}{s^2(k,u)}\left[\frac{du^2}{s^2(k,u)} + d\Omega^2\right]. \quad (17)$$

As follows from (16), with $n = +1$ we have $k > 0$, hence there is a single branch, whereas for a phantom scalar the solution splits into three branches according to (15), with qualitatively different properties. Their detailed descriptions may be found, e.g., in [43, 44].

3.1. The Canonical Sector

According to the above-said, the Jordan-frame metric and the scalar field ϕ in the theory (2) with $V(\phi) = 0$ and $\phi < 0$ ($n = +1$) may be presented as [26]

$$ds_J^2 = \cosh^2\psi\bigg\{\text{e}^{-2hu}dt^2$$
$$-\frac{k^2\text{e}^{2hu}}{\sinh^2 ku}\left[\frac{k^2 du^2}{\sinh^2 ku} + d\Omega^2\right]\bigg\}, \qquad (18)$$

$$\phi(u) = \tanh^2\psi, \quad \psi := \bar{\phi}/\sqrt{6} = Cu + \psi_0, \quad (19)$$

where the notation $\psi = \bar{\phi}/\sqrt{6}$ has been introduced for convenience, $C = \bar{C}/\sqrt{6}$, $\psi_0 = \bar{\phi}_0/\sqrt{6}$, and the constant k is expressed via C and h:

$$k = \sqrt{h^2 + 3C^2}. \qquad (20)$$

The solution is defined at $u > 0$ so that $u = 0$ corresponds to spatial infinity, near it, the spherical radius $r = \sqrt{-g_{22}}$ behaves as $r \sim 1/u$, and the Schwarzschild mass is[2]

$$m_J = h - C\tanh\psi_0. \qquad (21)$$

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[2] If we write the general static, spherically symmetric metric in the form (14) with an arbitrary radial coordinate u, this metric is asymptotically flat at some $u = u_0$ if [43]

$$\text{e}^{\beta(u)} \equiv r(u) \xrightarrow[u \to u_0]{} \infty, \quad |\gamma(u_0)| < \infty,$$

$$\text{e}^{\beta-\alpha}|\beta'| \xrightarrow[u \to u_0]{} 1.$$

Then, comparing (14) with the Schwarzschild metric, it is easy to obtain a general expression for the Schwarzschild mass at $u = u_0$:

$$m = \lim_{u \to u_0} \text{e}^{\beta}\gamma'/\beta'.$$

In particular, for the (anti-)Fisher metric (17) we have $m = h$ at $u_0 = 0$.

At the other end of the u range, as $u \to \infty$, there are three kinds of behavior:

- $C < h$: we have $g_{00} \to 0$, and $r \to \infty$. It is a naked attracting singularity located beyond a throat, the kind of singularity called a "space pocket" by P. Jordan [45].

- $C > h$: in this case, $g_{00} \to \infty$ and $r \to 0$, so this is a naked singularity at the center, repulsive for test particles.

- $C = h > 0$: both g_{00} and r tend to finite limits, so that $u = \infty$ is a regular sphere, and a continuation beyond it is necessary.

To extend the solution for $C = h$ beyond $u = \infty$, let us put

$$y = \coth hu, \quad u = \frac{1}{2h}\ln\frac{y+1}{y-1}. \qquad (22)$$

The metric becomes

$$ds_J^2 = \frac{(y+y_1)^2}{1-y_1^2}\left[\frac{dt^2}{(y+1)^2}\right.$$
$$\left. - \frac{h^2}{y^4}(y+1)^2(dy^2 + y^2 d\Omega^2)\right], \qquad (23)$$

where $y_1 = \coth(\psi_0/C)$. The sphere $u = \infty \leftrightarrow y = 1$ is now manifestly regular. We obtain:

- $y \to \infty$ is flat spatial infinity.

- $y_1 < 0 \Rightarrow y = -y_1 > 0$ is a *naked attracting singularity* at the center ($r \to 0$).

- $y_1 > 0 \Rightarrow y \to 0$ is one more flat infinity, and the whole configuration is a *traversable wormhole*.

- $y_1 = 0 \Rightarrow y = 0$ is a double horizon. Passing on to the coordinate $r = h(y+1)$, we obtain

$$ds_J^2 = \left(1 - \frac{h}{r}\right)^2 dt^2$$
$$- \left(1 - \frac{h}{r}\right)^{-2} dr^2 - r^2 d\Omega^2, \qquad (24)$$

which is the well-known *black hole solution with a scalar charge and a conformal scalar field* [27, 28], sometimes called the BBMB black hole solution.

One can notice that the substitution (22) loses its meaning at $y < 1$. Accordingly, the relation (8), that is, $\phi = -\tanh^2\psi$, is also meaningless at $\phi < -1$. Instead, after the conformal continuation, we have [25] in the Einstein frame another copy of the

Fisher solution, where, instead of (8), $\phi = -\coth^2\psi$, and the conformal factor in (18) is $\sinh^2\psi$. At the transition surface $y = 1$, the field ϕ crosses the critical value $\phi = -1$, and beyond it, at $\phi < -1$, there is an "antigravitational" region, with a negative effective gravitational constant, where, in other words, the graviton becomes a ghost [46].

We see that the solutions with $V \equiv 0$ and $n = +1$ generically contain naked singularities, while the only existing black hole and wormhole solutions are special, emerge due to conformal continuations, and each of them contains an "antigravitational" region.

3.2. The Phantom Sector

Assuming $\phi > 0$, $n = -1$, we obtain, quite similarly to (18),

$$ds_J^2 = \cos^2\psi\left\{e^{-2hu}dt^2\right.$$
$$\left. - \frac{e^{2hu}}{s^2(k,u)}\left[\frac{du^2}{s^2(k,u)} + d\Omega^2\right]\right\}, \qquad (25)$$

$$\phi(u) = \tan^2\psi, \quad \psi := \bar\phi/\sqrt{6} = Cu + \psi_0. \qquad (26)$$

For the integration constants k, h, and C we now have

$$k^2\mathrm{sign}k = h^2 - 3C^2. \qquad (27)$$

We see that $\mathrm{sign}k$ is not fixed, and accordingly the solution splits into three branches.

Let us assume, without loss of generality, $|\psi_0| < \pi/2$. Then, as before, the metric (25) is asymptotically flat at $u = 0$ ($r \equiv \sqrt{-g_{22}} \to \infty$ where $r \sim 1/u$),[3] and the Schwarzschild mass is

$$m = h + C\tan\psi_0. \qquad (28)$$

Other properties of the solution depend on the sign of k, taking into account the definition of $s(k,u)$ in (15).

Branch A: $k > 0$. The metric reads

$$ds_J^2 = \cos^2\psi\left\{e^{-2hu}dt^2\right.$$
$$\left. - \frac{k^2 e^{2hu}}{\sinh^2 ku}\left[\frac{k^2 du^2}{\sinh^2 ku} + d\Omega^2\right]\right\},$$
$$\psi = \psi_0 + Cu. \quad h^2 = 3C^2 + k^2. \qquad (29)$$

The only difference from (18) is the conformal factor $\cos^2(Cu + \psi_0)$ instead of $\cosh^2(Cu + \psi_0)$, which drastically changes the metric behavior. Indeed, as u grows from zero, $\psi(u)$ ultimately reaches the value where $\cos\psi = 0$ where, according to Eq. (9), $\phi \to \infty$.

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[3] The conformal factor $\cos^2\psi$ is not normalized to unity at $u = 0$ if $\psi_0 \neq 0$, which, however, does not affect the further description.

Fig. 1. The behaviors C1, C2, C3 of the metric (31) is illustrated by the corresponding curves. We assume $|k| = 1$; curves C1–C3 plot $\cos\psi$ for $C = 0.7$ and different ψ_0. Curve C1 corresponds to a naked singularity, C2 to a wormhole, and C3 to a black hole.

This happens where $Cu + \psi_0 \to \pi/2$ if $C > 0$ and where $Cu + \psi_0 \to -\pi/2$ if $C < 0$. Other quantities involved in the metric are there evidently finite. Thus it is a naked central (since $r \to 0$) singularity, and it is attractive for test particles due to $g_{00} \to 0$.

Branch B: $k = 0$. In this case, the solution has the form

$$ds_J^2 = \cos^2\psi\left[e^{-2hu}dt^2 - e^{2hu}\left(\frac{du^2}{u^4} + \frac{d\Omega^2}{u^2}\right)\right],$$
$$\psi = \psi_0 + Cu, \quad h^2 = 3C^2. \qquad (30)$$

As in Branch A, the coordinate u ranges from zero to the value where $\cos\psi = 0$ (say, $\psi = \pi/2$) and $\phi = \infty$, and we observe a central attractive singularity.

Branch C: $k < 0$. Now the solution reads

$$ds_J^2 = \cos^2\psi\left[e^{-2hu}dt^2 - \frac{k^2 e^{2hu}}{\sin^2 ku}\right.$$
$$\left. \times \left(\frac{k^2 du^2}{\sin^2 ku} + d\Omega^2\right)\right],$$
$$\psi = \psi_0 + Cu, \quad h^2 = 3C^2 - k^2. \qquad (31)$$

The solution behavior crucially depends on ψ_0 at given k, C and depends on which of the quantities $\sin|k|u$ or $\cos\psi$ will be the first to vanish as u grows beginning from zero. For asymptotic flatness we should assume that at $u = 0$ the factor $\cos^2\psi$ is nonzero, hence $|\psi_0| < \pi/2$ without loss of generality. Then, three possible behaviors should be singled out, see Fig. 1 (we assume for certainty $C > 0$).

C1: $(\pi/2 - \psi_0)/C < \pi/|k|$. The solution terminates at $u = u_s = (\pi/2 - \psi_0)/C$, where $\cos\psi = 0$, and $u = u_s$ is a naked central singularity quite similar to the one in branches A and B.

C2: $(\pi/2 - \psi_0)/C > \pi/|k|$. The solution terminates at $u_* = \pi/|k|$ where $\sin ku = 0$, corresponding

Fig. 2. Carter–Penrose diagram for a regular black hole with the metric (31). The diagram infinitely extends up and down.

to the second flat spatial infinity, where the radius r infinitely grows while g_{tt} and ϕ remain finite, and the Schwarzschild mass is there equal to

$$m_* = -e^{hu_*}(h\cos\psi_* + C\sin\psi_*) \qquad (32)$$

($\psi_* = \psi_0 + C\pi/|k| < \pi/2$ is the value of ψ at $u = u_*$). Such a wormhole solution is only quantitatively different from its anti-Fisher and Brans-Dicke analogs, see, e.g., [26, 29].

C3: $(\pi/2 - \psi_0)/C = \pi/|k|$. In this intermediate case, at $u = u_1 = \pi/|k|$ vanish both $\sin|k|u$ and $\cos\psi$, the spherical radius $r = \sqrt{-g_{\theta\theta}}$ is finite but $\phi = \infty$. Near $u = u_1$, the metric behaves as

$$ds_J^2 = C^2 \left[e^{-2hu_1}\Delta u^2 dt^2 - e^{2hu_1}\frac{du^2}{\Delta u^2} \right.$$
$$\left. - e^{2hu_1}d\Omega^2 \right], \qquad (33)$$

where $\Delta u = u_1 - u$. Consequently, $u = u_1$ is a double (extremal) horizon, and the metric should be continued beyond it.

The condition $(\pi/2 - \psi_0)/C = \pi/|k|$ leads to $\psi_0 = \pi(1/2 - C/|k|)$, thus $C < |k|$, hence the plot of $\cos\psi$ is wider than that of $\sin ku$. Therefore, as u further grows (describing the region beyond the horizon), the next zero of $\sin ku$ ($u = u_2 = 2\pi/|k|$) is reached before a zero of $\cos\psi$. The value $u = u_2$ corresponds to the second flat spatial infinity, in full similarity with wormhole solutions. This space-time is also globally regular but is now only one-side traversable due to emergence of the horizon. Figure 2 shows the corresponding Carter–Penrose diagram.

This black hole solution has much in common with the one with the metric (24) [26–28]. Both black holes are described by special solutions to the Einstein-scalar equations, both are asymptotically flat and extremal (with zero Hawking temperature), and in both cases the supporting scalar fields turn to infinity on the horizon, whereas the effective stress-energy tensors T_μ^ν are finite there (as is evident from finiteness of the Einstein tensor components G_μ^ν). Also, in both cases the scalar curvature is zero in the whole space, and the solutions are obtained from their Einstein-frame counterparts using conformal continuations [25, 26].

However, the solution (24) has a singular center $r = 0$ (the geometry is the same as that of the extreme Reissner–Nordström space-time), while the space-time (31) is globally regular and has no center at all.

It happens that none of the static, spherically symmetric solutions of the theory (2) with $V \equiv 0$ have simple horizons with finite Hawking temperature, which contradicts the results announced in [17].

A geometry with infinitely many horizons. There is a one-parameter family of solutions of interest obtained if we put

$$C = |k|, \quad \psi_0 = -\pi/2 \qquad (34)$$

(thus abandoning the asymptotic flatness requirement). We then have $\cos^2\psi = \sin^2 ku$, and the Jordan-frame metric reads

$$ds_J^2 = \sin^2 ku\, e^{-2hu}dt^2$$
$$- k^2 e^{2hu}\left(\frac{k^2 du^2}{\sin^2 ku} + d\Omega^2\right), \qquad (35)$$

where $h = \pm k$ according to (27). This metric, with $u \in \mathbb{R}$, describes a space-time unifying an infinite number of static regions (each described by a half-wave of the function $\sin ku$), separated by double horizons located at each $u = \pi n/|k|$, with any integer n. In this case, the Jordan-frame manifold $\mathbb{M}_J$ unifies a countable number of Einstein-frame manifolds $\mathbb{M}_E$, each of the latter representing an anti-Fisher wormhole whose both infinities turn into horizons in $\mathbb{M}_J$. Another example of a manifold obtained by infinitely many conformal continuations was obtained in [25], using a solution for a conformally coupled scalar field ϕ with a nonzero potential $U(\phi)$ and the normal sign of kinetic energy. In that example, the transition from one region to another occurred through ordinary surfaces S_{trans} of finite radius, there were no horizons, and the whole $\mathbb{M}_J$ was either completely static (shaped as an infinitely long tube with a periodically changing radius) or completely cosmological (forming a $(2+1)$ cosmology with a periodically changing scale factor). In the present case, all transitions surfaces S_{trans} are double horizons, and the structure is aperiodic due to the factors $e^{\pm hu}$ in (35). The corresponding global structure is shown in Fig. 3.

Fig. 3. Carter−Penrose diagram of the manifold $\mathbb{M}_J$ with the metric (35). The diagram occupies the whole plane. The thick broken line shows a horizon corresponding to a particular value of $u = n\pi/|k|$, $n \in \mathbb{N}$.

Another example of a manifold with infinitely many horizons [51] has been obtained for a family of phantom dilaton-Einstein−Maxwell black holes.

4. SOLUTIONS FOR $V(\phi) \not\equiv 0$

Before discussing particular examples, it makes sense to recall some general theorems concerning scalar-vacuum space-times with $V(\phi) \not\equiv 0$. Owing to such theorems, one can say much about the possible behavior of solutions to the field equations even being unable to solve them analytically. Many of these results concern the properties of minimally coupled scalar fields, for example, no-hair theorems (see, e.g., [48] for a recent review) indicating the conditions that exclude the existence of horizons, and global structure theorems [49] telling us about possible regular solutions and a maximum possible number of horizons. In particular, if $V(\phi) \geq 0$, an asymptotically flat black hole with a nontrivial canonical scalar field is impossible [50]. On the other hand, by [49], spherically symmetric scalar-vacuum space-times cannot contain more than two horizons, and this number is only one for asymptotically flat configurations. This result holds for both canonical and phantom scalars with any $V(\phi)$.

For Jordan-frame space-times, conformal to those with minimally coupled scalar fields, most of the theorems are preserved without changes if the conformal factor $f(\phi)$ in (6) is everywhere finite and regular since (at transitions to either side) a flat infinity maps to a flat infinity, a horizon maps to a horizon, and the potential $V(\phi)$ preserves its sign. The situation changes if $f(\phi)$ is somewhere zero or infinite, then the mapping can change the nature of singularities, if any, and conformal continuations can emerge. The

above examples show that such continuations can be numerous, up to an infinite number, as we saw in the Jordan-frame metric (35). In particular, the number and nature of horizons in $\mathbb{M}_J$ may be different from that in $\mathbb{M}_E$, including the possible number of simple horizons. It is, however, important that conformal continuations can only emerge at special values of integration constants [25].

As in the above massless case ($V \equiv 0$), HMPG solutions with nonzero potentials split into the canonical and phantom sectors, in which the conformal factors $1/f(\phi) = 1/(1 + \phi)$ have the forms $\cosh^2 \psi$ and $\cos^2 \psi$, respectively. The first one is able to blow up and the second one to vanish, so in both cases the nature of solutions in $\mathbb{M}_J$ can be quite different from that in $\mathbb{M}_E$. We will briefly analyze the behavior of such solutions in some particular cases of $V(\phi)$ admitting analytical solutions, known from the literature [34, 35, 37].

4.1. $V \not\equiv 0$, the Canonical Sector

Example 1 . This solution (in the Einstein frame) has been obtained by the inverse problem method [34] for a minimally coupled scalar field in the metric

$$ds_E^2 = A(x)dt^2 - \frac{dx^2}{A(x)} - r^2(x)d\Omega^2 \quad (36)$$

(that is, (14) under the so-called quasiglobal coordinate condition $\alpha + \gamma = 0$) by assuming

$$r(x) = \sqrt{x^2 - a^2}, \quad (37)$$

where a plays the role of a length scale. Let us assume $a = 1$, thus expressing all quantities in terms of this arbitrary length scale. Since one of the Einstein equations for the action (7) reads

$$2r''/r = -n\bar{\phi}'^2, \quad (38)$$

and since now $r''/r = -(x^2 - 1)^{-2} < 0$, we are dealing with a canonical scalar field, $n = +1$. Using, as before, $\psi = \bar{\phi}/\sqrt{6}$, we obtain the asymptotically flat (as $x \to \infty$) solution in the form [34]

$$A(x) = 1 - 3mx + \frac{3}{2}(x^2 - 1) \ln \frac{x+1}{x-1}, \quad (39)$$

$$\psi(x) = \frac{1}{2\sqrt{3}} \ln \frac{x+1}{x-1} + \psi_0, \quad \psi_0 = \text{const}, \quad (40)$$

$$U(\psi) = \frac{3m}{x^2 - 1} \left[6x + (3x - 1) \ln \frac{x+1}{x-1} \right], \quad (41)$$

where $U(\psi)$ is the Einstein-frame potential according to (7):

$$U(\psi) = V(\phi)/(1 + \phi)^2 \quad (42)$$

Fig. 4. The function $A(x)$ according to (39) for $m = 0.1, 0.2, 1/3, 0.5, 0.7$ (upside down). It has a simple zero if $m > 1/3$, the solution then describes a black hole. All solutions have a singularity at $x = -1$, where $r = 0$.

(in (41) it is a function of x, but its ψ dependence is easily restored by substituting $x = x(\psi)$ determined from (40)). In this solution, $x \to \infty$ is flat spatial infinity, m is the Schwarzschild mass, and the value $x = 1$ corresponds to the spherical radius $r = 0$, which is a naked central singularity if $m \leq 1/3$ and a singularity hidden under an event horizon if $m > 1/3$, see Fig. 4.

In the "massless" case, $m = 0$, we have $A \equiv 1$, $U \equiv 0$, and the present solution coincides with the case $h = 0$ of the solution (18), (19), though expressed using another radial coordinate.

The potential U is proportional to m, it is everywhere negative, singular at $x = 1$ and rapidly vanishes at infinity:

$$U(x) \sim \frac{\ln(x-1)}{x-1} \quad \text{as} \quad x \to 1,$$

$$U(x) \approx -\frac{8m}{5x^5} \quad \text{as} \quad x \to \infty. \qquad (43)$$

The Jordan-frame metric is

$$ds_J^2 = \cosh^2 \psi \, ds_E^2. \qquad (44)$$

The conformal factor $\cosh^2 \psi$ is well-behaved at $x > 1$, tends to a constant at large x (so that the metric ds_J^2 is also asymptotically flat), and blows up as $x \to 1$:

$$\cosh^2 \psi \approx \cosh^2 \psi_0 + \frac{\sinh(2\psi_0)}{\sqrt{3}x}, \quad x \to \infty,$$

$$\cosh^2 \psi \sim (x-1)^{-1/\sqrt{3}}, \quad x \to 1. \qquad (45)$$

Meanwhile, in the same limit $x \to 1$,

$$A(x) \approx 1 - 3m + 3m(x-1)[\ln 2 - 1 - \ln(x-1)].$$

Thus the conformal factor cannot regularize the metric at $x = 1$: it enhances the singularity if $m \neq 1/3$

(for example, g_{tt} remains finite in the Einstein frame but blows up in Jordan's) and only modifies it if $m = 1/3$. As a whole, the conformal factor only deforms the metric at $x > 1$ but does not change it qualitatively. We conclude that the HMPG solution with the potential according to (42), that is,

$$V(\phi) = U(\psi)f^2(\phi) = U(\psi)/\cosh^4 \psi \qquad (46)$$

describes a black hole with a simple horizon in the case $m > 1/3$. The black hole mass is equal to m in the Einstein frame, while in Jordan's we have

$$m_J = m \cosh \psi_0 - \frac{1}{\sqrt{3}} \sinh \psi_0. \qquad (47)$$

They coincide if $\psi_0 = 0$.

Example 2. Another Einstein-frame solution with the metric (36) has been obtained in [37] with the so-called separability approach but can also be found in full similarity with (37)–(41) by assuming

$$r(x) = \sqrt{x(x+a)}, \qquad (48)$$

where a again plays the role of a length scale and can be put equal to unity. The solution now reads

$$A(x) = 1 - 6m(2x+1)$$
$$+ 12mx(x+1) \ln \frac{x+1}{x}, \qquad (49)$$

$$\psi(x) = \frac{1}{2} \ln \frac{x+1}{x} + \psi_0, \qquad (50)$$

$$U(\psi) = -\frac{12m}{x(x+1)} \Big[-3(1+2x)$$
$$+ (1 + 6x + 6x^2) \ln \frac{x+1}{x} \Big]. \qquad (51)$$

The properties of this solution are quite similar to those of (37)–(41). The canonical nature of the scalar field is assured by the fact that $r''/r = -1/[4x^2(x+1)^2] < 0$. The solution has a naked singularity at $x = 0$ if $m \leq 1/6$ and describes a black hole with a simple horizon if $m > 1/6$, and the plots of $A(x)$ for different m look almost the same as in Fig. 4. The conformal factor $\cosh^2 \psi$ deforms the metric but does not remove the singulatities.

4.2. $V \not\equiv 0$, the Phantom Sector

Consider an analytic solution for a minimally coupled phantom scalar ψ [35], also obtained by the inverse problem method. Now we assume

$$r(x) = \sqrt{x^2 + a^2}, \qquad (52)$$

and, as before, put $a = 1$ as an arbitrary length scale. The inequality $r''/r = (x^2 + 1)^{-2} > 0$ confirms the phantom nature of ψ and hence the original scalar ϕ.

Fig. 5. The metric function $A(x)$ (a) and the potential $U(\phi(x))$ (b) for the Einstein-frame solution (36), (53)–(55), $m = -0.1, -0.05, 0.05, 0.1$ (bottom-up for $A(x)$, upside down for $U(x)$). As $x \to -\infty$, the metric is asymptotically de Sitter if $m > 0$ and AdS if $m < 0$.

Then, with the Einstein-frame metric (36), we have the solution [35]

$$A(x) = 1 + 3mx - 3m(x^2 + 1)\text{arccot}x, \quad (53)$$

$$\psi(x) = \frac{1}{\sqrt{3}}\arctan x + \psi_0, \quad (54)$$

$$U(\psi) = \frac{6m}{x^2 + 1}\left[-3x + (3x^2 + 1)\text{arccot}x\right]. \quad (55)$$

In this solution, $x \in \mathbb{R}$, $x \to \infty$ is flat infinity, and m has the meaning of the Schwarzschild mass. The behavior of the solution as $x \to -\infty$ is different, depending on the sign of m:

- $m < 0$: $A \sim x^2$, $U \to \text{const} < 0$—the solution describes a wormhole with an AdS limit at the "far end."

- $m = 0$: $A \to 1$, $U \equiv 0$—it is the simplest (Ellis) twice asymptotically flat wormhole with zero mass [26, 42] ($A \equiv 1, U \equiv 0$).

- $m > 0$: $A \sim -x^2$, $U \to \text{const} > 0$—we obtain a regular black hole with a de Sitter expansion far beyond the horizon instead of a singularity (a "*black universe*" [35, 36]).

The behavior of $A(x)$ and $U(x)$ is shown in Fig. 5.

In Jordan's frame we have the metric

$$ds_J^2 = \cos^2\psi\left[A(x)dt^2 - \frac{dx^2}{A(x)} - r^2(x)d\Omega^2\right], \quad (56)$$

and the geometry crucially depends on the value of ψ_0. The range of ψ is

$$\text{Range}(\psi) = \left(\psi_0 - \frac{\pi}{2\sqrt{3}}, \psi_0 + \frac{\pi}{2\sqrt{3}}\right), \quad (57)$$

its length is $\pi/\sqrt{3} < \pi$, smaller than π, length of the segment where $\cos\psi > 0$. The spatial asymptotic value $x \to \infty$ corresponds to $\psi = \psi_1 = \psi_0 + \pi/(2\sqrt{3})$, and the Schwarzschild mass is there

$$m_J = m\cos\psi_1 - \frac{1}{\sqrt{3}}\sin\psi_1. \quad (58)$$

If $\cos\psi \neq 0$ in the whole range (57) (for example, if $|\psi_0| < \pi(\sqrt{3} - 1)/(2\sqrt{3}) \approx 0.663$), then the conformal factor $\cos^2\psi$ is everywhere positive and regular, it then only deforms the metric ds_E^2 but does not change it qualitatively.

Otherwise, the conformal factor $\cos^2\psi$, in general, creates a singularity at some finite $x = x_s$. It destroys wormhole solutions, producing a central attracting singularity instead of their regular far end; a similar singularity is created instead of a horizon in black universe solutions if $x = x_s$ belongs to a static region or is precisely a horizon; it is like a big bang (or crunch) if it happens to be in a nonstatic region beyond the horizon. Lastly, if $\psi_0 = \pi/2 - \pi/(2\sqrt{3})$, so that $\psi_1 = \pi/2$, then the conformal factor destroys the asymptotic flatness of the solution.

We conclude that in the phantom sector all three kinds of solutions are generic: black hole ones, wormhole ones and those with naked singularities. The black hole solutions can be regular (black-universe type) or singular beyond the horizon, depending on the value of ψ_0.

5. STABILITY

Since the transition (6) to the Einstein frame may be viewed as simply a change of variables in the differential equations, it can be applied to the perturbed field equations on equal grounds with those for static configurations. This enables us to use the

existing results of the studies of small perturbations of scalar-vacuum space-times with minimally coupled scalar fields, see, e.g., [44, 52–59]. The Jordan frame perturbations obey the same equations as in the Einstein frame, but only expressed in other variables. However, the Jordan-frame stability inferences may be different since the boundary conditions should now be formulated according to the physical requirements inherent to $\mathbb{M}_J$.

Let us discuss the stability properties of the solutions described above with respect to purely radial (monopole) perturbations. The experience indicates that these perturbations are, in a clear sense, the most dangerous for configurations with scalar fields: if a system is unstable, it is most probably a monopole mode that implements this instability. A physical reason for that is that the effective potentials for all other perturbations contain centrifugal barriers which are positive and therefore favorable for stability.

5.1. Perturbation Equations

It is well known that in static, spherically symmetric scalar-vacuum space-times, monopole perturbations of the whole system are governed by the scalar field perturbations $\delta\phi(u,t)$, or those of the Einstein-frame field, $\delta\psi(u,t)$, representing the only dynamic degree of freedom. These perturbations obey a single linear equation whose coefficients depend on the parameters of the background static system, while the metric perturbations $\delta\alpha, \delta\beta, \delta\gamma$ (in terms of the metric (14)) can be found from the solutions of the "master equation" for $\delta\psi$. The master equation for a spectral component of the perturbation, $\delta\psi = \Psi(u)e^{i\omega t}$, in the Schrödinger-like canonical form is,

$$\frac{d^2Y}{dz^2} + \left(\omega^2 - W_{\text{eff}}(z)\right)Y = 0. \qquad (59)$$

In this equation, z is the so-called tortoise radial coordinate such that $du/dz = e^{\gamma-\alpha}$, where u is an arbitrary radial coordinate in the metric (14). The unknown function in (59) is $Y(z) = \Psi(u)e^\beta$, while the effective potential $W(z)$ has the form [44, 57, 58]

$$W_{\text{eff}}(z) = e^{2\gamma}\left[\frac{3n\psi'^2}{\beta'^2}(U - 2e^{-2\beta}) + \frac{\psi'}{\beta'}U_\psi\right.$$
$$\left. + \frac{n}{12}U_{\psi\psi}\right] + e^{2\gamma-2\alpha}[\beta'' + \beta'(\beta' + \gamma' - \alpha')], \quad (60)$$

where the index ψ denotes $d/d\psi$, the prime denotes d/du (u is again an arbitrary coordinate in (14)), and $U = U(\psi) = V(\phi)/(1 + \phi)^2$, see (42). It should be stressed here that the notations α, β, γ refer to the metric (14) written in the Einstein frame.

Solving this equation with appropriate boundary conditions, we find a spectrum of eigenvalues ω^2 of

this boundary-value problem, and, as usual, if there are $\omega^2 < 0$, we can conclude that the background configuration is unstable under linear monopole perturbations since there is a time-dependent perturbation growing as $e^{|\omega|t}$. To assert that the instability is inherent to the configuration itself rather than caused by energy pumping from outside, it is also necessary to verify that there is no energy flow into the system through the boundaries. However, this requirement does not lead to any new restrictions for our system: indeed, quite similarly to the reasoning in [52], at flat infinity the energy flux is zero for any admissible solution to (59), while at the other end the flow direction is controlled by the arbitrary sign in a solution to (59) and can always be chosen so that the energy leaks outward.

We will discuss the stability properties of the configurations enumerated above, using as much as possible the previous results available in the literature.

At flat spatial infinity, where both fields ϕ and ψ are regular, we naturally require both $\delta\phi \to 0$ and $\delta\psi \to 0$. In what follows we assume that this requirement is always applied, and focus on boundary conditions on the other end of the range of the radial coordinate.

5.2. Stability: the Canonical Sector

1. $V(\phi) \equiv 0$, solution (18), (19), the conformally mapped Fisher solution in the general case ($C \neq h$). At the singularity $u \to \infty$ we have $\phi \to -1$, and due to its fixed value it is natural to require $\delta\phi \to 0$ for meaningful perturbations. Since $\phi = -\tanh^2\psi$, for ψ this requirement transforms to $\delta\psi = o(e^{2\psi})$, where $\psi \to \infty$, whereas the instability of Fisher's solution was established under the much stronger boundary condition $|\delta\psi/\psi| < \infty$ [52]. The present condition $\delta\psi = o(e^{2\psi})$ is much weaker, therefore, the perturbations which grow with time in Fisher's solution, thus implementing its instability, manifestly satisfy the new, weaker boundary conditions, and we conclude that the solution (18)–(20) is unstable.

2. $V(\phi) \equiv 0$, solution (23), $y_0 > 0$ (a wormhole). This solution is unstable as proved in [60, 61]. The instability is related to the existence of a negative pole $W_{\text{eff}}(z) \approx -1/(4z^2)$ at the transition sphere $z = 0$ ($u = 1$) of the conformal continuation. This leads to the existence of negative eigenvalues ω^2 of the corresponding boundary-value problem.

3. $V(\phi) \equiv 0$, solution (23), $y_0 < 0$ (a naked singularity beyond the transition surface of conformal continuation).The instability conclusion follows from the same reasoning as in the previous case.

4. $V(\phi) \equiv 0$, solution (24) (black hole). This solution is stable as proved in [63], although previously [62] the opposite result was announced.

Fig. 6. The potential $W_{\text{eff}}(x)$ for the solution (39)–(41) with naked singularities, $m = 0.1, 0.25, 1/3$. The inset shows a more detailed behavior of $W_{\text{eff}}(x)$ for $m = 1/3$ near $x = 1$.

5. $V(\phi) \not\equiv 0$, solution (36), (39)–(41), $m \leq 1/3$ (a naked singularity). For this case, the shape of the effective potential (60) is shown in Fig. 6, and it asymptotically behaves as follows:

$$W_{\text{eff}} = -\frac{2m}{x^3} + \frac{-3 + 4m^2}{x^4} + O(x^{-5}),$$
$$x \to \infty, \qquad (61)$$

$$W_{\text{eff}} = -\frac{(3m-1)^3}{4(x-1)^2} + \frac{27m^2 - 3}{4(x-1)} + O\left(\ln^2(x-1)\right),$$
$$x \to 1. \qquad (62)$$

Thus for $m < 1/3$ the potential $W_{\text{eff}} \to -\infty$, but $W_{\text{eff}} \sim \ln(x-1)^2$ for $m = 1/3$. On the other hand, the "tortoise" coordinate $z = \int dx/A(x)$ is found as follows for $x \to 1$:

$$z \approx \frac{x-1}{1-3m}, \quad m < 1/3,$$
$$z \approx -\ln|\ln(x-1)| \to -\infty, \quad m = 1/3. \qquad (63)$$

For $m < 1/3$ we obtain that $W_{\text{eff}} \approx -1/(4z^2)$ as $z \to 0$; it is precisely the same behavior as for Fisher's solution discussed above in item 1. Since the appropriate boundary condition as $z \to 0$ is here also the same, we can conclude that this solution with a naked singularity in HMPG is unstable.

The case $m = 1/3$ is the most complicated. In this solution, again, $\phi \to -1$ as $x \to 1$, hence the boundary condition for perturbations is

$$\delta\phi = o(1) \Rightarrow \delta\psi = o\left((x-1)^{-1/\sqrt{3}}\right),$$

$$Y = o\left((x-1)^{1-1/\sqrt{3}}\right) \to 0. \qquad (64)$$

On the other hand, in the limit $x \to 1$ we obtain $z \to -\infty$, more precisely,

$$z \approx -\ln|\ln(x-1)| \Rightarrow x - 1 \approx \exp(-e^{|z|}),$$

$$W_{\text{eff}} \approx \frac{1}{4}\ln^2(x-1) \approx \frac{1}{4}e^{2|z|}. \qquad (65)$$

Solving Eq. (59) with this asymptotic form of W_{eff} under the assumption $\omega^2 = -S^2 < 0$, we obtain a linear combination of modified Bessel functions:

$$Y(z) \approx C_1 I_S(e^{|z|}/2) + C_2 I_{-S}(e^{|z|}/2), \qquad (66)$$

where both terms grow at large negative z as $\exp(-|z|/2 + e^{|z|/2}) \to \infty$. It follows that perturbations with imaginary frequencies ($\omega^2 < 0$) cannot satisfy the boundary condition (64), and consequently this solution is stable.

6. $V(\phi) \not\equiv 0$, solution (36), (39)–(41), $m > 1/3$ (a black hole). As follows from (39) and (60), at $m > 1/3$ there is a simple horizon at some $x = x_h > 1$, where $A = 0$, $A' > 0$, and $W_{\text{eff}} = 0$, see Fig. 7. Moreover, it turns out that $W_{\text{eff}} < 0$ at $x > x_h$. Meanwhile, $z(x_h) = -\infty$ since the integral $\int dx/A(x)$ logarithmically diverges there. It follows that $W_{\text{eff}} < 0$ in the whole range of z, which inevitably leads to the existence of eigenvalues $E < 0$ of the quantum-mechanical eigenvalue problem with Eq. (59), where it is required that $Y(x)$ should be quadratically integrable.

Let us determine the boundary condition for the function Y in Eq. (59) at $x = x_h$ in our HMPG model. The horizon $x = x_h$ is some intermediate point, where $-1 < \phi < 0$, so we have no reason to require there anything more than finiteness of $\delta\phi$, and since $\phi = -\tanh^2 \psi$, finiteness of $\delta\psi$. Furthermore, since $Y =$

Fig. 7. Generic behavior of $W_{\mathrm{eff}}(x)$ for the black hole solution (39)–(41), $m > 1/3$ (a), and comparison of $W_{\mathrm{eff}}(x)$ with $A(x)$ (b, c).

$e^\beta \delta\psi$ where $e^\beta = r(x_h)$ is finite, the boundary condition at $x = x_h$ ($z \to -\infty$) is simply $|Y| < \infty$, much weaker than would follow from quadratic integrability of Y. It is therefore clear that the "wave function" corresponding to $\omega^2 < 0$ as a quantum-mechanical "energy level," satisfies our boundary conditions and can implement instability of the black hole models under study.

5.3. Stability: the Phantom Sector

1. $V(\phi) \equiv 0$, solution (25)–(27) (the conformally mapped "anti-Fisher" solution). All branches A−C of the solution in $\mathbb{M}_{\mathrm{E}}$ contain throats $z = z_0$ (where $\beta' = 0$) even though not all of them correspond to wormholes, and, due to β' in the denominator, the potential $W_{\mathrm{eff}}(z)$ for all of them contains a pole, where $W_{\mathrm{eff}}(z) \approx 2/(z - z_0)^2$. This singularity admits regularization by a suitable Darboux transformation, after which $W_{\mathrm{eff}}(z)$ is replaced by a new potential $W_{\mathrm{reg}}(z)$ that is finite and regular in the whole range of u (or z) and is thus suitable for studying boundary-value problems for Eq. (59), as described in detail in [44, 54, 56, 59].

The potential $W_{\mathrm{reg}}(z)$ has different forms for different branches of the solution (25)–(27). We will not present them here, referring to [44] for details. It has turned out that all branches of the anti-Fisher solution are unstable [44, 54], as a result of the existence of a potential well in $W_{\mathrm{reg}}(z)$. To make clear whether or not this conclusion can be extended to the Jordan frame (hence to HMPG), for which $W_{\mathrm{reg}}(z)$ is the same, we must determine the corresponding boundary conditions and compare them with those applicable in the Einstein frame.

In branches A and B ($k \geq 0$), the solution in $\mathbb{M}_{\mathrm{E}}$ exists in the range $u > 0$ that corresponds to $z \in \mathbb{R}$, and an unstable mode is found [44] under the boundary conditions $\delta\psi \to 0$ as $z \to \pm\infty$. However, in $\mathbb{M}_{\mathrm{J}}$, owing to the factor $\cos^2\psi$ in the metric, the range of u only extends from zero to a singular point u_s such that

$\psi = Cu_s + \psi_0 = \pi/2$, and the range of z is truncated at $z_s = z(u_s)$ and reduces to $z \in (z_s, \infty)$. Therefore, the instability conclusion cannot be directly extended to $\mathbb{M}_{\mathrm{J}}$, and a separate new study is necessary, which is beyond the scope of this paper. We can forecast that the results will depend on the solution parameters, including ψ_0. Let us only try to formulate the appropriate boundary condition at the singularity. We have there $\cos\psi = 0$, and, since it is a regular point in $\mathbb{M}_{\mathrm{E}}$, we have in its neighborhood

$$\cos\psi \sim |u - u_s| \sim |z - z_s|,$$
$$\phi = \tan^2\psi \sim (z - z_s)^{-2}.$$

Next, since $\phi \to \infty$ at the boundary, a reasonable condition seems to be $|\delta\phi/\phi| < \infty$, so $\delta\phi$ is allowed to behave as $(z - z_s)^{-2}$. But since $d\phi/d\psi = 2\sin\psi/\cos^3\psi$, we have $\delta\phi \sim \delta\psi/\cos^3\psi$, and our boundary condition further translates to

$$|\delta\psi| \sim \cos\psi \sim z - z_s \Rightarrow \frac{|Y|}{z - z_s} < \infty. \tag{67}$$

In obtaining that, we took into account that u_s is a regular point in the solution in $\mathbb{M}_{\mathrm{E}}$, where, in particular, e^β is finite, hence $Y \sim e^\beta\delta\psi \sim \delta\psi$. Thus it is the condition (67) that should be applied in the boundary-value problem for Eq. (59).

The same situation is found for branch C1, in the cases where a singularity also occurs due to $\cos\psi = 0$.

In the wormhole case C2, the Jordan-frame solution is simply a finite deformation of its counterpart in $\mathbb{M}_{\mathrm{E}}$, therefore, all boundary conditions for perturbations are the same, and the instability conclusion from [44, 54] extends to our HMPG model.

In the black hole case C3, we must formulate the condition on the horizon, which now corresponds to $e^\beta \sim |z| \to \infty$ in $\mathbb{M}_{\mathrm{E}}$, and simultaneously $\cos\psi \sim 1/|z| \to 0$, $\phi \to \infty$. Therefore, if we again require $|\delta\phi/\phi| < \infty$, we obtain then, as in (67), $|\delta\psi/\cos\psi| \sim |z\delta\psi| < \infty$. In its turn, it follows $|Y| \sim e^\beta\delta\psi \sim$

Table 1. HMPG solutions: Stability under monopole perturbations

Solution	Description	Results
Canonical, $V \equiv 0$, (18), (19)	Mapped Fisher's solution, naked singularity	Unstable
Canonical, $V \equiv 0$, (23), $y_0 > 0$	Wormhole with a conformal scalar field	Unstable
Canonical, $V \equiv 0$, (23), $y_0 < 0$	Naked singularity after a conformal continuation	Unstable
Canonical, $V \equiv 0$, (24)	Black hole with a conformal scalar field	Stable
Canonical, $V \not\equiv 0$, (36)–(41), $m < 1/3$	Naked singularity, similar to Fisher's	Unstable
Canonical, $V \not\equiv 0$, (36)–(41), $m = 1/3$	Naked singularity of special kind	Stable
Canonical, $V \not\equiv 0$, (36)–(41), $m > 1/3$	Black hole with a simple horizon	Unstable
Phantom, $V \equiv 0$, (25)–(27), A, B, C1	Naked singularity due to $\cos\psi = 0$	Uncertain*
Phantom, $V \equiv 0$, (25)–(27), C2	Wormhole with a phantom conformal scalar field	Unstable
Phantom, $V \equiv 0$, (25)–(27), C3	Black hole with a phantom conformal scalar field	Unstable
Phantom, $V \equiv 0$, (34), (35)	A single static region among infinitely many horizons	Unstable
Phantom, $V \not\equiv 0$, (53)–(56)	Naked singularity due to $\cos\psi = 0$	Uncertain*
Phantom, $V \not\equiv 0$, (53)–(56)	Wormhole with an AdS or Minkowski far end	Unstable
Phantom, $V \not\equiv 0$, (53)–(56)	Black universe, generic configuration	Unstable
Phantom, $V \not\equiv 0$, (53)–(56)	Black universe, horizon at minimum of $r(x)$	Stable

* See comments around Eq. (67).

$|z\delta\psi| < \infty$. Thus, again, the boundary conditions in $\mathbb{M}_J$ turn out to be less restrictive than they were in $\mathbb{M}_E$ where the instability was established, and we conclude that this result is extended to our HMPG black hole model.

Lastly, in the solution (34), (35) with infinitely many horizons, any region between adjacent horizons is bounded by the same kind of surfaces as just discussed, with the corresponding "weakened" boundary conditions, and it is straightforward to conclude that it is also unstable.

2. $V(\phi) \not\equiv 0$, solution (53)–(56) (the conformally mapped solution from [35] describing wormholes and black universes). The following situations are possible.

(i) Solutions in which the conformal factor $\cos^2\psi$ only deforms the Einstein-frame solution in a regular manner. In these cases, all the stability results obtained for $\mathbb{M}_E$ remain valid in $\mathbb{M}_J$. More specifically: all wormhole solutions are unstable, while among the black-universe solutions there is a stable subset, in which the horizon coincides with the sphere of minimum radius (that is, $x_h = 0$, where $x = x_h$ is the horizon), in all other cases the external static region $x > x_h$ of a black universe is unstable [56]. These instabilities exist due to potential wells of finite depth in $W_{\rm reg}(z)$, see the beginning of Subsection 5.3.

(ii) Solutions describing black universes in $\mathbb{M}_E$, "spoiled" by a singularity $x = x_s < x_h$ due to $\cos\psi = 0$, i.e., there is a big-bang-like singularity located in the T-region beyond the horizon. The stability results for the external region $x > x_h$ remain the same as in item (i).

(iii) Solutions with naked singularities $x = x_s$ in a static region due to $\cos\psi = 0$, which are possible at any value of x in wormhole solutions or with any $x_s > x_h$ in black-universe solutions in $\mathbb{M}_E$. In all such cases, the situation looks the same as previously discussed for $V \equiv 0$: we have a truncated range $x_s < x < \infty$, with the boundary condition (67) at $x = x_s$, and a separate study is necessary to find out the exact (in)stability conditions.

The stability results are summarized in Table 1.

6. CONCLUDING REMARKS

We have considered exact analytical vacuum static, spherically symmetric asymptotically flat solutions of HMPG, using its scalar-tensor representation, with both zero and nonzero potentials $V(\phi)$, on the basis of known solutions of GR with minimally and conformally coupled scalar fields. All configurations split into two large classes, one corresponding to a canonical scalar field ($-1 < \phi < 0$), the other to a phantom one ($\phi > 0$).

It has been stated that in the case $V \equiv 0$ most of the HMPG space-times contain naked singularities, and a generic family of solutions in the phantom sector, as could be expected, describes traversable wormholes. As to possible black holes, it turns out that that there are only two special families (one in the canonical sector and another in the phantom one) that describe extremal black holes (hence having zero Hawking temperature), and the one with a phantom scalar is globally regular. Such results substantially disagree with those of [17], where the same problem was studied numerically with equations written in the Jordan frame, and black hole solutions with simple (finite-temperature) horizons were found. The reason for this disagreement is yet to be understood.

To obtain examples of exact solutions with $V \not\equiv 0$, we have used the previously obtained solutions of GR in which black hole subsets (this time with simple horizons) are generic. Naturally, in the canonical sector this can only happen with at least partly negative potential $V(\phi)$ since the well-known no-hair theorem from GR [50] (on nonexistence of black holes with variable minimally coupled scalar fields with nonnegative potentials) directly extends to the Jordan frame as long as the corresponding conformal factor is well-behaved. Here we again disagree with [17] where a number of HMPG black hole solutions were obtained numerically, and some of them with $V > 0$. Further studies are probably necessary in order to explain this contradiction.

In the phantom sector, generic black hole solutions are of black-universe type [35, 36], there are also wormholes with flat or AdS asymptotics at the far end. And, with both zero and nonzero potentials, there emerge a new kind of singularities due to vanishing of the conformal factor $\cos^2 \psi$; depending on the solution parameters, such singularities may be located in a static region (it is then a singular attracting center) or beyond a black hole horizon (it is then like a big bang or big crunch).

It has turned out that most of the solutions under study are unstable under spherically symmetric monopole perturbations. Some of these instability results have been extended from their counterparts known in GR (but certainly taking into account the boundary conditions formulated in the Jordan frame), some others have been obtained anew, see their summary in Table 1. Only some special solutions prove to be stable, including the well-known black hole with a massless conformal scalar field [27, 28, 63] and a conformally mapped black universe with a horizon at the minimum radius [35, 56].

In conclusion, let us mention some possible directions of continuation or extensions of the present study. First of all, in the case of zero potential ($V = 0$) it is straightforward to obtain similar solutions with electromagnetic fields $F_{\mu\nu}$, by analogy with previous studies in scalar-tensor theories [26, 32]. With nonzero potentials, similar configurations with electromagnetic fields can also be treated both analytically and numerically, e.g., on the basis of known GR solutions [47, 57]. Another trend of interest is a consideration of similar problems in the so-called extended HMPG containing functions $f(R, \mathcal{R})$ of two curvatures [12, 13, 64], whose scalar-tensor representation contains two interacting scalar fields.

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